A New Multihop Lightwave Network Based on the Generalized De-Bruijn Graph

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Abstract

Lightwave networks can be built by embedding virtual topologies over physical topologies. The optical passive star enables such embeddings easily. The Shuffle-net and the De-Bruijn graph are two popular virtual topologies proposed in the past for lightwave networks. Both however suffer from the lack of flexibility in scaling network sizes. In this paper, we present a generalization of the De-Bruijn network which overcomes the limitation of the strict relationships between the network size parameters seen in the Shuffle-net and the De-Bruijn networks. A generalization for the Shuffle-net is also possible with this idea. We emphasize the support of Time and Wavelength Division Multiplexed media access protocols for such architectures and present several properties of the proposed network with respect to the same.

1 Introduction

Optical passive stars [4, 8] provide a simple medium to connect nodes in a local or metropolitan area network. Each node is connected to the star via a pair of unidirectional fibers. The light signals entering the star are evenly divided among all the outgoing fibers such that a transmission from any node is receivable by all the nodes in the network. Passive stars present the advantage of smaller power losses as compared to linear optical busses [2]. This leads to greater network sizes. Moreover, the operation of the network is completely passive which provides greater reliability. The broadcast nature of the optical star can be exploited to build virtual topologies with smaller average delays [16]. Fig. 1 shows an optical passive star network with N nodes.

Several multihop lightwave networks have been proposed which use different regular virtual topologies such as a re-circulating multistage p-Shuffle [6], the de Bruijn graph [13] and the Bus-Mesh [3]. Regular virtual topologies present several advantages including simple routing, predictable path lengths, balanced loads, enhanced maximum throughput and the ability to cross embed other regular topologies. Multihop lightwave networks based on regular virtual topologies using TWDM schemes for media access offer the possibility of tapping the vast bandwidth of the fiber optic medium. Regular virtual topologies supported on optical passive stars are preferable because of the properties outlined above. Use of a small number of fixed wavelength devices further simplifies design and reduces costs.

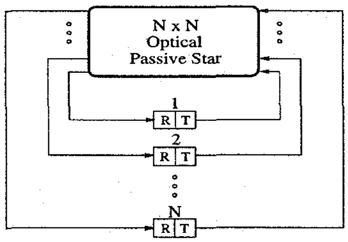

Figure 1: A passive star connected network : R = receiver, T = transmitter

The previously proposed lightwave network topologies such as the De-Bruijn graph [13] and the Shuffle-net [6] are not flexible enough to support networks of arbitrary sizes. These topologies severely constrain network sizes by imposing a strict relationship between their network size parameters. In this paper, we propose a generalized De-Bruijn network topology to eliminate these size constraints. Both the De-Bruijn graph and the Shuffle-net can be viewed as specific instances of the proposed topology. We highlight several properties of the generalized De-Bruijn network concerning support of a TWDM media access protocol for such networks.

In section 2 we present a brief background on the De-Bruijn graph and the Shuffle-net lightwave networks. We present the generalized De-Bruijn topology in detail in section 3. Section 4 presents a TWDM media access protocol for the generalized De-Bruijn network. We highlight several design trade-offs and properties of TWDM protocols over the proposed topology. In section 5 we evaluate the performance of the generalized De-Bruijn and compare it with the Shuffle-net and the De-Bruijn graphs. We conclude in section 6.

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2 Background and Previous Work

The Shuffle-net [6] and the De-Bruijn graph [13] are two popular regular topologies, proposed in the past for multihop lightwave networks. In both networks, M interconnections are provided where $M << N \cdot (N-1)$, i.e., a small subset of the set of interconnections needed for an N node graph to be completely connected. Thus a packet may have to make multiple hops to reach the destination.

2.1 The Shuffle-net Multihop Network

In the (p,k) Shuffle-net [6], $N = k \cdot p^k$ nodes are arranged in k columns[1] of p^k nodes each. Each column is connected to the next one by p^{k+1} directed links in a fixed p-shuffle pattern with the last column connected to the first one in a wrapped around fashion. Fig. 2 shows a Shuffle-net interconnected network with 18 nodes where $p = 3$ and $k = 2$. In this figure, the first column is repeated on the right to show the wrapped around nature of the connection.

