1. We could get the same effect as splaying (moving the accessed item to the root) using only zig and zag steps, not bothering with the more complicated zig-zig, zig-zag, zag-zig, and zag-zag steps. If we do that, we only need Case 1 on pages 4–5 in the notes. What is the amortized cost of an access with such simple splaying?

2. Consider a new operation on splay trees, \texttt{cut}\((i_1, i_2, T)\) that deletes all items \(i_1 \leq i \leq i_2\) from the splay tree \(T\). Implement the \texttt{cut} operation so that the amortized cost of a \texttt{cut} is logarithmic and analyze the amortized cost as in the table at the bottom of page 10 in the notes. \texttt{Hint:} This problem is easy.

3. In this exercise we prove that a splay tree is as good, to within a constant multiple, of any static binary search tree.
   
   (a) Given any binary search tree, let \(d(x)\) be the depth of (an internal) node \(x\), that is, the number of internal nodes on the path from the root to \(x\), so \(d(\text{root}) = 1\) and the cost of accessing \(x\) in the given tree is \(O(d(x))\). Prove the pseudo-Kraft inequality \(\sum_{\text{internal nodes} x \in T} 3^{-d(x)} \leq 1\) for any finite binary tree. Prove that the sum is exactly 1 for the complete infinite binary tree.
   
   (b) Define the weight of a node \(x\) in a splay tree as \(w(x) = 3^{-d(x)}\). Prove that the amortized cost of accessing item \(x\) in a splay tree is \(O(1 + d(x))\).

4. Derive the bound for \texttt{insert} given in the table at the top of page 11 of the splay tree notes.

5. PhD Qualifying Exam Section Problem 5.
   
   Redefine the potential of a splay tree to be
   \[
   \Phi(T) = \sum_{\text{nodes } x \in T} \alpha r(x),
   \]
   where \(\alpha = 3/\lg(27/4) \approx 1.089\).

   (a) Using this new potential function, prove that the amortized cost of the \texttt{ZIG-ZIG} step at \(x\) is at most \(3\alpha [r'(x) - r(x)]\). \texttt{Hint:} In the notes we proved that \(2 + r'(x) + r'(y) + r'(z) - r(x) - r(y) - r(z) \leq 3[r'(x) - r(x)];\) multiply both sides of this inequality by \(\alpha\).

   (b) Consider the more complicated rule, the \texttt{ZAG-ZIG-ZIG} step which uses 3 rotations (at \(w\), then at \(z\), then at \(y\)) to transform
We want to derive an upper bound of $3\alpha[r'(x) - r(x)]$ on the amortized cost of such a step.

i. Prove that the amortized cost of the $ZAG-ZIG-ZIG$ step is

$$\hat{c} = 3 + \alpha[r'(x) + r'(y) + r'(z) + r'(w) - r(x) - r(y) - r(z) - r(w)]$$

and that we can get the desired bound on the amortized cost of such a step by proving that

$$\alpha[3r'(x) - 2r(x) - r'(y) + r(y) - r'(z) - r'(w) + r(w)] \geq 3.$$ 

ii. Using the fact that $r(y) \geq r(w) \geq r(x)$, show that this last inequality follows from proving that

$$3r'(x) - r'(y) - r'(z) - r'(w) \geq 3/\alpha,$$

which follows in turn from

$$-2\log \frac{S'(y)}{S'(x)} - \log \frac{S'(w)}{S'(x)} \geq 3/\alpha.$$

Explain why this inequality must hold.