Tighter Approximation Bounds for Minimum CDS in Wireless Ad Hoc Networks⋆

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Abstract. Connected dominating set (CDS) has a wide range of applications in wireless ad hoc networks. A number of approximation algorithms for constructing a small CDS in wireless ad hoc networks have been proposed in the literature. The majority of these algorithms follow a general two-phased approach. The first phase constructs a dominating set, and the second phase selects additional nodes to interconnect the nodes in the dominating set. In the performance analyses of these two-phased algorithms, the relation between the independence number α and the connected domination number γ_c of a unit-disk graph plays the key role. The best-known relation between them is $\alpha \leq 3\frac{2}{3}\gamma_c + 1$. In this paper, we prove that $\alpha \leq 3.4306\gamma_c + 4.8185$. This relation leads to tighter upper bounds on the approximation ratios of two approximation algorithms proposed in the literature.

1 Introduction

Connected dominating set (CDS) has a wide range of applications in wireless ad hoc networks (cf. a recent survey [3] and references therein). Consider a wireless ad hoc network with undirected communication topology $G = (V, E)$. A CDS of G is a subset $U \subset V$ satisfying that each node in $V \setminus U$ is adjacent to at least one node in U and the subgraph of G induced by U is connected. A number of distributed algorithms for constructing a small CDS in wireless ad hoc networks have been proposed in the literature. The majority of these distributed algorithms follow a general two-phased approach [1,2,4,8,10,11,12]. The first phase constructs a dominating set, and the nodes in the dominating set are called dominators. The second phase selects additional nodes, called connectors, which together with the dominators induce a connected topology. The algorithms in [1,2,4,8,10,11] differ in how to select the dominators and connectors. For example, the algorithm in [2] selects the dominators using the Chvatal's greedy

algorithm [5] for Set Cover, the algorithms in [1,10] select an arbitrary maximal independent set (MIS) as the dominating set, and all the algorithms in [4,8,11,12] choose a special MIS with 2-hop separation property as the dominating set.

The approximation ratios of these two-phased algorithms [1,2,4,8,10,11] have been analyzed when the communication topology is a unit-disk graph (UDG). For a wireless ad hoc network in which all nodes lie in a plane and have equal maximum transmission radii normalized to one, its communication topology $G = (V, E)$ is often modelled by a UDG in which there is an edge between two nodes if and only if their Euclidean distance is at most one. Except the algorithms in [2,10] which have logarithmic and linear approximations ratios respectively, all other algorithms in [1,4,8,11,12] have constant approximation ratios. The algorithm in [1] targets at distributed construction of CDS in linear time and linear messages. With this objective, it trades the size of the CDS with the time complexity, and thus its approximation ratio is a large constant (but less than 192). The analyses of the algorithms in [4,8,11,12] rely on the relation between the independence number (the size of a maximum independent set) α and the connected domination number (the size of a minimum connected dominating set) γ_c of a connected UDG G. A loose relation $\alpha \leq 4\gamma_c + 1$ was obtained in [11], which implies an upper bound of 8 on the approximation ratios of both algorithms in [4,11]. A refined relation $\alpha \leq 3.8\gamma_c + 1.2$ was discovered in [13]. With such a refined relation, the upper bound on the approximation ratios of both algorithms in [4,11] was reduced from 8 to 7.6, and an upper bound of $5.8 + \ln 5 \approx 7.41$ on the approximation ratio of the algorithms in [8] was derived (the bound $4.8 + \ln 5 \approx 6.41$ in [8] was incorrect). The best-known relation $\alpha \leq 3\frac{2}{3}\gamma_c + 1$ if G has at least two nodes was recently proven in [12]. As a result, the upper bound on the approximation ratio of the algorithm in [11] was further reduced to $7\frac{1}{3}$ in [12]. Another greedy approximation algorithm was also proposed in [12] and its approximation ratio was proven to be bounded by $6\frac{7}{18}$.

In this paper, we first prove a further improved relation $\alpha \leq 3.4306\gamma_c + 4.8185$ in Section 3. The proof for this bound employs an integrated area and length argument, and involves some other interesting extreme geometric problems which are studied in Section 2. Subsequently in Section 4, we provide tighter analyses of the approximation algorithm in [11] and the other greedy algorithm in [12]. We prove that the approximation ratio of the former algorithm is at most 6.862 and the approximation ratio of the latter algorithm is at most 6.075.