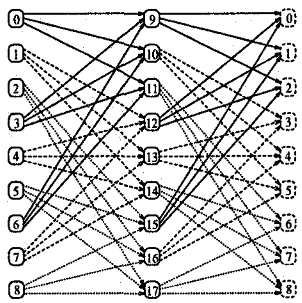

Figure 2: An 18 node Shuffle-net with $p = 3, k = 2$

If the nodes are numbered as shown in Fig. 2, then, the address of a node m, $(0 \leq m < k \cdot p^k)$, can be transformed into a 2-tuple, (c,r), where c, $(0 \leq c < k)$, denotes the column address and r, $(0 \leq r < p^k)$, denotes the row address of the node. We have $(c,r) = (\lfloor \frac{m}{p^k} \rfloor, m \bmod p^k)$. Letting $j = (i \cdot p) \bmod p^k$, a node with row address i has p arcs directed to nodes with row addresses $j, (j+1), \ldots, (j+p-1)$. The resulting interconnection pattern between two adjacent stages is called the p-shuffle, being a generalization of the perfect-shuffle $(p = 2)$ [9]. There are a total of $k \cdot p^{k+1}$ links in a Shuffle-net, with p incoming and p outgoing links at each node.

We observe the following :

- There is a strict relationship between the network size parameters, N, p and k, i.e., the number of nodes in the network, N, is always given by the relation $N = k \cdot p^k$.

[1] The terms *column* and *stage* are used interchangeably.

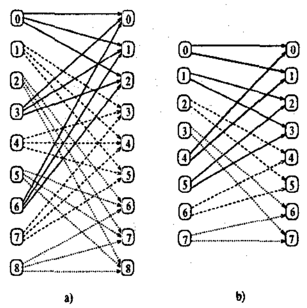

Figure 3: a) A (3,2) De-Bruijn Graph b) A (2,3) De-Bruijn Graph

- The diameter of the network is $2k - 1$ since a packet may have to make two passes through the network in order to reach all the nodes [6, 10].
- If channel sharing is not allowed, i.e., if each link represents a different wavelength, then, the total number of wavelengths, W, employed is, $k \cdot p^{k+1}$, or $N \cdot p$. In this case, p represents the number of transmitters and and the number of receivers at each node since each node transmits on p separate wavelengths and receives on p separate wavelengths.
- If channel sharing is allowed, then using a single fixed wavelength transmitter and a single fixed wavelength receiver at each node, the total number of wavelengths, W, reduces to, $k \cdot p^{k-1}$, or N/p. In this case, if a node transmits on a wavelength λ_i, then all the p nodes in the succeeding stage that are connected to the transmitting node should receive on wavelength λ_i. Similarly, if a node receives on a wavelength λ_j, then all the p nodes in the preceding stage connected to the receiving node must transmit on the same wavelength λ_j.

2.2 The De-Bruijn Multihop Network

The (p,k) De-Bruijn graph has $N = p^k$ nodes connected in a recirculating Shuffle-exchange pattern. A node with address i, $0 \leq i < p^k$, has p arcs directed to nodes with addresses $j, (j+1), \ldots, (j+p-1)$, where $j = (i \cdot p) \bmod p^k$. The interconnection function is thus the same as that in the Shuffle-net. However, there is only one stage in the De-Bruijn graph. In other words, if all stages in the (p,k) Shuffle-net represent the same set of nodes, then it is equivalent to a (p,k) De-Bruijn graph. Figs. 3a) and b) show a (3,2) De-Bruijn graph and a (2,3) De-Bruijn graph respectively.

We observe the following :

- Once again, the relationship $N = p^k$ has to be maintained. Thus, the network size can only be a small set of numbers.

- The diameter of the network is k.
- With no channel sharing, the number of wavelengths employed is $p^{k+1} = N \cdot p$. As with the Shuffle-net, p denotes the number of transmitters and the number of receivers at each node.
- With channel sharing, if p channels use the same wavelength then $p^{k-1} = N/p$ wavelengths will be used.

