On The Longest Edge of Relative Neighborhood Graphs in Wireless Ad Hoc Networks

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What is a wireless ad hoc network?

- a collection of wireless devices (transceivers) located in a geographic region
- each node is equipped with an omnidirectional antenna and has limited transmission power
- a communication session
  - a single-hop radio transmission
  - through relaying by intermediate devices
- no need for a fixed infrastructure
- can be flexibly deployed at low cost for varying missions
  - decision making in the battlefield
  - emergency disaster relief
  - environmental monitoring

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Maximal Transmission Radius

- each node is associated with a maximal transmission radius
- network topology is a graph
  - two nodes have an edge if within each other’s transmission range
- assume all nodes have the same maximal transmission radius $r$
  - induced network topology is exactly an $r$-dsik graph
- in many applications, ad hoc wireless devices are randomly deployed
- it is natural to represent the vertex set by a random point process
- the induced $r$-disk graphs are called random geometric graphs
Virtual Backbones

- constructed for routing packets within networks traditionally
- topology control
  - construction and maintenance of virtual backbones
  - major tasks in wireless ad hoc networks
- widely used ingredients for constructing virtual backbones
  - *Euclidean Minimal Spanning Trees* (EMST)
  - *Relative Neighbor Graphs* (RNG)
  - *Gabriel Graphs* (GG)
  - *Delauney Triangulations* (DT)
  - *Yao's Graphs* (YG)
Relative Neighbor Graphs (RNG)

- two nodes $u$ and $v$ have an edge between them if and only if no other nodes in
  \[ \text{Disk}(u, \|uv\|) \cap \text{Disk}(v, \|uv\|) \]
- assume all nodes have the same maximal transmission radius $r$
- to construct the RNG by only 1-hop information
  - $r$ should be large enough s.t. the RNG is a subgraph of the $r$-disk graph
  - $r$ is at least the maximal edge length of the RNG
- maximal edge length of the RNG is the critical transmission radius for construction the RNG by using only 1-hop information
- In this paper, we study the critical transmission radius of RNGs
Gilbert’s random geometric graph model (1961)

- devices are represented by an *infinite* random point process over the entire plane
- two devices are joined by an edge if and only if their distance is \( \leq r \)

Gupta and Kumar’s random geometric graph model (1998)

- devices are represented by a finite random uniform or Poisson point process over a disk
- two devices are joined by an edge if and only if their distance is \( \leq r \)
- if \( n \) nodes are placed in a unit-area disk, \( r(n) = \sqrt{\frac{\ln n + c_n}{\pi n}} \), then the resulting network is asymptotically connected *if and only if* \( c_n \rightarrow \infty \)
Related Works (cont.)

- Penrose (1997)
  - the probability of the event that the maximum edge length of the EMST is less than \( \sqrt{\frac{\ln n + \zeta}{\pi n}} \) for some constant \( \zeta \) is equal to \( \exp(-e^{-\zeta}) \) asymptotically.

- Kozma et al. (2004)
  - the maximal edge length of the DT of a uniform \( n \)-point process in a unit disk is \( O\left(\sqrt[3]{\frac{\ln n}{n}}\right) \).

- Wan et al. (2007)
  - derived the precise asymptotic distribution of the maximum edge length in the GG of a Poisson point process over a unit-area disk with density \( n \).
  - the probability of the event that the maximum edge length of the GG is at most \( 2\sqrt{\frac{\ln n + \zeta}{\pi n}} \) for some constant \( \zeta \) is equal to \( \exp(-2e^{-\zeta}) \) asymptotically.
Our Results

- assume a wireless ad hoc network is represented by a Poisson point process over the unit-area disk $\mathbb{D}$ with density $n$, which is denoted by $\mathcal{P}_n$
- all nodes have the same maximal transmission radius
- derived the precise asymptotic distribution of the maximum edge length in the RNG over $\mathcal{P}_n$
- the probability of the event that the maximum edge length of the RNG is at most $\beta_0 \sqrt{\frac{\ln n + \zeta}{\pi n}}$ for some constant $\zeta$ is equal to $\exp\left(-\frac{\beta_0^2}{2} e^{-\zeta}\right)$ asymptotically
  - where $\beta_0 = 1/\sqrt{\frac{2}{3} - \frac{\sqrt{3}}{2\pi}} \approx 1.6$
More precisely, we proved the following theorem

