Two-Phased Approximation Algorithms for Minimum CDS in Wireless Ad Hoc Networks

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***Abstract*— Connected dominating set (CDS) has a wide range of applications in wireless ad hoc networks. A number of distributed algorithms for constructing a small CDS in wireless ad hoc networks have been proposed in the literature. The majority of these distributed algorithms follow a general two-phased approach. The first phase constructs a dominating set, and the second phase selects additional nodes to interconnect the nodes in the dominating set. In this paper, we prove that the approximation ratio of the two-phased algorithm in [10] is at most $7\frac{1}{3}$, improving upon the previous best-known approximation ratio of 7.6 due to [12]. We also propose a new two-phased approximation algorithm and prove that its approximation ratio is at most $6\frac{7}{18}$. Our analyses exploit an improved upper bound on the number independent points that can be packed in the neighborhood of a connected finite planar set.**

I. Introduction

Connected dominating set (CDS) has a wide range of applications in wireless ad hoc networks (cf. a recent survey [3] and references therein). Consider a wireless ad hoc network with undirected communication topology $G = (V, E)$. A CDS of G is a subset $U \subset V$ satisfying that each node in $V \setminus U$ is adjacent to at least one node in U and the subgraph of G induced by U is connected. A number of distributed algorithms for constructing a small CDS in wireless ad hoc networks have been proposed in the literature. The majority of these distributed algorithms follow a general two-phased approach [1], [2], [4], [8], [9], [10]. The first phase constructs a dominating set, and the nodes in the dominating set are called dominators. The second phase selects additional nodes, called connectors, which together with the dominators induce a connected topology. The algorithms in [1], [2], [4], [8], [9], [10] differ in how to select the dominators and connectors. For example, the algorithm in [2] selects the dominators using the Chvatal's greedy algorithm [5] for Set Cover, both algorithms in [1], [9] selects an arbitrary maximal independent set (MIS) as the dominating set, and all the algorithms in [4], [8], [10] choose a special MIS with 2-hop separation property as the dominating set.

The approximation ratios of these two-phased algorithms [1], [2], [4], [8], [9], [10] have been analyzed when the communication topology is a unit-disk graph (UDG). For a wireless ad hoc network in which all nodes lie in a plane and have equal maximum transmission radii normalized to one, its communication topology $G = (V, E)$ is often modelled by a UDG in which there is an edge between two nodes if and only if their Euclidean distance is at most one. Except the algorithms in [2], [9] which have logarithmic and linear approximations ratios respectively, all other algorithms in [1], [4], [8], [10] have constant approximation ratios. The algorithm in [1] targets at distributed construction of CDS in linear time and linear messages. With this objective, it trades the size of the CDS with the time complexity, and thus its approximation ratio is a large constant (but less than 192). The analyses of the algorithms in [4], [8], [10] rely on the relation between the independence number (the size of a maximum independent set) $\alpha(G)$ and the connected domination number (the size of a minimum connected dominating set) $\gamma_c(G)$ of a UDG G. A loose relation

$$\alpha(G) \leq 4\gamma_c(G) + 1$$

was obtained in [10], which implies the an upper bound of 8 on the approximation ratios of both algorithms in [4], [10]. A refined relation

$$\alpha(G) \leq 3.8\gamma_c(G) + 1.2$$

was discovered in [12]. With such refined relation, the upper bound on the approximation ratios of both algorithms in [4], [10] was reduced from 8 to 7.6, and an upper bound of $5.8 + \ln 5 \approx 7.41$ on the approximation ratio of the algorithms in [8] was derived (the bound $4.8 + \ln 5 \approx 6.41$ in [8] was incorrect).

In this paper, we first prove a tighter relation between the independence number and the connected domination number in Section II: for every connected UDG G with at least two nodes,

$$\alpha(G) \leq 3\frac{2}{3}\gamma_c(G) + 1.$$

We then obtain an improved upper bound of $7\frac{1}{3}$ on the approximation ratio of the algorithm in [10] in Section III. In addition, we propose a new two-phased algorithm in Section IV. The first phase of this new algorithm selects the dominators as that in [10], but the second phase selects the connectors in a natural greedy manner. We prove that the approximation ratio of this new algorithm is at most $6\frac{7}{18}$.

We remark that in a recent paper [7] claimed that for any connected UDG G,

$$\alpha(G) \leq 3.453\gamma_c(G) + 8.291.$$

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However, the proof for a key geometric extreme property underlying such claim was missing, and such proof is far from being apparent or easy. We will discuss this in Section V. Consequently, we regard this bound as a conjecture rather than a proven result.

In the remaining of this section, we introduce some terms and notations. A unit-disk (respectively, unit-circle, unit-arc) refers to a disk (respectively, circle, arc) of radius one. For any point u, we use D_u to denote the unit-disk centered at u, and ∂D_u to denote the boundary circle of D_u. The neighborhood of a point set S is defined to be $\cup_{u \in S} D_u$. A finite planar set of points are said to be independent if their pairwise distances are all greater than one, and are said to be connected if it induces a connected unit-disk graph. A finite planar set S is called a *star* if there is a point $v \in S$ such that $S \subset D_v$. Clearly every star contains at least one point. A star consisting of k points is referred to as a k-star.

II. Bound on Independence Number

Throughout this section, we use I to represent an arbitrary set of independent points in the plane. For any planar point u and finite planar set U, define

$$I(u) = I \cap D_u,$$
$$I(U) = \cup_{u \in U} I(u).$$

It's trivial that $|I(u)| \leq 5$ for any planar point u, and $|I(u) \setminus I(o)| \leq 4$ for two points o and u with $ou \leq 1$. Thus, if $ou \leq 1$, then $|I(o) \triangle I(u)| \leq 8$. The next lemma states that the bound 8 cannot be achieved.

Lemma 1: If $ou \leq 1$, then $|I(o) \triangle I(u)| \leq 7$.

The proof of this lemma is delayed to the appendix. Now suppose that $\{u_1, u_2, u_3\} \subset D_o$. Then

$$\left|\left(\cup_{j=1}^{3} I(u_j)\right) \setminus I(o)\right| \leq \sum_{j=1}^{3} |I(u_j) \setminus I(o)| \leq 12.$$

The next lemma gives a condition under which the bound 12 cannot be achieved.

Lemma 2: Suppose that $\{u_1, u_2, u_3\} \subset D_o$. If

$$(I(o) \setminus \{o\}) \setminus \cup_{j=1}^{3} I(u_j) \neq \emptyset,$$

then

$$\left|\left(\cup_{j=1}^{3} I(u_j)\right) \setminus I(o)\right| \leq 11.$$

The proof of this lemma is delayed to the appendix. For any positive integer n, denote

$$\phi_n = \begin{cases} 3n+2, & \text{if } n \leq 2; \\ \min\{3n+3, 21\}, & \text{if } n \geq 3. \end{cases}$$

It's easy to verify that $\phi_n \leq 11n/3 + 1$ for $n \geq 2$.

