1. Write down the generating functions for the sequences whose *n*th terms are

   i) the number of partitions of *n* into parts equal to 3 or 5.
   ii) the number of partitions of *n* in which only parts which are powers of 2 can appear more than once.

Solution:

i) $\dfrac{1}{(1-x^3)(1-x^5)}$

ii) $\prod_{i\_not\_a\_power\_of\_2} (1+x^i)\left(\prod_i (1-x^{(2i)})\right)^{-1}$

2. By multiplying the relevant power series, determine the coefficient of x^9 in

$$\frac{1}{(1-x)(1-x^2)(1-x^3)}$$

Interpret your result as the number of partitions of a certain kind, and check your answer by listing the partitions explicitly.

Solution:

We have:

$$(1-x)^{-1}(1-x^2)^{-1}(1-x^3)^{-1} = (1+x+x^2+\ldots)(1+x^2+x^4+\ldots)(1+x^3+x^6+\ldots)$$

Expanding this we see that the coefficient of x^9 is 12. Equivalently, the number of partitions of 9 in which every part is 1, 2, or 3 is 12. They can be listed as

5 with no 3's : $2^4 1$, $2^3 1^3$, $2^2 1^5$, 21^7, 1^9 ;
4 with one 3 : 32^3, $32^2 1^2$, 321^4, 31^6;
2 with two 3's : $3^2 21$, $3^2 1^3$;
1 with three 3's: 3^3

3. Using a method based on generating functions prove that the number of partitions of n in which no even number occurs more than once as a part is equal to the number of partitions of n in which no part occurs more than three times.

Solution:

The generating function for the number of partitions of n such that no even number occurs more than once as a part is

$$G(x) = \frac{(1+x^2)(1+x^4)(1+x^6)\ldots}{(1-x)(1-x^3)(1-x^5)\ldots}$$

The generating function for the number of partitions of n such that each part occurs at most 3 times is

$$H(x) = (1+x+x^2+x^3)(1+x^2+x^4+x^6)(1+x^3+x^6+x^9)\ldots$$

We have that $(1+y+y^2+y^3) = (1-y^4)/(1-y)$, and hence

$$H(x) = \frac{(1-x^4)(1-x^8)(1-x^{12})\ldots}{(1-x)(1-x^2)(1-x^3)\ldots}$$

Now write $1-x^4 = (1+x^2)(1-x^2), 1-x^8 = (1+x^4)(1-x^4)$, etc.., and cancel the terms occurring on the top and bottom.

4. Show that the generating function for the sequence defined by

$$u_0 = 1, \quad u_{n+1} - 2u_n = 4^n \quad (\text{for } n \geq 0)$$

is

$$U(x) = \frac{1-3x}{(1-2x)(1-4x)}$$

Hence obtain a formula for u_n.

Solution:

If the recurrence were homogeneous, we know from the general theory that $(1 - 2x)\, U(x)$ would be a polynomial. So the recommended approach is to work out $(1 - 2x)\, U(x)$ anyway:

$$
\begin{aligned}
(1-2x)U(x) &= (1-2x)(u_0+u_1x+u_2x^2+\ldots) \\
&= u_0+(u_1-2u_0)x+(u_2-2u_1)x^2+(u_3-2u_2)x^3+\ldots \\
&= 1+x+4x^2+4^2x^3+\ldots \\
&= 1+x(1+4x+(4x)^2+\ldots) \\
&= 1+x(1-4x)^{-1} \\
&= \frac{1-3x}{1-4x}
\end{aligned}
$$

Hence

$$U(x)=\frac{1-3x}{(1-2x)(1-4x)}$$

In partial fractions

$$U(x)=\frac{1}{2}\left(\frac{1}{1-2x}+\frac{1}{1-4x}\right)$$

Using the negative binomial theorem, the coefficient of x^n is

$$\frac{1}{2}\left(2^n+4^n\right)=2^{n-1}+2^{2n-1}$$