CS 538 Combinatorial Optimization **Fall Semester, 2020**

Discussion of Integer Linear Programs

Was an assignment in some previous year

The two problems below ask to formulate integer programs for a maximization combinatorial optimization problem and for estimating the "integrality ratio" (precise definition below) of the linear program relaxation. Precisely, you must:

1. Show how to construct an INTEGER PROGRAMMING instance I_{IP} for every instance I of the combinatorial otimization problem.

2. Prove that I_{IP} has a feasible solution if and only I has a feasible solution

3. Let $Z^*(I)$ be the value of the optimum feasible solution for I and $Z^*_{IP}(I)$ be the value of the optimum feasible solution for I_{IP}. Prove $Z^*(I) = Z^*_{IP}(I)$ for every feasible instance I.

4. Consider the LINEAR PROGRAMMING instance I_{LP} which is obtained from I_{IP} by dropping the integrality constrains, and let $Z^*_{LP}(I)$ be the value of the optimum feasible solution for I_{LP}. Since we have maximization problems, it is true that for any instance I we have $Z^*_{LP}(I) \geq Z^*_{IP}(I)$.

5. Give examples of instances I where $Z^*_{LP}(I)/Z^*(I)$ is "large", making this ratio as large as you can for the given combinatorial optimization problem. As both KNAPSACK and MAXIMUM INDEPENDENT SET are NP-Hard you should be able to find instances I with $Z^*_{LP}(I)/Z^*(I) > 1$. The ratio $Z^*_{LP}(I)/Z^*(I)$ is a function of I, and to make it large it may be necessary to consider an infinite sequence of intances I_j, with j being either the number of items or vertices, and the ratio a function of j.

6. For KNAPSACK, it is possible to show $Z^*_{LP}(I)/Z^*(I) \leq 2$. To do this, you must use the condition $S \geq s_i$ for all $i = 1, 2, \ldots, n$, and a greedy algorithm typically covered in CS 430 for solving the linear program (known as the FRACTIONAL KNAPSACK problem). Then, from the solution to the linear program, you must somehow obtain an integer solution of half $Z^*_{LP}(I)$.

Problem 1 Consider the KNAPSACK problem, defined as follows (the textbook uses another definition; here you use this one). An instance consists of n items $1, 2, \ldots, n$ where item i has size s_i and profit p_i, and a knapsack size S with $S \geq s_i$ for all $i = 1, 2, \ldots, n$. A feasible solution consists of a subset B of $\{1, 2, \ldots, n\}$ such that $\sum_{i \in B} s_i \leq S$. The objective is to maximize the total profit of B - that is $\sum_{i \in B} p_i$.

1. formulate KNAPSACK as an Integer Program (IP)

2. consider the Linear Program (LP) relaxation that can be obtained from IP by droping the integrality constraints. Find upper and lower bounds for the ratio of the optimum values of IP and LP. Make these bounds as close to each other as you can.

Problem 2 Consider the Maximum Independent Set Problem: Given a graph, find a maximum size set of vertices no two of which are joined by an edge.

1. formulate Maximum Independent Set as an Integer Program (IP)
2. write the Linear Program (LP) relaxation that can be obtained from IP by droping the integrality constraints. Find instances (graphs) where the optimum of LP is very large compared to the IP optimum.