We remark that a recent paper [7] claimed that for any connected UDG G,

$$\alpha \leq 3.453\gamma_c + 8.291.$$

However, as discovered in [12], the proof for a key geometric extreme property underlying such claim was missing, and such proof is far from being apparent or easy. Such property is rigorously proved in Lemma 5. Consequently, the bound claimed in [7] can be treated at most as a conjecture at the time of its publication rather than a proven result.

In the remaining of this section, we introduce some terms and notations. For any point u and any $r > 0$, we use $disk_r(u)$ to denote the closed disk of radius r centered at u, and $circle_r(u)$ to denote the boundary circle of $disk_r(u)$. A path

or a polygon is said to be inscribed in a circle if all its vertices lie on the circle. The Lebesgue measure (or area) of a measurable set $A \subset \mathbb{R}^2$ is denoted by $|A|$. The topological boundary of a set $A \subset \mathbb{R}^2$ is denoted by ∂A. For the simplicity of presentation, the line segment between two points u and v and its length are both denoted by uv by slightly abusing the notation, but the actual meaning can be clearly told from the context.

2 Canonical Polygons and Inscribed Polygons

Suppose that s, o and t are three points from the left to the right on a horizontal line with $os = 1$ and $ot = 0.5$. For any pair of points u and v on $circle_{1/\sqrt{3}}(o)$, let $\widehat{uv}$ be the arc in $circle_{1/\sqrt{3}}(o)$ from u to v in the counterclockwise manner. Denote by ϕ the radian of $\widehat{uv}$, and let $k = \lceil \phi/(\pi/3) \rceil$. We construct a path Q of k edges from u to v with all vertices on $\widehat{uv}$ as follows: If ϕ is an integer multiple of $\frac{\pi}{3}$, then all edges of Q are tangent to $circle_{0.5}(o)$; otherwise, all edges except the $\lceil k/2 \rceil$-th edge are tangent to $circle_{1/\sqrt{3}}(o)$ (we remark that in this case, the $\lceil k/2 \rceil$-th edge is disjoint from $circle_{1/\sqrt{3}}(o)$). The path Q is referred to as the *canonical path* inscribed in $circle_{1/\sqrt{3}}(o)$ from u to v.

For any point u which lies on the right side the the vertical line through t, we construct a polygon P as follows: let u_1 and u_2 be the two points on $circle_{1/\sqrt{3}}(o)$ such that the two line segments $u_1 u$ and $u_2 u$ are tangent to $circle_{1/\sqrt{3}}(o)$ and u_1 is above the line st. Then, P is surrounded by $u_1 u$, $u_2 u$ and the canonical path from u_1 to u_2. The polygon P is referred to as the *canonical polygon* of u. The point u is called the *base vertex* of P, and the angle $\theta = \arccos \frac{1}{2ou}$ is called the *base angle* of P. Note that if u is on the ray ot, then P is symmetric with respect to the line ot, and the area of $P \cap disk_{1.5}(s)$ is a function of the base angle θ, which is denoted by $f(\theta)$. In this section, we will derive the explicit expression of $f(\theta)$ and explore some useful properties of the function $f(\theta)$. We will also prove that for any canonical polygon P, $|P \cap disk_{1.5}(s)| \geq f(\theta)$ where θ is the base angle of P.

We first introduce a geometric function g on $[0, \pi]$ defined as follows. For any $\theta \in [0, \pi]$, let v be a point on $circle_{1/2}(o)$ satisfying that $\angle tov = \theta$ and v is above st. Let w be the point on $circle_{1.5}(s)$ satisfying that vw is tangent to $circle_{1/2}(o)$ and w lies to the right of v. Then, $g(\theta)$ is defined to be the area of the region surrounded by arc tw and the three line segments ot, ov and vw (see Figure 1). The next lemma presents the explicit expressions of $g(\theta)$ and its first and second order derivatives.

Due to space limit, we omit some of the proofs in this version.