Theorem 3: Suppose that S is an n-star $n \geq 1$. Then $|I(S)| \leq \phi_n$. If $n \leq 4$ and $\max_{v \in S} |I(v)| \leq 4$, then $|I(S)| \leq \phi_n - 1$.

Proof: Theorem 3 is trivial for $n = 1$. The well-known Wegner Theorem [11] on finite circle packing implies that any disk of radius two contains at most 21 points whose pairwise distances are all at least one. As the result, $|I(S)| \leq 21$. Since $3n + 3 \geq 21$ when $n \geq 6$, Theorem 3 holds when $n \geq 6$. Hence, we only have to prove Theorem 3 for $2 \leq n \leq 5$. Let $S = \{o, u_1, u_2, \cdots, u_{n-1}\}$. For the clarity of presentation, we denote $I_0 = I(o)$, $I_j = I(u_j)$ for each $1 \leq j < n$, and $I_j^* = \cup_{0 \leq i < n, i \neq j} I_i$ for each $0 \leq j < n$. Then, for each $0 \leq j < n$,

$$|I(S)| = \left|I_j^*\right| + \left|I_j \setminus I_j^*\right| = |I_j| + \left|I_j^* \setminus I_j\right|.$$

If $|I_0| = 5$, then

$$|I(S)| \leq |I_0| + \sum_{j=1}^{n-1} |I_j \setminus I_0| \leq 5 + 3(n-1) = 3n + 2 \leq \phi_n.$$

If $|I_0| \leq 1$, then

$$|I(S)| \leq |I_0| + \sum_{j=1}^{n-1} |I_j \setminus I_0| \leq 1 + 4(n-1) = 4n - 3 \leq \phi_n - 1.$$

Thus, the theorem holds if $|I_0| = 5$ or $|I_0| \leq 1$. So, we assume that $2 \leq |I_0| \leq 4$. Then, $o \notin I$.

Case 1: $n = 2$. By Lemma 1, $|I_0 \triangle I_1| \leq 7$. Hence either $|I_1 \setminus I_0| \leq 3$ or $|I_0 \setminus I_1| \leq 3$. By symmetry, assume the former occurs.

$$|I(S)| = |I_0| + |I_1 \setminus I_0| \leq 5 + 3 = 8.$$

If $\max_{0 \leq j \leq 1} |I_j| \leq 4$, then $|I_0 \cup I_1| \leq 7$.

Case 2: $n = 3$. Then,

$$|I(S)| = |I_0 \cup I_1| + |I_2 \setminus (I_0 \cup I_1)| \leq 8 + 4 = 12.$$

If $\max_{0 \leq j \leq 2} |I_j| \leq 4$. Then, $|I_0 \cup I_1| \leq 7$ and hence $|I(S)| \leq 11$.

Case 3: $n = 4$. If $I_0 \setminus I_0^* = \emptyset$, then $|I(S)| = |I_0^*| \leq 3 \cdot 5 = 15$. If $I_0 \setminus I_0^* \neq \emptyset$, then $|I_0^* \setminus I_0| \leq 11$ by Lemma 2, and hence

$$|I(S)| = |I_0| + |I_0^* \setminus I_0| \leq 4 + 11 = 15.$$

Now, we assume that $\max_{0 \leq j \leq 3} |I_j| \leq 4$. If $I_0 \setminus I_0^* = \emptyset$, then $|I(S)| = |I_0^*| \leq 3 \cdot 4 = 12$. So we assume that $I_0 \setminus I_0^* \neq \emptyset$. Then, $|I_0^* \setminus I_0| \leq 11$. If $|I_0| \leq 3$, then

$$|I(S)| = |I_0| + |I_0^* \setminus I_0| \leq 3 + 11 = 14.$$

So we further assume $|I_0| = 4$. Then for each $1 \leq j < 4$,

$$|I_j \setminus I_0| = |I_j \cup I_0| - |I_0| \leq 7 - 4 = 3$$

we have $|I_j \setminus I_0| \leq 7 - 4 = 3$, and hence $|I_0^* \setminus I_0| \leq 3 \cdot 3 = 9$. So,

$$|I(S)| = |I_0| + |I_0^* \setminus I_0| \leq 4 + 9 = 13.$$

Case 4: $n = 5$. We prove $|I(S)| \leq 18$ by contradiction. Assume to the contrary that $|I(S)| \geq 19$. Then, for each $1 \leq j \leq 4$,

$$19 \leq |I(S)| = \left|I_j^*\right| + \left|I_j \setminus I_j^*\right| \leq 15 + 4 = 19,$$

which implies that $|I(S)| = 19$, $\left|I_j^*\right| = 15$ and $\left|I_j \setminus I_j^*\right| = 4$ for each $1 \leq j \leq 4$. Thus, for each $1 \leq j \leq 4$, we have $|I_j \setminus I_0| = 4$ since

$$4 = \left|I_j \setminus I_j^*\right| \leq |I_j \setminus I_0| \leq 4.$$

Furthermore, $|I_0| = 3$ as

$$|I_0| \le |I(S)| - \sum_{j=1}^{4} \left| I_j \setminus I_j^* \right| = 3$$

and

$$|I_0| \ge |I(S)| - \sum_{j=1}^{4} |I_j \setminus I_0| = 3.$$

For each $1 \le i \le 4$,

$$|(\cup_{1 \le j \le 4, j \ne i} I_j) \setminus I_0| \ge \sum_{1 \le j \le 4, j \ne i} \left| I_j \setminus I_j^* \right| = 12,$$

and consequently by Lemma 2,

$$I_0 \subseteq \cup_{1 \le j \le 4, j \ne i} I_j.$$

On the other hand, for each $1 \le j \le 4$,

$$|I_j \cap I_0| = |I_j| - |I_j \setminus I_0| \le 5 - 4 = 1.$$

Let $I_0 = \{x_1, x_2, x_3\}$. Since $I_0 \subset \cup_{j=1}^{3} I_j$, we have $|I_j \cap I_0| = 1$ for each $1 \le j \le 3$. By symmetry, we assume that $x_j \in I_j$ for each $1 \le j \le 3$. Since $I_0 \subset \cup_{j=2}^{4} I_j$, we must have $x_1 \in I_4$. But then, $x_3 \notin I_1 \cup I_2 \cup I_4$, which is a contradiction. Thus, Theorem 3 holds when $n = 5$. This completes the proof of theorem 3. ■

In the next, we extend Theorem 3 from a star to an arbitrary connected planar set. A *star-decomposition* of a V is a partition of V into stars. A star-decomposition is said to be *nontrivial* if none of the its stars is a singleton.

