Activity/Homework: Meanings of Programs; Satisfaction; Trivial Triples

A. Why?

To understand how programs work, we must understand what they mean. To know if a program meets its specification, we have to know when predicates are satisfied.

B. Outcomes

By the end of the activity you should

- Be able to calculate $\mathcal{M}[\![S]\!](\sigma)$ for short programs S.
- Be able to check whether or not a predicate is satisfied in a state. (I.e., is $\sigma \vDash p$?)

C. Questions

We'll do some of these questions as an activity; do the rest as homework.
For the activity, let's do questions 1, 2.

Meanings of Programs

1. If $\sigma_0(x) = 5$ and $\sigma_0(y) = 9$, what is $\mathcal{M}[\![x := x{+}1 \ ; \ y := y * x]\!](\sigma_0)$?

Group 6: $\mathcal{M}[\![x{:=}x{+}1;\ y{:=}y^*x]\!](\sigma_0)$

$= \sigma_0[x := \alpha{+}1;\ y := y^*\alpha]$ where $\alpha = \sigma_0(x)$

$= \sigma_0[x := \sigma_0(x){+}1;\ y := y^*\sigma_0(x)]$

$= \sigma_0[x := 5{+}1;\ y := 9^*6]$

$= \sigma_0[x := 6;\ y := 54]$

$\mathcal{M}[\![x{:=}6;\ y{:=}54]\!]$

So $\sigma_0[x := \alpha{+}1;\ y := y^*\alpha]$ means you're updating twice?

$\sigma_0[x := \alpha{+}1][y := y^*\alpha]$

An update needs to be like ...[y := value], not ...[y := expr]

[I'd write this as;]

$\mathcal{M}[\![x{:=}x{+}1;\ y{:=}y^*x]\!](\sigma_0) = \{\ \sigma_0[x{:=}6][y{:=}54]\ \}$

$\mathcal{M}[\![x{:=}x{+}1;\ y{:=}y^*x]\!](\sigma_0)$

$= \mathcal{M}[\![y{:=}y^*x]\!](\mathcal{M}[\![x{:=}x{+}1]\!](\sigma_0))$

$= \mathcal{M}[\![y{:=}y^*x]\!](\{\sigma_0[x := \sigma_0(x{+}1)]\})$

$= \mathcal{M}[\![y{:=}y^*x]\!](\{\sigma_0[x := 6]\})$

$= \{\sigma_0[x := 6][y{:=}\ \sigma_0[x := 6](y^*x)]\}$

$= \{\sigma_0[x := 6][y{:=}\ 54]\}$

because

$$\sigma_0[x := 6](y * x) = \sigma_0[x := 6](y) * \sigma_0[x := 6](x)$$
$$= \sigma_0(y) * 6 = 9*6 = 54$$

2. For the same σ_0, what is $\mathcal{M}[\![y := y * x;\ x := x+1]\!](\sigma_0)$?

Group 5: $\sigma_0[y := \sigma_0(y) * \sigma_0(x)][x := \sigma_0(x)+1]$
$= \sigma_0[y := 9*5][x := 5+1]$
$= \sigma_0[y := 45][x := 6]$

Technically we need curly braces (for singleton set), but again, not a big deal.

3. Let σ_1 be a state and let $S \equiv$ **if** $v > w$ then $w := w*2$ **else** $v := v*3$ **fi**. What is $\mathcal{M}[\![S]\!](\sigma_1)$? Note: you'll have two cases, depending on the relationship between $\sigma_1(v)$ and $\sigma_1(w)$. The new values of v and w will involve $\sigma_1(v)$ and $\sigma_1(w)$.
4. Let $\Omega \equiv$ **while true do skip od**. For any σ, what is the sequence of test states (τ_0, τ_1, ….) for $\mathcal{M}[\![\Omega]\!](\sigma)$? What set is the value of $\mathcal{M}[\![\Omega]\!](\sigma)$?
5. Let $W \equiv$ **while** $x \neq 0$ **do** $x := x\text{-}1$ **od**. Let $\sigma'(x) = 3$. What is the sequence of test states for $\mathcal{M}[\![W]\!](\sigma')$? What is the value of $\mathcal{M}[\![W]\!](\sigma')$?
6. Let $\sigma''(x) = -4$. What is the sequence of test states for $\mathcal{M}[\![W]\!](\sigma'')$? [Same W as in the previous question.] What is the value of $\mathcal{M}[\![W]\!](\sigma'')$?

Predicate Satisfaction

7. Give an example of a state that $\models 0 < m < n$. Also, give an example of a state that $\not\models 0 < m < n$. (The state should be "proper" — define type-correct values for m and n — even though those values don't satisfy $0 < m < n$.)
8. For an existential, $\sigma \models \exists x \in T\ .\ p$ iff there's some value (of type T) for x that makes σ updated at x with α satisfy p. E.g., $\sigma \models \exists x\ .\ x^2 < 1$ (where x ranges over the integers); we can use 0 for the "witness" α and find $\sigma[x := 0] \models x^2 < 1$. In general, it's possible to have many witnesses. What are the possible witnesses α that can be used to show $\sigma \models \exists x\ .\ x^2 > 1$?
9. If we have a set of states X, then "X satisfies p" (written $X \models p$) means that every state in X satisfies p. (And if $X = \varnothing$, then $X \models p$ automatically.) Now, in general, $\sigma \models p \rightarrow q$ iff $\sigma \models p$ implies that $\sigma \models q$. Question: Say $\sigma \models p \rightarrow q$ for all states σ. If $X \models p$, then does $X \models q$ also?

Trivial Partial Correctness Triples

10. Remember, $\sigma \models \{p\}\ S\ \{q\}$ iff ($\sigma \not\models p$) or ($M[\![S]\!](\sigma) = \emptyset$) or $M[\![S]\!](\sigma) \models q$. There are three cases in which $\sigma \models \{p\}\ S\ \{q\}$ kind of trivially:
    - (a) When $\sigma \not\models p$ for every σ.
    - (b) When $M[\![S]\!](\sigma) = \emptyset$ for every σ.
    - (c) When $\tau \models q$ for every state τ (because then the ones $\in M[\![S]\!](\sigma)$ must $\models q$ too).

Give an example of a p that fits case (a). Give an example of an S that fits case (b). Give an example of a q that fits case (c).