Lemma 1. *Let* $\beta = \theta - \arccos \frac{1+2\cos\theta}{3}$. *Then,*

$$g(\theta) = \frac{9}{8}\beta - \frac{3}{4}\sin\beta + \frac{\sqrt{6}}{4}\sin\frac{\beta}{2},$$
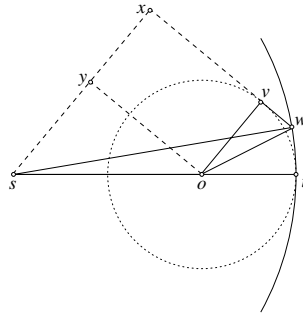
$$g'(\theta) = \frac{13}{8} - 1.5\cos\beta,$$

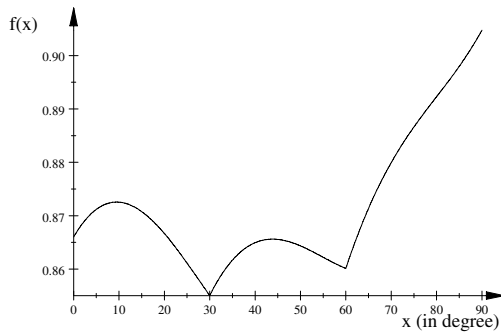**Fig. 1.** Calucaltion of $\angle wst$ and $g(\theta)$

Fig. 2. The curve of f on $[0\,°, 90\,°)$

$$g''(\theta) = 1.5 \left(1 - \sqrt{1 - \frac{1}{2 + \cos\theta}}\right) \sin\beta.$$

In addition, g is increasing and convex on $[0, \pi]$, while both g' and g'' are increasing on $\left[0, \frac{\pi}{2}\right]$.

It is easy to show that $f(\theta) = 2g(\theta) + h(\theta)$, where

$$h(\theta) = \frac{1}{4\sqrt{3}} \left\lfloor 6 - \frac{\theta}{\frac{\pi}{6}} \right\rfloor + \frac{1}{6} \sin 2 \left(\left(6 - \left\lfloor 6 - \frac{\theta}{\frac{\pi}{6}} \right\rfloor\right) \frac{\pi}{6} - \theta\right).$$

Figure 2 is the curve of f on $[0\,°, 90\,°)$. We observe and will prove later that f is increasing on $[60\,°, 90\,°)$. However, on either of the two intervals $[0\,°, 30\,°]$ and $[30\,°, 60\,°]$, f is neither monotone, nor concave, and nor convex. Fortunately, on either of these two intervals f has the following weak but still nice *quasi-concave* property: f is said to be quasi-concave on an interval $[a, b] \subset [0\,°, 90\,°)$ if for each triple of increasing values $\theta_1, \theta_2, \theta_3$ in $[a, b]$, $f(\theta_2) \geq \min\{f(\theta_1), f(\theta_3)\}$.

Lemma 2. *f is quasi-concave on $\left[0, \frac{\pi}{6}\right]$ and $\left[\frac{\pi}{6}, \frac{\pi}{3}\right]$ respectively, and increasing on $\left[\frac{\pi}{3}, \frac{\pi}{2}\right)$.*

Denote $f\left(\frac{\pi}{6}\right)$ by σ. Then,

$$\sigma = \frac{\sqrt{3}}{6} - \frac{1}{2} + \frac{\sqrt{8 + 2\sqrt{3}}}{4} + \frac{3\pi}{8} - \frac{9}{4}\arccos\frac{1 + \sqrt{3}}{3} \approx 0.855\,053\,28.$$

It is easy to verify that $f(0) = \sqrt{3}/2$ and $f(32°) < f\left(\frac{\pi}{3}\right) < f(34°)$. So, by Lemma 2 we have the following corollary.

Corollary 1. *The minimum of f on the interval $[0°, 90°)$ (respectively, $[32°, 90°)$ and $[34°, 90°))$ is achieved at $30°$ (respectively, $32°$ and $60°$).*

Finally, we prove the following extreme property of the canonical polygons.

Lemma 3. *For any canonical polygon P with base angle θ, $|P \cap disk_{1.5}(s)| \geq f(\theta)$.*

Next, we prove the following lemma about inscribed polygons.

Lemma 4. *Suppose that P is a polygon inscribed in $circle_{1/\sqrt{3}}(o)$ satisfying that $disk_{0.5}(o) \subseteq P$. Then,*

$$|P| > \sqrt{3}/2,$$
$$|P \cap disk_{1.5}(s)| \geq \sigma.$$

3 Independence Number vs. Connected Domination Number

In this section, we present an improved upper bound on the independence number in terms of the connected domination number.