Lemma 4: Any connected planar set of at least points has a non-trivial star-decomposition.

Proof: We prove the lemma by induction. When $n = 2$, V itself is a star and the claim holds trivially. Now suppose that the lemma holds for any connected planar set V with $2 \le |V| \le n-1$. Consider a connected planar set V of $n \ge 3$ points.. Let G be the UDG induced by V. Pick an arbitrary node $v \in V$. By induction hypothesis, all non-singleton (connected) components of $G - \{v\}$ allows nontrivial star-decompositions. Let $\mathcal{S}$ denote the collection of stars in the nontrivial star-decompositions of all non-singleton components of $G - \{v\}$. We consider two cases:

Case 1: $G - \{v\}$ has at least one singleton component. Then all these singleton components are adjacent to v.They together with v form a star denoted by S. Then, $\mathcal{S} \cup \{S\}$ is a nontrivial star-decomposition of V.

Case 2: $G - \{v\}$ has no singleton components. Let u be an arbitrary neighbor of v in G, and S be the star in $\mathcal{S}$ containing u. If $S \subset D_u$ then $(\mathcal{S} \setminus \{S\}) \cup \{S \cup \{u\}\}$ is a nontrivial star-decomposition of V; otherwise, $|S| \ge 3$ and hence $(\mathcal{S} \setminus \{S\}) \cup \{S \setminus \{u\}, \{u, v\}\}$ is a nontrivial star-decomposition of V. ■

Lemma 5: Let $\mathcal{S}$ be a star-decomposition of a finite connected planar set V, and S is a star in $\mathcal{S}$. If none of the stars in $\mathcal{S} \setminus \{S\}$ is singleton, then $|I(V) \setminus I(S)| \le \frac{11}{3} |V \setminus S|$.

Proof: Suppose that $\mathcal{S}$ consists of k sets. Consider an ordering of the sets in $\mathcal{S}$ satisfying that $V_1 = S$ and for each $2 \le i \le k$ the set V_i is adjacent to $\cup_{j=1}^{i-1} V_j$. For each $2 \le i \le k$, let $u_i \in \cup_{j=1}^{i-1} V_j$ which is adjacent to V_i. Then, $\left(I(V_i) \setminus I\left(\cup_{j=1}^{i-1} V_j\right)\right) \cup \{u_i\}$ is independent. By Theorem 3,

$$\left| I(V_i) \setminus I\left(\cup_{j=1}^{i-1} V_j\right) \right| + 1 \le \frac{11}{3} |V_i| + 1$$

and consequently

$$\left| I(V_i) \setminus I\left(\cup_{j=1}^{i-1} V_j\right) \right| \le \frac{11}{3} |V_i|.$$

Thus,

$$\begin{aligned} |I(V) \setminus I(S)| &= \sum_{i=2}^{k} \left| I(V_i) \setminus I\left(\cup_{j=1}^{i-1} V_j\right) \right| \\ &\le \frac{11}{3} \sum_{i=2}^{k} |V_i| = \frac{11}{3} |V \setminus S|. \end{aligned}$$

■

Theorem 6: Suppose that V is a connected planar set of $n \ge 2$ points. Then $|I(V)| \le 11n/3 + 1$. If $\max_{v \in V} |I(v)| \le 4$, then$|I(V)| \le 11n/3$. If $V \cap I \ne \emptyset$, then $|I(V)| \le 11n/3 - 1$.

Proof: Let $\mathcal{S}$ be a nontrivial star-decomposition $\mathcal{S}$ of V. Pick an arbitrary star $S \in \mathcal{S}$. By Theorem 3, $|I(S)| \le \frac{11}{3} |S| + 1$. By Lemma 5,

$$|I(V) \setminus I(S)| \le \frac{11}{3} |V \setminus S|.$$

Hence,

$$|I(V)| = |I(S)| + |I(V) \setminus I(S)| \le \frac{11}{3} |V| + 1.$$

If $\max_{v \in V} |I(v)| \le 4$, then $|I(S)| \le \frac{11}{3} |S|$ and consequently $|I(V)| \le \frac{11}{3} |V|$.

Now suppose that $V \cap I \ne \emptyset$. Let $v \in V \cap I$, and let l be the number of singleton components of $G - \{v\}$. Then, $l \le 5$. These l singleton nodes together with v induces a star S of $l + 1$ nodes. Then, $|I(S)| \le 4l + 1$. By Lemma 4, each non-singleton component of $G - \{v\}$ has a nontrivial star-decomposition. Hence, By Lemma 5 we have

$$|I(V) \setminus I(S)| \le \frac{11}{3} |V \setminus S| = \frac{11}{3} (n - l - 1).$$

Thus,

$$\begin{aligned} |I(V)| &\le (4l + 1) + \frac{11}{3} (n - l - 1) \\ &= \left(\frac{11}{3} n - 1 \right) + \frac{l - 5}{3} \le \frac{11}{3} n - 1. \end{aligned}$$

■

Theorem 6 immediately implies the following relation between the independence number and connected domination number of any connected UDG with at least two nodes.

Corollary 7: For every connected UDG G with at least two nodes, $\alpha(G) \le 3\frac{2}{3} \gamma_c(G) + 1$.

III. Improved Approximation Ratio of The Algorithm in [10]

In this section, we derive a tighter bound on the approximation ratio of the distributed algorithm proposed in [10]. Let G be a unit-disk graph. For convenience of presentation, we omit the parameter of G in $\alpha(G)$ and $\gamma_c(G)$. Let OPT be a a minimum CDS of G. The CDS produced by the algorithm in [10] consists of a maximal independent set I and a set C of connectors. Specifically, let T be an arbitrary rooted spanning tree of G. The set I is selected in the first-fit manner in the breadth-first-search ordering in T. Let s be the neighbor of the root of T which is adjacent to the largest number of nodes in I. Then, C consists of s and the parents (in T) of the nodes in $I \diagdown I(s)$. It was proved in [10] that $I \cup C$ is a CDS and $|I \cup C| \leq 8\gamma_c - 1$. Later on, it was proved in [12] that $|I \cup C| \leq 7.6\gamma_c + 1.4$. The next theorem further improves the bound on $|I \cup C|$.

Theorem 8: $|I \cup C| \leq 7\frac{1}{3}\gamma_c$.

