1 Minimum Spanning Trees Definitions

$G = (V, E)$ is an undirected graph whose edges have weight w. A subgraph of G is called *spanning* if it has V as its vertex set. A spanning subgraph of G is identified with its set of edges.

Let A be a set of edges $A \subseteq E$. We say that edge e is **safe** for A if the following property holds: if A is contained in some minimum spanning tree, then $A \cup \{e\}$ is contained in some minimum spanning tree.

A **cut** $(S, \bar{S})$ of a graph is a partition of V into two nonempty sets S and $\bar{S} = V \setminus S$.

We say that an edge e **crosses** a cut $(S, \bar{S})$ if one of the endpoints of e is in S and the other endpoint is in $\bar{S}$.

We say that a cut **respects** a set of edges A if no edge of A crosses the cut.

An edge is a **light** edge crossing a cut if its weight is the minimum of any edge crossing the cut.

Theorem 1.1 Blue Rule. *If the edge e is light for some cut which respects the set of edges A, e is safe for A.*

Proof Sketch. (some details missing). Assume that there exists a minimum spanning tree T (viewed as a set of edges) which contains A, that $(S, \bar{S})$ is a cut that respects A, and that e is a minimum-weight edge crossing $(S, \bar{S})$. If $e \in T$, we are done, so let us assume $e \notin T$. Let u and v be the endpoints of e, with $u \in S$ and $v \in \bar{S}$. The unique path in T from u to v must have an edge e' such that e' crosses $(S, \bar{S})$ and $e' \notin A$ since $(S, \bar{S})$ respects A.

We have that $T \setminus \{e'\} \cup \{e\}$ is another tree since it has the same number of edges and it is connected. And its weight is no more than the weight of T, since e is a minimum-weight edge crossing $(S, \bar{S})$. Then $T \setminus \{e'\} \cup \{e\}$ is also a minimum spanning tree, and it contains $A \cup \{e\}$.