Theorem 1. *Let α and γ_c be the independence number and connected domination number of a connected UDG G. Then,*

$$\alpha \leq 3.4306\gamma_c + 4.8185.$$

We prove the above theorem by an integrated area and length argument. Let U be a minimum CDS of G, and define

$$\Omega = \bigcup_{u \in U} disk_{1.5}(u).$$

Consider a maximum independent set I of G. We construct the Voronoi diagram defined by I. For each $o \in I$, we use $Vor(o)$ to denote its Voronoi cell and call the set $Vor(o) \cap \Omega$ as the *truncated Voronoi cell* of o. Clearly, $|\Omega|$ is the total area of truncated Voronoi cells of all nodes in I. We partition I into two subsets I_1 and I_2 defined by

$$I_1 = \left\{o \in I : disk_{1/\sqrt{3}}(o) \subset \Omega\right\},$$
$$I_2 = I \setminus I_1.$$

Denote by α_1 and α_2 the size of I_1 and I_2 respectively. The next lemma provides a lower bound on each truncated Voronoi cell.
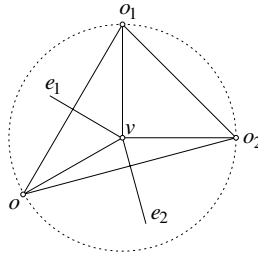
Fig. 3. Any vertex of $Vor(o)$ is apart from o by at least $1/\sqrt{3}$

Lemma 5. *For each o in I_1 (respectively, I_2), the area of its truncated Voronoi cell is at least $\sqrt{3}/2$ (respectively, σ).*

Proof. Since the pairwise distances of the points in I are at least one, the distance between o and each side of $Vor(o)$ is at least 0.5 and consequently $disk_{0.5}(o) \subset Vor(o)$. Next, we show that no vertex of $Vor(o)$ is inside $disk_{1/\sqrt{3}}(o)$. Let v be a vertex of $Vor(o)$, and e_1 and e_2 be the two sides of $Vor(o)$ incident to v (see figure 3). Let o_1 (respectively, o_2) be the point which is symmetric to o with respect to e_1 (respectively, e_2). Then, both o_1 and o_2 belong to I, and hence the three sides of $\triangle oo_1o_2$ are all at least 2. Clearly, v is the center of $\triangle oo_1o_2$. Since at least one of the three central angles of $\triangle oo_1o_2$ is at most $120°$, the circumscribing radius of $\triangle oo_1o_2$ is at least $1/\sqrt{3}$. Thus, $ov \geq 1/\sqrt{3}$.

Let s be the node in the MCDS U closest to o. Then, $o \in disk_1(s)$. If $disk_{1/\sqrt{3}}(o) \subseteq Vor(o)$, then $|Vor(o) \cap \Omega| \geq \left| disk_{1/\sqrt{3}}(o) \cap disk_{1.5}(s) \right| > \sqrt{3}/2$. So, we assume $disk_{1/\sqrt{3}}(o)$ is not fully contained in $Vor(o)$. Then $Vor(o)$ intersects $circle_{1/\sqrt{3}}(o)$. We construct a polygon $P \subseteq Vor(o)$ satisfying that P is inscribed in $circle_{1/\sqrt{3}}(o)$ and $disk_{0.5}(o) \subseteq P \subseteq Vor(o)$. Let Q be the sequence of intersecting points between $Vor(o)$ and $circle_{1/\sqrt{3}}(o)$ in the counterclockwise order. For each pair of successive u and v in Q, if $\angle uov \leq \frac{\pi}{3}$, we add to P a side between u and v; otherwise, we add to P a path inscribed in the arc from u to v satisfying that each edge in this path is either tangent to or disjoint from $circle_{1/\sqrt{3}}(o)$ (see Figure 4). The resulting polygon P meets the requirement. By Lemma 4, $|P| \geq \sqrt{3}/2$.

If $o \in I_1$, then

$$P \subseteq Vor(o) \cap disk_{1/\sqrt{3}}(o) \subseteq Vor(o) \cap \Omega,$$

hence

$$|Vor(o) \cap \Omega| \geq |P| \geq \sqrt{3}/2.$$

Now, we assume that $o \in I_2$. Note that $|P \cap disk_{1.5}(s)|$ grows when moving o away from s along a fixed radius of $disk_{1.5}(s)$. By Lemma 4, $|P \cap disk_{1.5}(s)| \geq \sigma$. Since

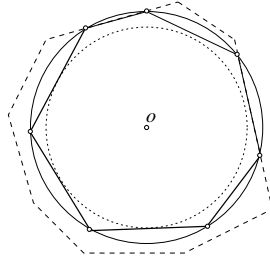$$P \cap disk_{1.5}(s) \subseteq Vor(o) \cap \Omega,$$

Fig. 4. Inserting operations

we have

$$|Vor\,(o) \cap \Omega| \geq |P \cap disk_{1.5}\,(s)| \geq \sigma.$$

We define

$$\Omega' = \bigcup_{v \in U} disk_{1.5-1/\sqrt{3}}\,(v)\,.$$

The next lemma gives an upper bound on the length of $\partial\Omega'$.