Proof: If $\gamma_c = 1$, then $|I| \leq 5$ and $|C| = 1$, hence $|I \cup C| \leq 6 < 7\frac{1}{3} - 1$. Thus, the theorem holds if $\gamma_c = 1$. From now on, we assume that $\gamma_c \geq 2$. Then, $|I| \geq |I(s)| \geq 2$. Clearly, $|C| \leq 1 + |I \diagdown I(s)| = |I| - |I(s)| + 1$. Thus, $|I \cup C| \leq 2|I| + 1 - |I(s)|$. If $I \cap OPT \neq \emptyset$, then $|I| \leq \frac{11}{3}\gamma_c - 1$ by Theorem 6, and $|I \cup C| \leq 2|I| - 1 \leq 7\frac{1}{3}\gamma_c - 3$. So we further assume that $I \cap OPT = \emptyset$. If $|I(s)| \geq 3$, then $|C| \leq |I| - 2$ and hence

$$|I \cup C| \leq 2|I| - 3 \leq 2\left(\frac{11}{3}\gamma_c + 1\right) - 2 = 7\frac{1}{3}\gamma_c.$$

So we assume that $|I(s)| = 2$. Then, $|I \cup C| \leq 2|I| - 1$. Pick an arbitrary $v \in OPT$ which is adjacent to the leader. Then,

$$1 \leq |I(v)| \leq |I(s)| = 2.$$

Let l be the number of singleton components of the UDG over $OPT \setminus \{v\}$. Then $l \leq 5$. These l singleton nodes together with v induces a star S of $l+1$ nodes and we denote $m = |I(S)|$. By Lemma 4, each non-singleton component of $G - \{v\}$ has a nontrivial star-decomposition. Hence, By Lemma 5 we have

$$|I \setminus I(S)| \leq \frac{11}{3}|OPT \setminus S| = \frac{11}{3}(\gamma_c - 1 - l).$$

Then, $|I| \leq m + \frac{11}{3}(\gamma_c - 1 - l)$. Thus,

$$|I \cup C| \leq 2|I| - 1 \leq 2\left(m + \frac{11}{3}(\gamma_c - 1 - l)\right) - 1$$
$$= 7\frac{1}{3}\gamma_c - 2\left(\frac{11}{3}(1 + l) + \frac{1}{2} - m\right).$$

Denote $\delta = \frac{11}{3}(1 + l) + \frac{1}{2} - m$. In the next, we prove $\delta \geq 0$, from which the theorem follows immediately.

If $l = 0$, then $m = |I(v)| \leq 2$ and

$$\delta \geq \frac{11}{3} + \frac{1}{2} - 2 > 0.$$

If $l = 5$, then $m \leq 21$ and

$$\delta \geq \frac{11}{3} \cdot 6 + \frac{1}{2} - 21 = 1.$$

Next, we assume that $1 \leq l \leq 4$. It's easy to see that $m \leq |I(v)| + 4l \leq 2 + 4l$. Thus,

$$\delta \geq \frac{11}{3}(1 + l) + \frac{1}{2} - (2 + 4l) = \frac{13}{6} - \frac{l}{3} \geq \frac{13}{6} - \frac{4}{3} > 0.$$

■

We remark that with a more subtle analysis, we can actually show that $|I \cup C| \leq 7\frac{1}{3}\gamma_c - 1$.

IV. A New Approximation Algorithm

In this section, we present a new two-phased approximation algorithm and prove that its approximation ratio is at most $6\frac{7}{18}$. The first phase of this algorithm is the same as the algorithm in [10], and we let I be the selected maximal independent set. But the second phase selects the connectors in a more economic way. Before we describe the algorithm for the second phase, we introduce some terms and notations. For any subset $U \subseteq V \setminus I$, let $q(U)$ be the number of connected components in $G[I \cup U]$. For any $U \subseteq V \setminus I$ and any $w \in V \setminus I$, define

$$\triangle_w q(U) = q(U) - q(U \cup \{x\}).$$

The value $\triangle_w q(U)$ is referred to as the gain of w with respect to U. Clearly, if $w \in I \cup U$, then $\triangle_w q(U) = 0$; if $w \notin I \cup U$, then $\triangle_w q(U)$ is one less than the number of connected components in $G[I \cup U]$ adjacent to w since I is a maximal independent set.

The next lemma is essential to both the correctness and the performance analysis of our new algorithm.

Lemma 9: Suppose that there are $q(U) > 1$ for some $U \subseteq V \setminus I$. Then, there exists a $w \in V \setminus (I \cup U)$ such that $\triangle_w q(U) \geq \max\{1, \lceil q(U)/\gamma_c \rceil - 1\}$.

Proof: Since the set I has 2-hop separation property [10], there is a node w which is adjacent to at least two connected components of $G[I \cup U]$. For such node w, $\triangle_w q(U) \geq 1$. Now, let d_i be the number of components adjacent to the i-th node in OPT for $1 \leq i \leq \gamma_c$. Then, $\sum_{i=1}^{\gamma_c} d_i \geq q(U)$ because each component of of $G[I \cup U]$ must be adjacent to some node in OPT. So, $\max_{1 \leq i \leq \gamma_c} d_i \geq \lceil q(U)/\gamma_c \rceil$. Let w be the node in OPT which is adjacent to the largest number of connected components in $G[I \cup U]$. Then, $\triangle_w q(U) \geq \lceil q(U)/\gamma_c \rceil - 1$. ■

The above lemma implies that for any set $U \subseteq V \setminus I$, the set $I \cup U$ is a CDS if and only if $q(U) = 1$, which holds further if and only if every node has zero gain. So, we propose the following greedy algorithm for the second phase. We use C to denote the sequence of selected connectors. Initially C is empty. While $q(C) > 1$, choose a node $w \in V \setminus (I \cup C)$ with *maximum* gain with respect to C and add w to C. When $q(C) = 1$, then $I \cup C$ is a CDS. Let C be the output of the second phase. We have the following bound on $|I \cup C|$.

Theorem 10: $|I \cup C| \leq 6\frac{7}{18}\gamma_c$.

Proof: If $\gamma_c = 1$, then $|I| \leq 5$ and $|C| \leq 1$, hence $|I \cup C| \leq 6$. Thus, the theorem holds trivially if $\gamma_c = 1$. From now on, we assume that $\gamma_c \geq 2$. By Corollary 7, $|I| \leq \lfloor 3\frac{2}{3}\gamma_c \rfloor + 1$. We break C into three contiguous (and possibly empty) subsequences C_1, C_2 and C_3 as follows. C_1

is the shortest prefix of C satisfying that $q(C_1) \le \lfloor 3\frac{2}{3}\gamma_c \rfloor - 3$, and $C_1 \cup C_2$ is the shortest prefix of C satisfying that $q(C_1 \cup C_2) \le 2\gamma_c + 1$. We show that $|C_1| \le 1$, $|C_2| \le \frac{13}{18}\gamma_c - 1$, and $|C_3| \le 2\gamma_c - 1$, from which the theorem follows immediately. In the following, the gain of a node $u \in C$ is always with respect to the prefix of C just before u.