2.3 Motivation for the Proposed Topology

Because of the strict relations between N, p and k, the Shuffle-net and the De-Bruijn graph can only realize a small number of network sizes. In this paper, we propose a multihop network based on the generalized De-Bruijn interconnection that requires only the trivial condition, $N \geq p$. Thus the size restriction is overcome. This network also has a diameter of $\lceil \log_p N \rceil$. We propose a single stage design with the generalized De-Bruijn interconnection. The idea can easily be extended for a multistage network to yield a generalized Shuffle-net. The proposed network uses a simple routing algorithm to forward packets. The diameter and the average distance (in hops) of the proposed network is shown to be shorter than that of the Shuffle-net with the same p and N. The importance and benefits of TWDM media access schemes have already been highlighted. In this paper, we wish to focus on the support of a TWDM protocol for the generalized De-Bruijn network.

3 The Generalized De-Bruijn Network

3.1 Topology

We briefly revisit the p-shuffle interconnection. Let $N = p^k$ nodes be interconnected in a p-shuffle pattern. Let the address of a node in the network be represented by a p-ary number, k-digits wide. We can write the address of a node a, $(0 \leq a < p^k)$, as, $< a_{k-1}, a_{k-2}, \ldots, a_2, a_1, a_0 >$, $(0 \leq a_0, a_1, \ldots, a_{k-1} < p)$. A node in the p-shuffle interconnection network has p successors connected via p links. The p links are numbered as $0, 1, \ldots, (p-1)$, which we shall call *link indices.* The p-shuffle interconnection for a node with address a is then described by the following p interconnection functions :

$$\mathcal{F}_i(< a_{k-1}, a_{k-2}, \ldots, a_1, a_0 >) = < a_{k-2}, a_{k-3}, \ldots, a_1, a_0, i >, \quad (1)$$
$$\text{where } 0 \leq i < p \text{ and } 0 \leq a_j < p, \forall j, 0 \leq j < k.$$

In equation 1, i represents the link index. Thus, $\mathcal{F}_i(a)$ represents the function describing the connection of node a to its i^{th} successor $(0 \leq i < p)$. The interconnection functions in equation 1 can be generalized for any $p \leq N$ as follows [7] :

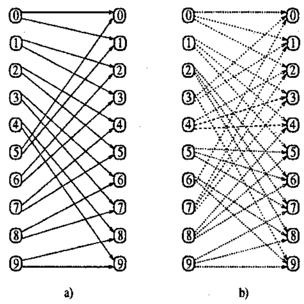

Figure 4: a) A (2,10) generalized De-Bruijn graph b) A (3,10) generalized De-bruijn graph

$$\mathcal{G}_i(a) = (a \cdot p + i) \bmod N, \quad (2)$$

where a and i are defined as before. Again, in equation 2, i represents the link index for node a. Thus $\mathcal{G}_i(a)$ gives the address of the node connected to node a via its i^{th} link. We call the network resulting from the interconnection function defined as equation 2, the *generalized De-Bruijn* network. It is seen that equation 1 is a special case of equation 2 with $N = p^k$. The generalized De-Bruijn network is thus characterized by only two network size parameters - N and p, for any p and N, as long as $p \leq N$ which is trivially maintained. We can thus denote such networks as (p, N) networks. Figs. 4a) and b) show the (2,10) and (3,10) generalized De-Bruijn networks respectively.

3.2 Routing Algorithm

A simple shortest path routing algorithm for the generalized De-Bruijn network is presented in this section. Let N_a^k denote the set of nodes which are reachable from node a in exactly k hops. Let $\mathcal{G}_{i_1, i_2, \ldots, i_k}(a)$ be the composition $\mathcal{G}_{i_k}(\mathcal{G}_{i_{k-1}}(\ldots(\mathcal{G}_{i_2}(\mathcal{G}_{i_1}(a)))\ldots))$. e.g. $\mathcal{G}_{i,j}(a) = \mathcal{G}_j(\mathcal{G}_i(a))$ and $\mathcal{G}_{i,j,k}(a) = \mathcal{G}_k(\mathcal{G}_j(\mathcal{G}_i(a)))$. We use the shorthand notation $\mathcal{G}^k(a)$ to denote the composite function $\mathcal{G}_{i_1, i_2, \ldots, i_k}(a)$. We define $\mathcal{G}^0(a) = a$. Thus we have, $N_a^k = \{b | \mathcal{G}^k(a) = b\}$, for $k \geq 0$.