Lemma 6. *The length of $\partial\Omega'$ is at most* $2\left(1 - 1/\sqrt{3}\right)\alpha_2$.

By Lemma 5,

$$|\Omega| \geq \frac{\sqrt{3}}{2}\alpha_1 + \sigma\alpha_2 = \frac{\sqrt{3}}{2}\alpha - \left(\frac{\sqrt{3}}{2} - \sigma\right)\alpha_2,$$

which implies

$$\alpha \leq \frac{|\Omega|}{\frac{\sqrt{3}}{2}} + \left(1 - \frac{\sigma}{\frac{\sqrt{3}}{2}}\right)\alpha_2. \tag{1}$$

It is easy to prove by induction on γ_c that

$$|\Omega| \leq \frac{9}{2}\left((\gamma_c - 1)\left(\arcsin\frac{1}{3} + \frac{\sqrt{8}}{9}\right) + \frac{\pi}{2}\right), \tag{2}$$

and the length of of $\partial\Omega'$ is at most

$$2\left(3 - \frac{2}{\sqrt{3}}\right)\left((\gamma_c - 1)\arcsin\frac{1}{3 - \frac{2}{\sqrt{3}}} + \frac{\pi}{2}\right).$$

By Lemma 6,

$$\alpha_2 \leq \frac{2\left(3 - \frac{2}{\sqrt{3}}\right)\left((\gamma_c - 1)\arcsin\frac{1}{3 - \frac{2}{\sqrt{3}}} + \frac{\pi}{2}\right)}{2\left(1 - \frac{1}{\sqrt{3}}\right)}$$

$$= \frac{\sqrt{3} + 7}{2}\left((\gamma_c - 1)\arcsin\frac{1}{3 - \frac{2}{\sqrt{3}}} + \frac{\pi}{2}\right). \tag{3}$$

The three inequalities (1), (2) and (3) imply altogether that α is at most

$$(\gamma_c - 1) \left(\frac{\sqrt{27} \left(\arcsin \frac{1}{3} + \frac{\sqrt{8}}{9} \right)}{+ \frac{\left(1 - \frac{\sigma}{\frac{\sqrt{3}}{2}}\right)(\sqrt{3}+7)}{2} \arcsin \frac{1}{3 - \frac{2}{\sqrt{3}}}} \right) + \frac{\pi}{2} \left(\sqrt{27} + \left(1 - \frac{\sigma}{\frac{\sqrt{3}}{2}}\right) \frac{\sqrt{3}+7}{2} \right)$$

$$\approx 3.4305176\gamma_c + 4.8184688.$$

Thus, Theorem 1 follows.

4   Tighter Approximation Ratios

In this section, we derive tighter bounds on the approximation ratio of the distributed algorithm proposed in [11] and the other greedy algorithm proposed in [12]. For the convenience of presentation, we call them **WAF** and **WWY** respectively. Let $G = (V, E)$ be a unit-disk graph. We denote by α and γ_c the independence number and connected domination number of G respectively. For any finite set S, we use $|S|$ to denote the cardinality of S.

   The CDS produced by the algorithm **WAF** consists of a maximal independent set I and a set C of connectors. Specifically, let T be an arbitrary rooted spanning tree of G. The set I is selected in the first-fit manner in the breadth-first-search ordering in T. Let s be the neighbor of the root of T which is adjacent to the largest number of nodes in I. Then, C consists of s and the parents (in T) of the nodes in $I \setminus I(s)$. It was proved in [11] that $I \cup C$ is a CDS and $|I \cup C| \leq 8\gamma_c - 1$. Later on, two progressively improved tighter bounds $7.6\gamma_c + 1.4$ and $7\frac{1}{3}\gamma_c$ were obtained in [13] and [12] respectively. The next theorem further improves the bound on $|I \cup C|$.

Theorem 2. *The CDS produced by the algorithm **WAF** has size at most* 6. $862\gamma_c + 8.637$.

Proof. Let I and C be the set of nodes selected by the algorithm **WAF** in the first phase and the second phase respectively. Since $|C| \leq |I| - 1$, we have

$$|I \cup C| \leq 2|I| - 1 \leq 2(3.4306\gamma_c + 4.8185) - 1 \leq 6.862\gamma_c + 8.637.$$

So, the theorem follows.

