Dynamic Table Slides

Mitsunori Ogihara
A **dynamic table** is a table of variable size, where an **expansion** (or a **contraction**) is caused when the load factor has become larger (or smaller) than a fixed threshold.

Let the expansion threshold be 1 and the expansion rate be 2; that is, *the table size is doubled when an item is to be inserted when the table is full.*

Let the contraction threshold be 1/4 and the contraction rate be 1/2; that is, *the table size is halved when an item is to be eliminated when the table is exactly 1/4 full.*
When these operations take place we create a new table and move all the elements from the old one to the new one.

Suppose that there are $n$ calls of insertion and deletion are made, what is the average cost of each operation?
If the size is kept the same the cost is $O(1)$.

If the size is doubled from $M$ to $2M$, the actual cost is $M + 1$. The time that it takes for the next table size change to occur is at least $M$ steps for doubling and at least $M/2$ steps for halving. So the actual cost can be spread over the next $M/2$ “normal” steps. This gives an amortized cost of $O(1)$.

If the size is halved from $M$ to $M/2$, the actual cost is $M/4$. The time that it takes for the next table size change to occur is at least $M/4$ steps for doubling and at least $M/8$ steps for halving. So the actual cost can be spread over the next $M/8$ steps to yield an amortized cost of $O(1)$. 
For each $i$, $1 \leq i \leq n$, define $c_i$ to be the number of insertions and deletions that are executed at the $i$-th operation, and define

$$\Phi_i = \begin{cases} 
2\text{num}_i - \text{size}_i & \text{if } \alpha_i \geq \frac{1}{2}, \\
\frac{\text{size}_i}{2} - \text{num}_i & \text{if } \alpha_i < \frac{1}{2},
\end{cases}$$

Here $\text{size}_i$ is the table size, $\text{num}_i$ is the number of elements in the table, and $\alpha_i$ is the ratio $\text{num}_i/\text{size}_i$ after the $i$-th operation.

Note that

- at time 0, the table is empty, so $\Phi_0 = 0$,
- for all $i$, $\Phi_i \geq 0$, and thus, $\Phi_n \geq \Phi_0$, and
- $\Phi_n \leq 2n - n = n$, so the contribution of the potential function to the amortized cost is at most 1.
Here \( m = \text{num}_{i-1} \) and \( s = \text{size}_{i-1} \)

(a) \( \alpha_{i-1} = 1 \): Here \( m = s \).

\[
\begin{array}{c|c|c|c}
    c_i & \Phi_i & \Phi_{i-1} & \hat{c}_i \\
    \hline
    m + 1 & 2(m + 1) - 2s & 2m - s & 3
\end{array}
\]

(b) \( \frac{1}{2} \leq \alpha_{i-1} < 1 \):

\[
\begin{array}{c|c|c|c}
    c_i & \Phi_i & \Phi_{i-1} & \hat{c}_i \\
    \hline
    1 & 2(m + 1) - s & 2m - s & 3
\end{array}
\]

(c) \( \alpha_i = \frac{1}{2} \): Here \( m + 1 = \frac{s}{2} \).

\[
\begin{array}{c|c|c|c}
    c_i & \Phi_i & \Phi_{i-1} & \hat{c}_i \\
    \hline
    1 & 2(m + 1) - s & s/2 - m & 0
\end{array}
\]

(d) \( \alpha_i < \frac{1}{2} \):

\[
\begin{array}{c|c|c|c}
    c_i & \Phi_i & \Phi_{i-1} & \hat{c}_i \\
    \hline
    1 & s/2 - m - 1 & s/2 - m & 0
\end{array}
\]

So the amortized cost of insertion is \( O(1) \).
(a) $\alpha_i \geq \frac{1}{2}$:

<table>
<thead>
<tr>
<th>$c_i$</th>
<th>$\Phi_i$</th>
<th>$\Phi_{i-1}$</th>
<th>$\hat{c}_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1$</td>
<td>$2(m-1) - s$</td>
<td>$2m - s$</td>
<td>$-1$</td>
</tr>
</tbody>
</table>

(b) $\alpha_{i-1} = \frac{1}{2}$: Here $2m = s$.

<table>
<thead>
<tr>
<th>$c_i$</th>
<th>$\Phi_i$</th>
<th>$\Phi_{i-1}$</th>
<th>$\hat{c}_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1$</td>
<td>$\frac{s}{2} - (m-1)$</td>
<td>$2m - s$</td>
<td>$2$</td>
</tr>
</tbody>
</table>

(c) $\frac{1}{4} < \alpha_{i-1} \leq \frac{1}{2}$:

<table>
<thead>
<tr>
<th>$c_i$</th>
<th>$\Phi_i$</th>
<th>$\Phi_{i-1}$</th>
<th>$\hat{c}_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1$</td>
<td>$s/2 - (m-1)$</td>
<td>$s/2 - m$</td>
<td>$2$</td>
</tr>
</tbody>
</table>

(d) $\alpha_{i-1} = \frac{1}{4}$: $m = \frac{s}{4}$ and $\alpha_i < \frac{1}{2}$.

<table>
<thead>
<tr>
<th>$c_i$</th>
<th>$\Phi_i$</th>
<th>$\Phi_{i-1}$</th>
<th>$\hat{c}_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m$</td>
<td>$s/4 - (m-1)$</td>
<td>$s/2 - m$</td>
<td>$1$</td>
</tr>
</tbody>
</table>

So the amortized cost of deletion is $O(1)$. 6