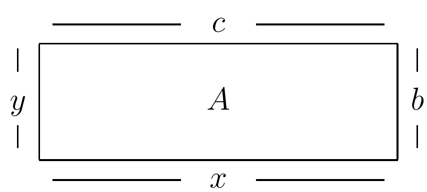
1 Linear Programming Duality (based on Karloff's book).

Here is how we build the dual in general. For each *constraint* in the primal, there is a *variable* in the dual. For each *variable* in the primal, there is a *constraint* in the dual.

	PRIMAL	DUAL		
row i	$\sum_j a_{ij}x_j = b_i$	y_i	$\lesseqgtr$	0
row i	$\sum_j a_{ij}x_j \geq b_i$	y_i	$\geq$	0
var j	$x_j \lesseqgtr 0$	$\sum_{i=1}^{m} y_i a_{ij}$	$=$	c_j
var j	$x_j \geq 0$	$\sum_i y_i a_{ij}$	$\leq$	c_j
min	$c^T x$	max		$y^T b$

To get the dual, a maximization problem, first turn any $\leq$ constraints in the primal into $\geq$'s by negating both sides. Put the coefficients of the objective function on the right-hand side of the dual. Read down the primal's columns and use the entries in one column to write down one dual constraint. A dual variable y_i is sign-constrained ($y_i \geq 0$) if and only if the corresponding primal constraint is an inequality; a dual constraint is an inequality if and only if the corresponding primal variable is sign-constrained ($x_j \geq 0$). Remember, inequality in one problem corresponds to inequality in the other. Equality in one corresponds to $\lesseqgtr$ in the other.



Theorem 1.1 *The dual of the dual is the primal.*

From our construction of the dual, a reasonable conjecture would be that if w is feasible in the primal and u is feasible in the dual, then $c^T w \geq u^T b$. It *is* true. (If A is $m \times n$, c and w are n-vectors and u and b are m-vectors.)

Theorem 1.2 *If w, u is a primal/dual feasible pair, $c^T w \geq u^T b$.*

Theorem 1.3 *If a linear program has an optimal solution, so does its dual, and their optimal costs are identical.*

A lot of mathematics (e.g., combinatorial optimization, mathematical economics) is based on this theorem!

The dual of an LP in Standard form is an LP in pseudo-packing form, as shown below.

	PRIMAL	DUAL	
min	$c^T x$	max	$b^T y$
s.t.	$Ax = b$	s.t.	$A^T y \leq c$
	$x \geq 0$		

Corollary 1.4 *If we run Simplex on LPS, at termination* $(B^{-1})^T c_B$ *is dual optimal (if an optimal point exists).*

If the primal is unbounded, the dual must infeasible. Otherwise, how could $c^T w \geq u^T b$, if $c^T w$ can be made arbitrarily small? Analogously, if the dual is unbounded, the primal is infeasible.

Corollary 1.5 *Exactly one of these three cases occurs:*

1. *Primal and dual are both infeasible.*
2. *One is unbounded and the other is infeasible.*
3. *Both have optimal points.*

Theorem 1.6 *(***Complementary Slackness***)*
Let P *be a linear program in general form:*

$$\begin{array}{rrcll} P: & \min c^T x & & & \\ & \textit{s.t. } A^i x & = & b_i, & 1 \leq i \leq h \\ & A^i x & \geq & b_i, & h+1 \leq i \leq m \\ & x_j & \geq & 0, & 1 \leq j \leq l \\ & x_j & \lesseqgtr & 0, & l+1 \leq j \leq n. \end{array}$$

$$\begin{array}{rrcll} \textit{Let its dual be } D: & \max y^T b & & & \\ & \textit{s.t. } y_i & \lesseqgtr & 0, & 1 \leq i \leq h \\ & y_i & \geq & 0, & h+1 \leq i \leq m \\ & A_j^T y & \leq & c_j, & 1 \leq j \leq l \\ & A_j^T y & = & c_j, & l+1 \leq j \leq n. \end{array}$$

Let w *be primal feasible and let* u *be dual feasible. Then* w *is primal optimal and* u *is dual optimal if and only if*

$$(A^i w - b_i) u_i = 0 \textit{ for } i = 1, 2, ..., m$$

and

$$w_j (c_j - A_j^T u) = 0 \textit{ for } j = 1, 2, ..., n.$$

(If a dual variable is nonzero, the corresponding primal constraint must be tight. If a primal variable is nonzero, the corresponding dual constraint must be tight.)

Farkas' Lemma is a remarkably simple characterization of those linear systems that have solutions. Via the Duality Theorem, its proof is trivial.

Theorem 1.7 $Ax \leq b$ *has a solution if and only if there is no nonnegative vector* y *satisfying* $A^T y = 0$ *and* $b^T y < 0$.

A similar result, proven in the same way, is also known as Farkas' Lemma:

Theorem 1.8 $Ax = b$, $x \geq 0$ *has a solution if and only if there is no vector* $y \lesseqgtr 0$ *satisfying* $A^T y \leq 0$ *and* $b^T y > 0$.