We first prove $|C_1| \le 1$. This is trivial if $|I| \le \lfloor 3\frac{2}{3}\gamma_c \rfloor - 2$. So we assume that $|I| \ge \lfloor 3\frac{2}{3}\gamma_c \rfloor - 1$. Let w be the first node in C. We claim that $q(\{w\}) \le \lfloor 3\frac{2}{3}\gamma_c \rfloor - 3$, which would imply $C_1 = \{w\}$ immediately.

Case 1: $|I| = \lfloor 3\frac{2}{3}\gamma_c \rfloor - 1$. Since $\lfloor 3\frac{2}{3}\gamma_c \rfloor - 1 \ge 3\gamma_c$, the gain of w is at least two by Lemma 9, and hence

$$q(\{w\}) \le \left\lfloor 3\frac{2}{3}\gamma_c \right\rfloor - 1 - 2 = \left\lfloor 3\frac{2}{3}\gamma_c \right\rfloor - 3.$$

Case 2: $|I| = \lfloor 3\frac{2}{3}\gamma_c \rfloor$. Since $\lfloor 3\frac{2}{3}\gamma_c \rfloor > 3\gamma_c$, the gain of w is at least three by Lemma 9, and hence

$$q(\{w\}) \le \left\lfloor 3\frac{2}{3}\gamma_c \right\rfloor - 3.$$

Case 3: $|I| = \lfloor 3\frac{2}{3}\gamma_c \rfloor + 1$. By Theorem 6 and the greedy choice of w, $|I(w)| = 5$. Thus, the gain of w is four, and hence

$$q(\{w\}) = \left\lfloor 3\frac{2}{3}\gamma_c \right\rfloor + 1 - 4 = \left\lfloor 3\frac{2}{3}\gamma_c \right\rfloor - 3.$$

Next, we prove $|C_3| \le 2\gamma_c - 1$. By Lemma 9since each node in C_3 has gain at least one. If $q(C_1 \cup C_2) \le 2\gamma_c$, then

$$|C_3| \le q(C_1 \cup C_2) - 1 \le 2\gamma_c - 1.$$

If $q(C_1 \cup C_2) = 2\gamma_c + 1$, then the first node in C_3 has gain at least two with respect to $C_1 \cup C_2$ by Lemma 9, and hence

$$2 + (|C_3| - 1) \le q(C_1 \cup C_2) - 1 = 2\gamma_c + 1 - 1,$$

which implies that $|C_3| \le 2\gamma_c - 1$.

Finally, we prove $|C_2| \le \frac{13}{18}\gamma_c - 1$. This inequality holds trivially if $C_2 = \emptyset$. So we assume that $C_2 \ne \emptyset$. Then, $\gamma_c > 2$, for otherwise $\lfloor 3\frac{2}{3}\gamma_c \rfloor - 3 = 2\gamma_c$ and hence $C_2 = \emptyset$. Since

$$\left\lfloor 3\frac{2}{3}\gamma_c \right\rfloor - 3 > 2\gamma_c + 1$$

when $\gamma_c > 2$, the gain of each node in C_2 is at least two by Lemma 9. Let v be the last node in C_2. Then, $q(C_1 \cup C_2 \setminus \{v\}) \ge 2\gamma_c + 2$. Thus,

$$\begin{aligned} &2(|C_2| - 1) \le q(C_1) - q(C_1 \cup C_2 \setminus \{v\}) \\ &\le \left\lfloor 3\frac{2}{3}\gamma_c \right\rfloor - 3 - (2\gamma_c + 2), \end{aligned}$$

which implies that

$$|C_2| \le \left\lfloor \left\lfloor \frac{5}{3}\gamma_c - 3 \right\rfloor /2 \right\rfloor.$$

It's easy to verify that when $3 \le \gamma_c \le 5$,

$$\left\lfloor \left\lfloor \frac{5}{3}\gamma_c - 3 \right\rfloor /2 \right\rfloor = \left\lfloor \frac{13}{18}\gamma_c \right\rfloor - 1.$$

Hence $|C_2| \le \frac{13}{18}\gamma_c - 1$ when $3 \le \gamma_c \le 5$. In the next, we assume that $\gamma_c \ge 6$. In this case,

$$\left\lfloor 3\frac{2}{3}\gamma_c \right\rfloor - 3 \ge 3\gamma_c + 1.$$

We break C_2 further into two contiguous (and possibly empty) subsequence C_2' and C_2'' satisfying that $C_1 \cup C_2'$ is the shortest prefix of C satisfying that $q(C_1 \cup C_2') \le 3\gamma_c + 2$. We consider two cases:

Case 1: $q(C_1 \cup C_2') \le 3\gamma_c + 1$. We prove that $|C_2'| \le \frac{2}{9}\gamma_c - 1$ and $|C_2''| \le \frac{\gamma_c}{2}$, would imply

$$|C_2| \le \frac{2}{9}\gamma_c - 1 + \frac{\gamma_c}{2} = \frac{13}{18}\gamma_c - 1.$$

We first show that $|C_2'| \le \frac{2}{9}\gamma_c - 1$.This inequality holds trivially if $C_2' = \emptyset$. So we assume that $C_2' \ne \emptyset$. By Lemma 9, each node in C_2' has gain at least three. Let w be the last node in C_2'. Then, $q(C_1 \cup C_2' \setminus \{w\}) \ge 3\gamma_c + 3$. So,

$$\begin{aligned} &3(|C_2'| - 1) \le q(C_1) - q(C_1 \cup C_2' \setminus \{w\}) \\ &\le \left\lfloor 3\frac{2}{3}\gamma_c \right\rfloor - 3 - (3\gamma_c + 3), \end{aligned}$$

which implies that $|C_2'| \le \frac{2}{9}\gamma_c - 1$. Next, we show that $|C_2''| \le \gamma_c/2$. This inequality holds trivially if $C_2'' = \emptyset$. So, we assume that $C_2'' \ne \emptyset$. If $m \le 3\gamma_c$, then

$$\begin{aligned} &2(|C_2''| - 1) \le q(C_1 \cup C_2') - q(C_1 \cup C_2 \setminus \{v\}) \\ &\le 3\gamma_c - (2\gamma_c + 2), \end{aligned}$$

which implies that $|C_2''| \le \gamma_c/2$. If $m = 3\gamma_c + 1$, then the first node in C_2'' has gain at least three and hence

$$\begin{aligned} &3 + 2(|C_2''| - 2) \le q(C_1 \cup C_2') - q(C_1 \cup C_2 \setminus \{v\}) \\ &\le 3\gamma_c + 1 - (2\gamma_c + 2), \end{aligned}$$

which also implies that $|C_2''| \le \gamma_c/2$. Therefore, $|C_2''| \le \gamma_c/2$.

