Satisfaction, Validity, and State Updates
CS 536, Mon Jan 23, 2012

A. Why
- A predicate is satisfied or unsatisfied relative to a state.
- A predicate is valid if it is satisfied in all states.
- State updates occur when we introduce new variables or change the values of existing variables.

B. Outcomes
At the end of today, you should
- Know how to check a predicate for satisfaction in a state.
- Know how to check a predicate for validity
- Know how to update a state.

C. Questions
1. Is $\{x = 1, x = 2\}$ a proper state?
2. Let $\sigma = \{x = 0, y = 5, b = (2, 4, 6, 8)\}$.
   a. Does $\sigma \models x+y < b[b[0]]$?
   b. Does $\sigma \models \forall\, i \,.\, 0 \le i < sizeof(b) \rightarrow x < b[i]$?
   c. Does $\sigma \models \exists\, i \,.\, 0 \le i < sizeof(b) \wedge b[i] > y$?
3. Is $0 \le i < sizeof(b) \rightarrow b[i] < 9$ valid? (If not, give a counterexample.)
4. Is $b[i] < 9$ valid? (If not, give a counterexample.)
5. Let $\sigma = \{x = 3, y = 4, b = (7, 9)\}$.
   a. Does $\sigma \models x > sizeof(b)$?
   b. Does $\sigma \models x < y \wedge b[x-2] < b[y-3]$?
6. Is $\models x = x+0$? (I.e., is it valid?) (If not, give a counterexample.)
7. Let $\sigma = \{x = 3, y = 4, b = (7, 9)\}$.
   a. What is $\sigma(x)$? $\sigma(y)$? $\sigma(z)$?
   b. What is $\sigma[x \mapsto 9]$? $\sigma[z \mapsto F]$? $\sigma[x \mapsto 9][y \mapsto 10][x \mapsto 4]$?
   c. What is $\sigma[b[1] \mapsto 6]$?
   d. Are $\sigma[x \mapsto 9][y \mapsto 10]$ and $\sigma[y \mapsto 10][x \mapsto 9]$ equal?

e. Say u and v are variables (possibly the same) and α and β are values (possibly equal). When is $\sigma[u \mapsto \alpha][v \mapsto \beta] = \sigma[v \mapsto \beta][u \mapsto \alpha]$?

8. Let $\sigma = \{x = 3, y = 4, b = (7, 9, 12, 16)\}$.

   a. Does $\sigma \models \exists\, k \,.\, b[k] < b[k+1]$? If so, what witness did you use? Are there other possible witnesses?

   b. What satisfactions would we have to check in order to see if $\sigma \models \forall\, j \,.\, 0 \le j \le 2 \rightarrow b[j] < b[j+1]$?

   c. Does $\sigma \models \exists\, z \,.\, b[z] < b[z+1]$? Does it matter that z is undefined in σ?

   d. Does $\sigma \models \exists\, x \,.\, b[x] < b[x+1]$? Does it matter that $\sigma(x)$ is defined or what value it has?

9. Is $\models \forall\, x \,.\, \exists\, y \,.\, y > x^2$? Does it matter [for purposes of satisfaction] that the witness for y would have to depend on the value of x?

10. Is $\models \exists\, x \,.\, \forall\, y \,.\, 0 \le y \wedge y \le 20 \rightarrow x \ge y^2$?

D. Solutions

1. No (we can't have > 1 binding for a variable).
2. Yes; Yes; Yes (witnesses for $i = 2$ or 3)
3. No. A counterexample could be
   {b = (0, 0)} -- oops! no, this state does ⊨ the predicate
   {b = (10, 11), i = 0} would be a counterexample.
4. No; same counterexample is okay.
5. Yes; No
6. Yes — For any value α, if we take the state $\{x = \alpha\}$, we get $\{x = \alpha\} \models x = x+0$
7. (a) 3, 4, undefined.
   (b.1) $\{x = 9, y = 4, b = (7, 9)\}$
   (b.2) $\{x = 3, y = 4, b = (7, 9), z = F\}$
   (b.3) $\{x = 4, y = 10, b = (7, 9)\}$:
   $\sigma = \{x = 3, y = 4, b = (7, 9)\}$, so
   $\sigma[x \mapsto 9] = \{x = 9, y = 4, b = (7, 9)\}$, so
   $\sigma[x \mapsto 9][y \mapsto 10] = \{x = 9, y = 10, b = (7, 9)\}$, so
   $\sigma[x \mapsto 9][y \mapsto 10][x \mapsto 4]$
   $= \{x = 4, y = 10, b = (7, 9)\}$.
   (c) $\{x = 3, y = 4, b = (7, 6)\}$

(d) $\sigma[x \mapsto 9][y \mapsto 10] = \{x = 9, y = 10, b = (7, 9)\}$
$= \sigma[y \mapsto 10][x \mapsto 9]$
$= \{x = 9, y = 10, b = (7, 9)\}$

