Decidability

For convenience we use languages to represent various computational problems because we have already set up terminology for dealing with languages. For mathematical object $B$ (be it a graph, DFA, CFG, TM, etc.) we use $\langle B \rangle$ to denote an encoding of the object as a string (so that we can give it as input to a Turing Machine). We say for $\langle B \rangle$ that $B$ is a graph (or DFA, etc.) if $\langle B \rangle$ is a valid encoding of a graph (or DFA, etc), and that graphs is called $B$.

For example, the acceptance problem for DFAs of testing whether a particular finite automaton accepts a given string can be expressed as a language, $A_{DFA}$. This language contains the encodings of all DFAs together with strings that the DFAs accept.

Let $A_{DFA} = \{ \langle B \rangle, w \mid B$ is a DFA that accepts input string $w \}$.

**Theorem 1 (Theorem 4.1, page 166)** $A_{DFA}$ is a decidable language.

We can also prove a similar theorem for nondeterministic finite automata. Let $A_{NFA} = \{ \langle B \rangle, w \mid B$ is a NFA that accepts input string $w \}$.

**Theorem 2 (Theorem 4.2, page 167)** $A_{NFA}$ is a decidable language.

Similarly, we can test whether a regular expression generates a given string. Let $A_{REX} = \{ \langle R \rangle, w \mid R$ is a regular expression that generates string $w \}$.

**Theorem 3 (Theorem 4.3, page 168)** $A_{REX}$ is a decidable language.
In the preceding theorems we had to determine whether a finite automaton accepts a particular string. In the next theorem we determine whether a finite automaton accepts any strings at all. Let $E_{DFA} = \{ \langle A \rangle \mid A \text{ is a DFA and } L(A) = \emptyset \}$.

**Theorem 4 (Theorem 4.4, page 168)** $E_{DFA}$ is a decidable language.

The next theorem states that determining whether two DFAs recognize the same language is decidable. Let $EQ_{DFA} = \{ \langle A \rangle, \langle B \rangle \mid A \text{ and } B \text{ are DFAs and } L(A) = L(B) \}$.

**Theorem 5 (Theorem 4.5, page 169)** $EQ_{DFA}$ is a decidable language.

In the proof, we construct a new DFA $C$ from $A$ and $B$, where $C$ accepts only those strings that are accepted by either $A$ or $B$ but not by both. The language of $C$ is

$$L(C) = (L(A) \cap \overline{L(B)}) \cup (\overline{L(A)} \cap L(B)).$$

This expression is sometimes called the symmetric difference of $L(A)$ and $L(B)$. Here $\overline{L(A)}$ is the complement of $L(A)$.

Here describe algorithms to determine whether a CFG generates a particular string and to determine whether the language of a CFG is empty. Let $A_{CFG} = \{ \langle G \rangle, w \mid G \text{ is a CFG that generates string } w \}$.

**Theorem 6 (Theorem 4.7, page 170)** $A_{CFG}$ is a decidable language.

As we did for DFAs, we can show that the problem of determining whether a CFG generates any strings at all is decidable. Let $E_{CFG} = \{ \langle G \rangle \mid G \text{ is a CFG and } L(G) = \emptyset \}$.

**Theorem 7 (Theorem 4.8, page 171)** $E_{CFG}$ is a decidable language.
Let \( EQ_{CFG} = \{ \langle G \rangle, \langle H \rangle \mid G \text{ and } H \text{ are CFGs and } L(G) = L(H) \} \).

Now we show that every context-free language is decidable by a Turing machine.

**Theorem 8 (Theorem 4.9, page 172)** Every context-free language is decidable.

Now we turn to our first theorem that establishes the undecidability of a specific language: the problem of testing whether a Turing machine accepts a given input string. We call it \( A_{TM} \) by analogy with \( A_{DFA} \) and \( A_{CFG} \). But, whereas \( A_{DFA} \) and \( A_{CFG} \) were decidable, \( A_{TM} \) is not. Let \( A_{TM} = \{ \langle M \rangle, w \mid M \text{ is a TM that accepts } w \} \). \( A_{TM} \) is sometimes called the halting problem.

**Theorem 9 (Theorem 4.11, page 174)** \( A_{TM} \) is Turing recognizable, and undecidable.

**Definition 10 (Definition 4.12, page 175)** Assume that we have two sets \( A \) and \( B \) and a function \( f \) from \( A \) to \( B \). Say that \( f \) is one-to-one if it never maps two different elements to the same place, that is, if \( f(a) \neq f(b) \) whenever \( a \neq b \). Say that \( f \) is onto if it hits every element of \( B \), that is, for every \( b \in B \) there is an \( a \in A \) such that \( f(a) = b \). Say that \( A \) and \( B \) are the same size if there is a one-to-one, onto function \( f : A \to B \). A function that is both one-to-one and onto is called a correspondence. In a correspondence every element of \( A \) maps to a unique element of \( B \) and each element of \( B \) has a unique element of \( A \) mapping to it. A correspondence is simply a way of pairing the elements of \( A \) with the element of \( B \).

Let \( \mathcal{N} \) be the set of natural numbers \( \{1, 2, 3, \ldots\} \).
Definition 11 (Definition 4.14, page 175) A set $A$ is countable if either it is finite or it has the same size as $\mathbb{N}$.

Theorem 12 (Examples, pages 174-7) The set of even positive integers is countable. The set of positive rational numbers $\mathbb{Q}$ is countable.

Some infinite sets no correspondence with $\mathbb{N}$ exists. These sets are called uncountable. Let $\mathcal{R}$ be the set of real numbers.

Theorem 13 (Theorem 4.17, page 177) $\mathcal{R}$ is uncountable.

Corollary 14 (Corollary 4.18, page 178) Some languages are not Turing-recognizable.

In the proof, an infinite binary sequence is an unending sequence of 0s and 1s. Let $\mathcal{B}$ be the set of all infinite binary sequences. Let $\mathcal{L}$ be the set of all languages over alphabet $\Sigma$. Let $\Sigma^* = \{s_1, s_2, s_3, \ldots\}$. Each language $A \in \mathcal{L}$ has a unique sequence in $\mathcal{B}$. The $i$th bit of that sequence is a 1 if $s_i \in A$ and a 0 if $s_i \notin A$, which is called the characteristic sequence of $A$.

We say a language is co-Turing-recognizable if it is the complement of a Turing-recognizable language.

Theorem 15 (Theorem 4.22, page 181) A language is decidable if and only if it is both Turing-recognizable and co-Turing-recognizable.

In other words, a language is decidable if and only if both it and its complement are Turing-recognizable.

Corollary 16 (Corollary 4.23, page 182) $\overline{A_{TM}}$ is not Turing-recognizable.