Case 2: $q(C_1 \cup C_2') = 3\gamma_c + 2$. We prove that $|C_2'| \le \frac{2}{9}\gamma_c - \frac{5}{3}$ and $|C_2''| \le \frac{\gamma_c + 1}{2}$, which would imply

$$|C_2| \le \frac{2}{9}\gamma_c - \frac{5}{3} + \frac{\gamma_c + 1}{2} = \frac{13}{18}\gamma_c - 1\frac{1}{6}.$$

Since each node in C_2' has gain at least three by Lemma 9, we have

$$3|C_2'| \le q(C_1) - q(C_1 \cup C_2') \le \left\lfloor 3\frac{2}{3}\gamma_c \right\rfloor - 3 - (3\gamma_c + 2),$$

which implies $|C_2'| \le \frac{2}{9}\gamma_c - \frac{5}{3}$. Since the first node in C_2'' has gain at least three by Lemma 9, we have

$$\begin{aligned} &3 + 2(|C_2''| - 2) \le q(C_1 \cup C_2') - q(C_1 \cup C_2 \setminus \{v\}) \\ &\le 3\gamma_c + 2 - (2\gamma_c + 2), \end{aligned}$$

which implies $|C_2''| \le \frac{\gamma_c + 1}{2}$. ∎

V. Discussions

The largest number of independent points that can be packed in the neighborhood of a connected planar set plays a key role in obtaining tight bounds on the approximation ratio of various algorithms for minimum connected dominating set. In this paper, we first prove that at most ϕ_n independent points can be packed in the neighborhood of an n-star, where ϕ_n is equal to $3n+2$ if $n \leq 2$, and $\min\{3(n+1), 21\}$ otherwise. Such bound is tight for $n \leq 3$, as shown by an instance illustrated in 1. Fix a node o at the origin, and let $u_1 = (1,0)$ and $u_2 = -u_1$. Denote by D_0, D_1 and $D_2$the unit-disks centered at o, u_1 and u_2 respectfully. Let ε be a very small positive parameter ε, and let $v_1 = (1/2, \varepsilon)$, $w_1 = (0, 1-\varepsilon)$, $v_2 = -v_1$, and $w_2 = -w_1$. When ε is sufficiently small, $I_0 = \{v_1, w_1, v_2, w_2\}$ is independent. Since the distances from v_1 and w_1 to the topmost point on ∂D_1 are both greater than one, there is a point $p_1 \in \partial D_1$ satisfying that (1) p_1 lies on the proper left side of the vertical diameter of D_1, and (2) both v_1p_1 and w_1p_1 are greater than one. Let $p_2 \in \partial D_1$ which is symmetric to p_1 about the line ou_1. Then, $v_1p_2 > v_1p_1 > 1$ and $w_2p_2 = w_1p_1 > 1$. Let q_1 and q_2 be the two points evenly on the the major arc between p_1 and p_2. Then $I_1 = \{p_1, q_1, p_2, q_2\}$ is independent. In addition, $I_0 \cup I_1$ is independent set. Let I_2 be the symmetric image of I_1 with respect to the vertical diameter of D_0. Then $I_0 \cup I_1 \cup I_2$ is independent.

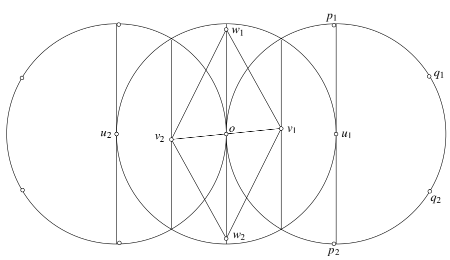

Fig. 1. The neighborhood of a 2-star (resp. 3-star) may contain 8 (resp. 12) independent points.

Using the technique of star-decomposition, we also obtain an upper bound of $3\frac{2}{3}n + 1$ on the maximum number of independent points that can be packed in the neighborhood of a connected planar set of n points. On the other hand, we can generalize the instance in Figure 1 to an instance in Figure 2, which shows that the number of independent points in the neighborhood of $n \geq 3$ linear points with consecutive distance equal to one may contain $3(n+1)$ independent points. We conjecture that this is a worst instance, and that $3(n+1)$ is the maximum number of independent points that can be packed in the neighborhood of a connected planar of $n \geq 3$ points. If this conjecture is true, we can show that the approximation ratio of the algorithm in [10] is at most 6, and the approximation ratio of the algorithm given in Section IV is at most 5.5.

A recent paper [7] claimed an upper bound of $3.453n + 8.291$ on the maximum number of independent points that can be packed in the neighborhood of a connected planar set

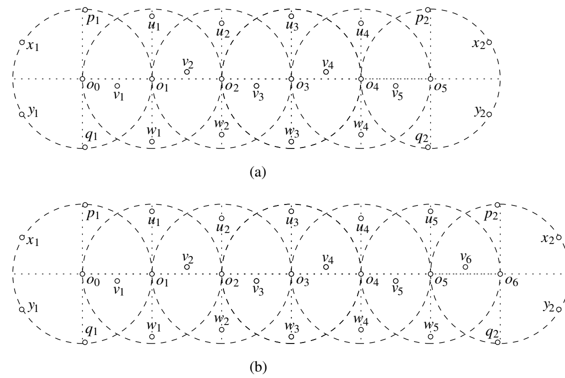

Fig. 2. The neighborhood of n linear points with consecitive distance equal to one may contain $3n+3$ independent points: (a) n is even; (b) n is odd.

V of n points using a simple area-argument. Specifically, let I be a set of independent points in the neighborhood of V, and let Ω denote the union of disks of radius 1.5 centered at V. Construct the Voronoi diagram defined by I, and let $Vor(u)$ denote the Voronoi cell of a point $u \in I$. Then,

$$|I| \leq \frac{area(\Omega)}{\min_{u \in I} area(Vor(u) \cap \Omega)}.$$

It is obvious that $area(\Omega)$ achieves maximum when all points in V are linear with consecutive distance equal to one. On the other hand, the area of $Vor(u) \cap \Omega$ was claimed in [7] to be at least the area of $H \cap \Omega$ where H is a regular hexagon centered at u with side equal to $1/\sqrt{3}$. The proof for such claim given in [7] was simply that "it follows immediately from the well-known result by Fejes Tóth [6], which proves that the densest packing of unit disks in the plane is attained by a hexagonal lattice". However, it is hardly possible to imply such claim by the classical and result of Fejes Tóth immediately. While the result of Fejes Tóth can be proved very easily, the proof for the previous claim is far from being apparent or easy. Consequently, we regard the bound $3.453n + 8.291$ claimed in [7] as a conjecture, rather than a proven result.

