1. For each set $A$ (and $B$), we define polynomial $f_A$ (and $f_B$) such that coefficients are either 0 or 1 depending on whether $i \in A$, where $a_i = 1$ or $i \notin A$, where $a_i = 0$ (and similarly, on whether $i \in B$, where $b_i = 1$ or $i \notin B$, where $b_i = 0$).

Now, representing $f_C := f_A \cdot f_B$ in coefficient form, we note that the $i^{th}$ coefficient $c_i$ is equal to the number of times it is a sum of an element from each of $A$ and $B$. Since we can perform the polynomial multiplication in time $O(n \lg n)$ (from what we learnt in class), we can get the final answer in time $O(n \lg n)$.

Since each coefficient can only be 1, the final coefficient is counting the total number of times $x^k$ becomes the sum of two elements from $A$ and $B$ (where $a_j \neq 0$ and $b_{k-j}$).

2. We must show,

$$DFT_{2n}(a \otimes b) = DFT_{2n}(a) \cdot DFT_{2n}(b)$$

Replacing $n$ with $2n$, and using the formula in the text, we get the following three equations

$$y_k^a = a_0w_{2n}^0 + a_1w_{2n}^2 + a_2w_{2n}^{2k} + \cdots + a_{2n-1}w_{2n}^{(2n-1)k}$$

$$y_k^b = b_0w_{2n}^0 + b_1w_{2n}^2 + b_2w_{2n}^{2k} + \cdots + b_{2n-1}w_{2n}^{(2n-1)k}$$

and

$$y_k^c = c_0w_{2n}^0 + c_1w_{2n}^2 + c_2w_{2n}^{2k} + \cdots + c_{2n-1}w_{2n}^{(2n-1)k}$$
Using $c_j = \sum_{k=0}^{j} a_k b_{j-k}$, we have

$$y_k^c = a_0 b_0 w_{2n}^0 + (a_0 b_1 + a_1 b_0) w_{2n}^1 + \cdots + \left( \sum_{i=0}^{2n-1} a_i b_{2n-1-i} \right) w_{2n}^{(2n-1)k} \quad (1)$$

Next, $y_k^a \times y_k^b =

\left( a_0 w_{2n}^0 + a_1 w_{2n}^k + a_2 w_{2n}^{2k} + \cdots + a_{2n-1} w_{2n}^{(2n-1)k} \right) \left( b_0 w_{2n}^0 + b_1 w_{2n}^k + b_2 w_{2n}^{2k} + \cdots + b_{2n-1} w_{2n}^{(2n-1)k} \right)

= a_0 b_0 w_{2n}^0 + (a_0 b_1 + a_1 b_0) w_{2n}^1 + \cdots + \left( \sum_{i=0}^{2n-1} a_i b_{2n-1-i} \right) w_{2n}^{(2n-1)k}

which is the same as equation (1).

4. For $n$ is a power of 3, the cube of $n$-th complex roots of unity are the $n/3$ complex $(n/3)$-th roots of unity. As a result, the halving lemma and cancellation lemma modified.

Cancellation lemma: $(\omega_n^k)^3 = (\omega_{n/3}^k)$

Halving lemma: $(\omega_n^{k+n/3})^3 = \omega_n^3 \omega_n^k = (\omega_{n/3}^k)^3$

Because $A(x) = \sum_{j=0}^{n-1} a_j x^j$, FFT applies a divide & conquer strategy. We define $A[i] = \sum_{j=0}^{n/3-1} x^j$ where $i = 0, 1, 2$.

$$A[0] = a_0 + a_3 x + a_6 x^2 + \cdots + a_{n-3} x^{n/3-1}$$
$$A[1] = a_1 + a_4 x + a_7 x^2 + \cdots + a_{n-2} x^{n/3-1}$$
$$A[2] = a_2 + a_5 x + a_8 x^2 + \cdots + a_{n-1} x^{n/3-1}$$

Thus, we get $A(x) = A[0](x^3) + x A[0](x^3) + x^2 A[2](x^3)$.

$$\implies T(n) = 3T(\frac{n}{3}) + \Theta(n) = 9T(\frac{n}{9}) + 3\Theta(\frac{n}{3}) + \Theta(n) = \cdots = \Theta(n \log_3 n) = \Theta(n \log n)$$