**Theorem**

For any constant $\xi$, we have

$$\lim_{n \to \infty} \Pr \left[ \lambda \left( \text{RNG} \left( \mathcal{P}_n \right) \right) \leq \beta_0 \sqrt{\frac{\ln n + \xi}{\pi n}} \right] = e^{-\frac{\beta_0^2}{2} e^{-\xi}}.$$

- $\text{RNG} \left( \mathcal{P}_n \right)$ denote the Relative Neighborhood Graph over $\mathcal{P}_n$
- $\lambda \left( \text{RNG} \left( \mathcal{P}_n \right) \right)$ denote the maximum edge length of the graph $\text{RNG} \left( \mathcal{P}_n \right)$
A brief overview on our approach to prove the theorem

- Let

\[ r_n = \beta_0 \sqrt{\frac{\ln n + \zeta}{\pi n}}, \quad R_n = \beta_0 \sqrt{\frac{\ln n + \zeta n}{\pi n}} \quad \text{and} \quad R'_n = 1.1 \beta_0 \sqrt{\frac{\ln n}{\pi n}}.\]

\[
M_n = \left| \left\{ e \in \text{RNG} (\mathcal{P}_n) : r_n < \| e \| \leq R_n \right\} \right|
\]

\[
M'_n = \left| \left\{ e \in \text{RNG} (\mathcal{P}_n) : R_n < \| e \| \leq R'_n \right\} \right|
\]

\[
M''_n = \left| \left\{ e \in \text{RNG} (\mathcal{P}_n) : R'_n < \| e \| < +\infty \right\} \right|
\]

- Then \( \lambda (\text{RNG} (\mathcal{P}_n)) \leq r_n \) if and only if \( M_n + M'_n + M''_n = 0 \) a.a.s.

- We proved the following asymptotical equalities using different techniques
  - \( M'_n = 0 \) a.a.s.
  - \( M''_n = 0 \) a.a.s.
  - \( M_n \) is asymptotically Poisson with mean \( \frac{\beta^2}{2} e^{-\zeta} \)
Techniques used to prove the results

- $M_n' = 0$ a.a.s.
  - Palm Theory on the Poisson point process
- $M_n'' = 0$ a.a.s.
  - a technique tool called minimal scan statistics
- $M_n$ is asymptotically Poisson with mean $\frac{\beta^2}{2} e^{-\xi}
  - Brun’s sieve theorem on the Poisson point process
Techniques used to prove the results (cont.)

- Palm Theory on the Poisson point process

**Theorem**

Suppose that \( h(U, V) \) is a bounded measurable function defined on all pairs of the form \((U, V)\) with \( V \) being a finite planar set and \( U \) being a subset of \( V \). Then any positive integer \( k \),

\[
\mathbb{E} \left[ \sum_{U \subseteq \mathcal{P}_n, |U|=k} h(U, \mathcal{P}_n) \right] = \frac{n^k}{k!} \mathbb{E} \left[ h(\mathcal{X}_k, \mathcal{X}_k \cup \mathcal{P}_n) \right].
\]
Techniques used to prove the results (cont.)

- Brun’s sieve theorem on the Poisson point process

**Theorem**

Suppose that $N$ is a non-negative integer random variable, and $B_1, \cdots, B_N$ are $N$ Bernoulli random variables. If there is a constant $\mu$ such that for every fixed positive integer $k$,

$$
\mathbb{E} \left[ \sum_{I \subseteq \{1, \cdots, N\}, |I| = k} \prod_{i \in I} B_i \right] \sim \frac{1}{k!} \mu^k,
$$

then $\sum_{i=1}^N B_i$ is asymptotically Poisson with mean $\mu$. 

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Thanks and Questions?