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APPENDIX: PROOFS OF LEMMA 1 AND LEMMA 2

A minor (respectively, major) arc is an arc of a circle having measure at most (respectively, at least) 180^o. An arc-polygon is a bounded region surrounded by a finite number of minor unit-arcs and line segments. A vertex of an arc-polygon is a point on the boundary which is not in the interior of any boundary arc or line segment. It's easy to show that the diameter of an arc-polygon is at most one if and only if the diameter if its vertex set is at most one. We first introduce two lemmas that will be used in both the proof of Lemma 1 and the proof of Lemma 2.

Lemma 11: Consider a convex quadrilateral $oupv$ with $ov = up$ (see Figure 3(a)). Then $\angle ovp + \angle upv \le 180^o$ if and only if $vp \ge ou$.

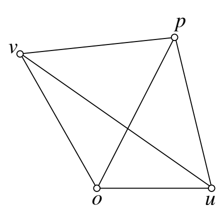

Fig. 3. If $ov = up$ and $vp \geq ou$, then $\angle ovp + \angle upv \leq 180^o$.

Lemma 12: Suppose that $0 < ou \le 1$, $a \in \partial D_o \cap \partial D_u$, $p \in \partial D_u$ satisfying that $ap \le 1 \le op$ (see Figure 4(a)), $v_1 \in \partial D_p \cap \partial D_o$ which is on the same side of op as a, $\partial D_p \cap \partial D_u = \{v_2, q\}$ with v_2 on the same side of up as a, and $q \in \partial D_p \cap \partial D_o$ which is on the same side of oq as a. Then $diam(\{v_1, v_2, p, s\}) = 1$.

Due to the space limitation, we omit the proofs of Lemma 11 and Lemma 12.

Now, we prove Lemma 1 by contradiction. Assume to the contrary that $|I(o) \triangle I(u)| \ge 8$. Then, $|I(o) \triangle I(u)| = 8$, and $|I(u) \setminus I(o)| = |I(o) \setminus I(u)| = 4$. By symmetry, assume that u lies straight right to o (see Figure 5). Let $\partial D_o \cap \partial D_u =$

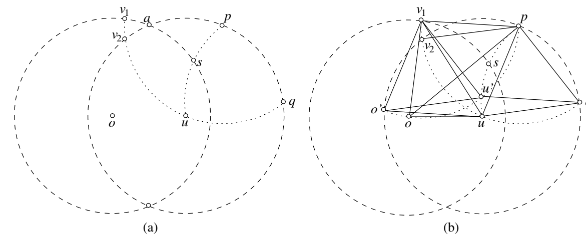

Fig. 4. $diam(\{v_1, v_2, p, s\}) = 1$.

$\{a, a'\}$ with a above ou. Sort of the four points in $I(u) \setminus I(o)$ (resp., $I(o) \setminus I(u)$) in the clockwise (resp., counterclockwise) order with respect to u (resp., o), and let uq_1 and uq_1' (resp., oq_2 and oq_2') be the radii of D_u (resp., D_o) through the second point and the third point respectively (see Figure 5). Let $p_1 \in \partial D_{q_1} \cap \partial D_u$ which is on the same side of uq_1 as a. Then $p_1 \notin D_o$. Let $s_1 \in \partial D_{q_1} \cap \partial D_o$ which is on the same side of oq_1 as a. Then, the arc triangle ap_1s_1 contains exactly one point in I. Similarly, we construct the other three arc triangles $a'p_1's_1'$, ap_2s_2, $a'p_2's_2'$ as shown in Figure 5. Each of them contains exactly one point in I. Thus, $diam(\{p_1, s_1, p_2, s_2\}) > 1$ and $diam(\{p_1', s_1', p_2', s_2'\}) > 1$. On the other hand, since $\angle q_1up_1 = \angle p_1'uq_1' = 60^o < \angle q_1'oq_1$, we have $\angle p_1up_1' < 180^o$. Similarly, $\angle p_2'up_2 < 180^o$. Thus, either $\angle uop_2 + \angle p_1uo < 180^o$ or $\angle p_2'ou + \angle oup_1' < 180^o$. By symmetry, we assume that the former inequality holds. Then by Lemma 11, $p_1p_2 < ou$. By Lemma 12, $diam(\{p_1, p_2, s_1\}) \le 1$ and $diam(\{p_1, p_2, s_2\}) \le 1$. Applying Lemma 12 again, $diam(\{p_1, s_1, p_2, s_2\}) \le 1$, which is a contradiction.

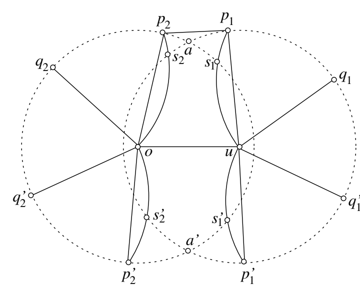

Fig. 5. $|I_o \triangle I_u| \leq 7$.

We proceed to prove Lemma 2 by establishing three additional lemmas.

Lemma 13: Suppose that $ou \le 1$, $a \in \partial D_o \cap \partial D_u$, and $v \in D_o \setminus D_u$ (see Figure 6). Let $p = a$ if $av \ge 1$ and otherwise the point on $\partial D_u \setminus D_o$ with $pv = 1$. Then, $\angle uov + \angle puo \ge 150^o$.

Lemma 14: Suppose that $ou \le 1$, $\partial D_o \cap \partial D_u = \{a, a'\}$, and $v \in I(o) \setminus I(u)$ (see Figure 7). Let $p = a$ if $av \ge 1$ and otherwise the point on $\partial D_u \setminus D_o$ with $pv = 1$, and let $q, q', p' \in \partial D_u$ be such that $pqq'p'$ is a semicircle not containing a and the four points p, q, q', p' are evenly distributed. Let $s' \in \partial D_{q'} \cap \partial D_o$ which is on the same side of oq as a'. Assume that $|I(u) \setminus I(o)| = 4$. Then $p' \notin D_o$ and the arc triangle

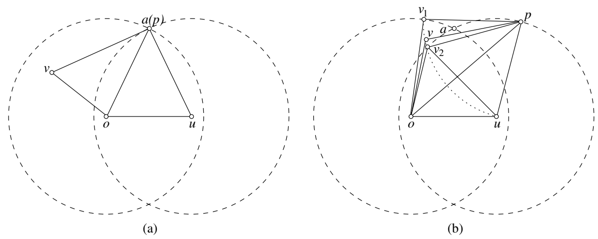

Fig. 6. $\angle uov + \angle puo \geq 150^o$: (a) $av \geq 1$; (b) $av < 1$.

