About Influences And Cliques

Suppose that there are n individuals $I_1, I_2, ..., I_n$ some of whom can influence each other in making decisions. We can represent the way individuals influence each other as digraph where I_i are vertices and the edges indicate who influences whom. If I_i influences I_j and I_j influences I_i then we will use a double arrow (↔) to indicate this.

The digraph in Figure 1 represents the influence relations between 5 members of a group.

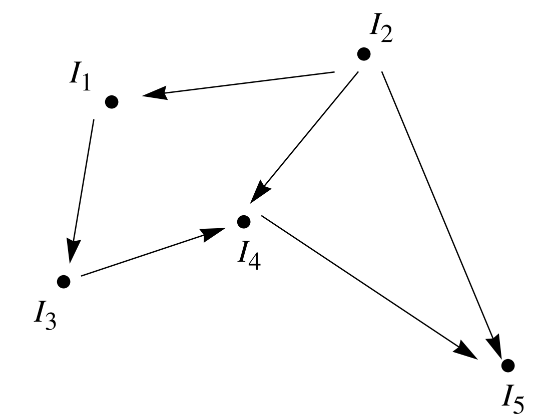

Figure. 1

Notice that, since no individual influences herself, the digraph has no cycles of length 1 (loops). I_2 is the *leader* of the group since nobody influences her.

Note also that the relation described in Figure 1 is not transitive. I_2 influences I_1 and I_1 influences I_3, yet I_1 does not influence I_3 (there is no edge from I_2 to I_3).

Can we model real life relations using a digraph as the one in Figure 1? I_2 may have no direct influence on I_3; however she has influence on I_1 which in turn has influence on I_3. As we all know this works in many cases. We say that I_2 has a *two-stage* influence on I_3.

We say that I_i has a two-stage influence on I_j, if there is a path of length two from I_i to I_j. Similarly, we can speak of a k-stage influence if there is a path of length k from I_i to I_j.

We are now interested to know how to find a path of length k in a digraph. Visual inspection works for small digraphs, but we want a more general method, something that can be automated and run on a computer.

Paths in Relations and Digraphs

Let R be a relation on a set S. A path of length n in R from a to b is a finite sequence π: a, $x_1, x_2, ..., x_{n-1}$, b, which begins with a and ends in b, such that $aRx_1, x_1Rx_2, ..., x_{n-1}Rb$.

Paths in a relation can be used to define new relations. Given some fixed positive integer n, we define the relation R^n on the set S as follows: aR^nb means that there is a path of length n from a to b.

Let's denote by $\mathbf{M}_R$ the matrix of a relation.

Theorem: If R is a relation on a set S, then the matrix of R^2, denoted by $\mathbf{M}_{R2}$ is $\mathbf{M}_{R2} = \mathbf{M}_R \bullet \mathbf{M}_R$, where $\bullet$ denotes the boolean product of two matrices.

The boolean product of **A** (of size m x p) and **B** (of size p x n), denoted by **A•B** (of size m x n), is the matrix **C**=[c_{ij}] defined by

- $c_{ij} = 1$ if $a_{ik} = 1$ and $b_{kj} = 1$ for some k, $1 \le k \le p$
- $c_{ij} = 0$ otherwise

This definition works as follows:

- Select row i of A and column j of B and arrange them sisde by side
- Compare corresponding entries; if even a single pair of corresponding entries consists of two 1's, then $c_{ij} = 1$. Otherwise $cij = 0$

Ex 1:

$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$ Then their boolean product is $C = A \bullet B = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

Cliques

The Webster's dictionary defines a clique as "a small exclusive group of people". This term is used when studying organizational structures to refer to subsets of people in which any pair of individuals is related. A clique in an influence digraph is a subset Q of vertices such that

- $|Q| \ge 3$
- if I_i and I_j are in Q, then I_i influences I_j and I_j influences I_i
- Q is the largest subset that satisfies the second condition

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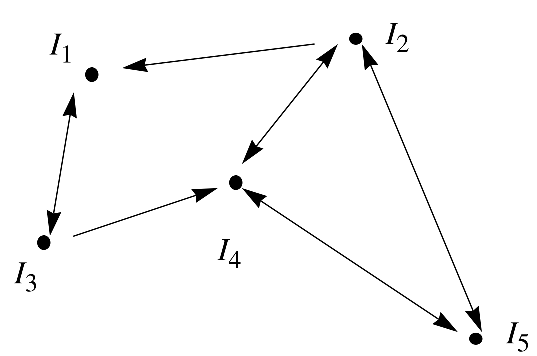

Figure 2

In Figure 2 there is only one clique, including I_2, I_4, and I_5. Note that I_1 and I_3 are not a clique since the size of this subset is 2 (the first condition in the definition requires $|Q| \ge 3$).

It is easy to find cliques in a digraph when the digraph is small. But, as with k-stage influences, we want to find a way to automate the job. The following algorithm solves the problem:

Algorithm Clique

- If **A** = [a_{ij}] is the matrix of the influence relation, construct the matrix **S**=[s_{ij}] as follows: s_{ij}=s_{ji}=1 if and only if *aij*=*aji*=1. Otherwise, s_{ij}=0
- Compute C = [c_{ij}] = **S•S•S**
- I_{i} belongs to a clique if and only if c_{ii} is positive.

End Clique