In the next, we study the algorithm **WWY**. The first phase of this algorithm is the same as the algorithm **WAF**, and we let I be the selected maximal independent set. But the second phase selects the connectors in a more economic way. For any subset $U \subseteq V \setminus I$, let $q(U)$ be the number of connected components in $G[I \cup U]$. For any $U \subseteq V \setminus I$ and any $w \in V \setminus I$, we define

$$\triangle_w q(U) = q(U) - q(U \cup \{x\}).$$

The value $\triangle_w q(U)$ is referred to as the *gain* of w with respect to U. The following lemma was proved in [12].

Lemma 7. *Suppose that $q(U) > 1$ for some $U \subseteq V \setminus I$. Then, there exists a $w \in V \setminus (I \cup U)$ such that $\triangle_w q(U) \geq \max\{1, \lceil q(U)/\gamma_c \rceil - 1\}$.*

The second phase of the algorithm **WWY** runs as follows. We use C to denote the sequence of selected connectors. Initially C is empty. While $q(C) > 1$, choose a node $w \in V \setminus (I \cup C)$ with *maximum* gain with respect to C and add w to C. When $q(C) = 1$, then $I \cup C$ is a CDS. It was proved in [12] that $|I \cup C| \leq 6\frac{7}{18}\gamma_c$. We derive a tighter bound on the output CDS in the theorem below.

Theorem 3. *The CDS produced by the algorithm **WWY** has size at most $6.075\gamma_c + 5.425$.*

Proof. Let I and C be the set of nodes selected by the algorithm **WWY** in the first phase and the second phase respectively. If $\gamma_c = 1$, then $|I| \leq 5$ and $|C| \leq 1$, hence $|I \cup C| \leq 6$. Thus, the theorem holds trivially if $\gamma_c = 1$. If $|I| \leq 3\gamma_c + 2$, then $|I \cup C| \leq 2|I| - 1 \leq 6\gamma_c + 3$, and the theorem also holds. From now on, we assume that $\gamma_c \geq 2$ and $|I| > 3\gamma_c + 2$.

We break C into three contiguous (and possibly empty) subsequences C_1, C_2 and C_3 as follows. C_1 is the shortest prefix of C satisfying that $q(C_1) \leq 3\gamma_c + 2$, and $C_1 \cup C_2$ is the shortest prefix of C satisfying that $q(C_1 \cup C_2) \leq 2\gamma_c + 1$. We can prove that

$$|C_1| \leq \begin{cases} \frac{|I|}{3} - \gamma_c & \text{if } q(C_1) \leq 3\gamma_c + 1, \\ \frac{|I|-2}{3} - \gamma_c & \text{if } q(C_1) = 3\gamma_c + 2; \end{cases}$$

$$|C_2| \leq \begin{cases} \frac{\gamma_c}{2} & \text{if } q(C_1) \leq 3\gamma_c + 1, \\ \frac{\gamma_c+1}{2} & \text{if } q(C_1) = 3\gamma_c + 2; \end{cases}$$

$$|C_3| \leq 2\gamma_c - 1.$$

From the first two inequalities, we have

$$|C_1 \cup C_2| \leq \frac{|I|}{3} - \frac{\gamma_c}{2}.$$

Using the third inequality, we have

$$|C| \leq \frac{|I|}{3} - \frac{\gamma_c}{2} + 2\gamma_c - 1 = \frac{|I|}{3} + \frac{3}{2}\gamma_c - 1.$$

So,

$$|I \cup C| \leq \frac{4|I|}{3} + \frac{3}{2}\gamma_c - 1$$
$$\leq \frac{4}{3}(3.4306\gamma_c + 4.8185) + \frac{3}{2}\gamma_c - 1$$
$$\leq 6.075\gamma_c + 5.425.$$

Thus, the theorem follows.

5   Discussions

In this paper, we obtained a tighter relation between the independence number and connected domination number of a connected UDG. We actually proved the following stronger result on packing. Let V be a set of n nodes of a connected dominating set, and Γ be the unions of unit-disks centered at V. Then, we can pack in Γ at most $3.4306n + 4.8185$ points whose pairwise distances are greater than or equal to one. We'd like to emphasize that here we allow two points packed in Γ to have distance equal to one. On the other hand, a packing of $3n + 3$ points in Γ whose pairwise distances are greater than one was presented in [12]. It was also conjectured $3n+3$ is the exact bound. Thus, there is still a gap between the bound $3.4306n + 4.8185$ derived in this paper and the conjectured bound $3n + 3$.

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