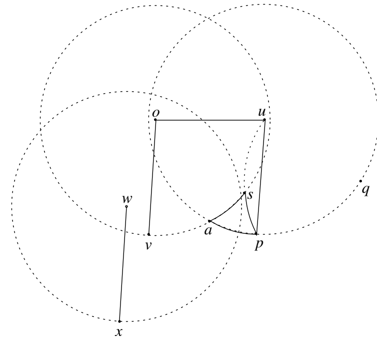

Fig. 8. Bound on the number of points in $(D_w \setminus (D_o \cup D_u)) \cap I$.

$a'p's'$ contains exactly one point in I. In addition, $ap < ou$ and $a'p' \leq ou$.

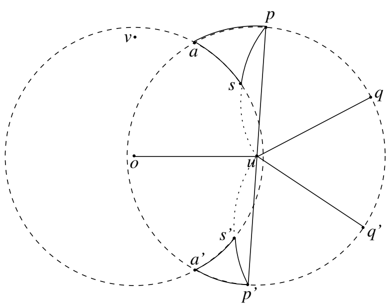

Fig. 7. $p' \notin D_o$, $a'p' \leq ou$, and arc triangle $a'p's'$ contains exactly one point in I.

Lemma 15: Suppose that $0 < ou \leq 1$, $a \in \partial D_o \cap \partial D_u$, $p \in \partial D_u \setminus D_o$ satisfying that $ap \leq ou$, $v \in \partial D_p \cap \partial D_o$ which is on the same side of op as a, $q \in \partial D_p \cap \partial D_u$ on the different side of pu from a, and $s \in \partial D_p \cap \partial D_u$ which is one the same side of oq as p (see Figure 8). Let $w \in D_u \setminus D_o$ which lies on the same side of ou as a but is not on the line ou, and x be such that $wxvo$ is a parallelogram. Assume that the arc triangle aps contains one point in I.

1) If $wp > 1 \geq wv$, then $x \notin D_o \cup D_u$ and $D_w \setminus (D_o \cup D_u)$ contains at most one point in I which lies on the same side of wx as a.
2) If $wp \leq 1$ or w lies on the same side of ov as a, then $D_w \setminus (D_o \cup D_u)$ contains at most three points in I.

The proof of Lemma 13 utilizes Lemma 11, and the proofs of Lemma 14 and Lemma 15 utilize Lemma 12. Due to the space limitation, these three proofs are omitted.

Finally, we can complete the proof of Lemma 2. For the clarity of presentation, we denote $I_0 = I(o)$, $I_j = I(u_j)$ for each $1 \leq j \leq 3$, and $I_j^* = \cup_{0 \leq i \leq 3, i \neq j} I_i$ for each $0 \leq j \leq 3$. We prove Lemma 2 by contradiction. Assume to the contrary that $|I_0^* \setminus I_0| > 11$. Then $\left|I_j \setminus I_j^*\right| = 4$ for each $1 \leq j \leq 3$. Thus, for any $1 \leq i < j \leq 3$, we have $|I_i \triangle I_j| = 8$, which implies that $u_i u_j > 1$ by Lemma 1. Suppose that u_1, u_2 and u_3 are in counterclockwise order with respect to o (see Figure 9). Let $v \in (I_0 \setminus \{o\}) \setminus I_0^*$. By symmetry, we assume that v is between ou_1 and ou_2. Then, the four points u_1, v, u_2, u_3 are independent. So their adjacent angle separations are all more than 60^o and less than 180^o. Hence, for $i = 1$ and 2, the point v is on the same side of ou_i as a_i while u_3 is on the same side of ou_i as a_i'. For $i = 1$ and 2, let p_i be the point a_i if $a_i v \geq 1$ and otherwise the point on $\partial D_i \setminus D_o$ such that $p_i v = 1$, and let $q_i, q_i', p_i' \in \partial D_i$ be such that $p_i q_i q_i' p_i'$ is a half circle not containing a_i and the four points p_i, q_i, q_i', p_i' are evenly distributed. Let $s_i' \in \partial D_{q_i'} \cap \partial D_o$ which is on the same side of oq_i as a_i'. By Lemma 14, $p_i' \notin D_o$, $a_i' p_i' \leq ou_i$ and the arc triangle $a_i' p_i' s_i'$ contains exactly one point in I. Let $v_i' \in \partial D_{p_i'} \cap \partial D_o$ which is on the same side of op_i' as a_i'. By Lemma 15(2), $u_3 p_i' > 1$ and u_3 lies on the different side of ov_i' from a_i'. In other words, ov_1' lies between ou_3 and ou_1, and ov_2' lies between ou_2 and ou_3. By Lemma 13, $\angle u_1 ov + \angle p_1 u_1 o \geq 150^o$. On the other hand, in the convex quadrilateral $ou_1 p_1' v_1'$, the side $ou_1 \leq 1$ and all other three sides equal to one. By Lemma 11, $\angle v_1' ou_1 + \angle ou_1 p_1' \geq 180^o$. Hence, $\angle v_1' ou_1 \geq 180^o - \angle ou_1 p_1' = \angle p_1 u_1 o$. Therefore, $\angle v_1' ou_1 + \angle u_1 ov_1 \geq 150^o$. Similarly, we can show that $\angle vou_2 + \angle u_2 ov_2 \geq 150^o$. Thus, $\angle v_2' ov_1' \leq 60^o$. So $u_3 v_i' \leq 1$. Let x_i be the point such that $u_3 x_i v_i' o$ is a parallelogram for $i = 1$ and 2. We split $D_3 \setminus (D_o \cup D_1 \cup D_2)$ by $u_3 x_1$ and $u_3 x_2$ into three regions. The middle region has a diameter of at most one and contains at most one point in I. By Lemma 15, each of the other two regions contains at most one point in I. So $|I_3 \setminus I_3^*| \leq 3$, which is a contradiction.

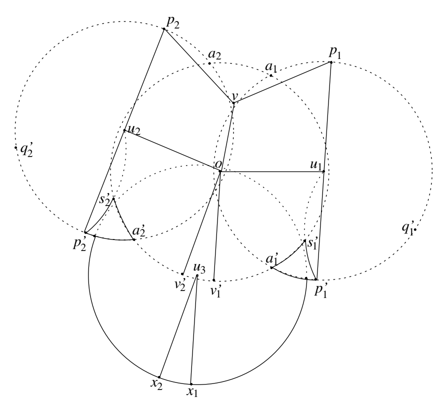

Fig. 9. At most 11 independent points can be packed in $\left(\cup_{i=1}^{3} D_i\right) \setminus D_o$.