For example, from Fig. 4a) we note that node 2 can be reached from node 3 in two hops following the path[2] $3 \rightarrow 6 \rightarrow 2$. So, $6 \in N_3^1$ and $2 \in N_3^2$. From equation 2, we observe that $\mathcal{G}_0(3) = (3 \cdot 2 + 0) \bmod 10 = 6$ and $\mathcal{G}_0(6) = (6 \cdot 2 + 0) \bmod 10 = 2$. Thus, $\mathcal{G}_0(\mathcal{G}_0(3)) = \mathcal{G}_{0,0}(3) = 2$. This can be expanded as shown below :

$$\begin{aligned} \mathcal{G}_0(\mathcal{G}_0(3)) &= (\mathcal{G}_0(3) \cdot 2) \bmod 10 \\ &= (((3 \cdot 2) + 0) \bmod 10) \cdot 2 + 0) \bmod 10 \\ &= (((3 \cdot 2) + 0) \cdot 2 + 0) \bmod 10 \\ &= (3 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0) \bmod 10 \quad (3) \\ &= 2. \end{aligned}$$

[2]We use the terms path and route interchangeably.

The part $0 \cdot 2^1 + 0 \cdot 2^0$ in equation 3 represents the path from node 3 to node 2 in the example above. The coefficient of 2^1 determines the link index (i.e. i_1) in the first application of function $\mathcal{G}$ (i.e $\mathcal{G}_{i_1}$) and the coefficient of 2^0 determines the link index (i.e. i_2) in the second (recursive) application of $\mathcal{G}$ (i.e., $\mathcal{G}_{i_2}$), on the node address 3.

If $b \in N_a^k$, then $\exists$ a k-hop path from node a to node b, $a \to b_1 \to b_2 \to \ldots \to b_{k-1} \to b_k = b$. We note that $b_k = \mathcal{G}^k(a)$ and $b_k = \mathcal{G}_{i_k}(b_{k-1}) = \mathcal{G}_{i_k}(\mathcal{G}^{k-1}(a))$. Equating the two we have the recurrence :

$$\begin{aligned} \mathcal{G}^k(a) &= \mathcal{G}_{i_k}(\mathcal{G}^{k-1}(a)) \\ &= (\mathcal{G}^{k-1}(a) \cdot p + i_k) \bmod N \end{aligned} \tag{4}$$

with $\mathcal{G}^0(a) = a$ defined as the boundary condition. We can recursively expand equation 4 to yield:

$$\begin{aligned} \mathcal{G}^k(a) &= (\mathcal{G}^{k-1}(a) \cdot p + i_k) \bmod N \\ &\vdots \\ &= (a \cdot p^k + \mathcal{R}_k(p)) \bmod N, \end{aligned} \tag{5}$$

where, $\mathcal{R}_k(p)$, $(0 \leq \mathcal{R}_k(p) < p^k)$, termed the *route polynomial*, is defined as :

$$\mathcal{R}_k(p) = \sum_{j=1}^{k} i_j \cdot p^{k-j} \tag{6}$$

Letting $\mathcal{I} =< i_1, i_2, \ldots, i_{k-1}, i_k >$, we see that $\mathcal{I}$ defines a k-hop route from node a to node b. $i_1, i_2, \ldots, i_k$ denote the indices of the links followed en-route to the destination, in the 1^{st},2^{nd},...,k^{th} hops respectively. Our goal is to find the smallest k for which equation 5 is satisfied, i.e., to find the shortest path from a given source to a given destination.

Let s and d be the source and destination addresses respectively ($s \neq d$). Presented below is a *shortest path routing algorithm* for the generalized De-Bruijn network. Algorithm *SHORTEST-PATH* returns three values, denoted by variables j, c and k. j denotes the value of the route polynomial for a shortest path route from s to d, c represents the number of shortest path routes between s and d and k denotes the number of hops in the minimum length path/s from s to d. We define a *gap* function as follows :

$$\begin{aligned} \text{gap}(s,d,k) &= \text{the min. } j,\ (j \geq 0), \text{ such that} \\ & \qquad d = (s \cdot p^k + j) \bmod N \end{aligned} \tag{7}$$

$$= j = (s \cdot p^k - d) \bmod N \tag{8}$$

```
0  SHORTEST-PATH(s, d)
1  begin
2      k = 1;
3      repeat
4          j = gap(s, d, k);
5          if (j < p^k) then
6          begin
7              find max. c, (c ≥ 1), such that
               j + (c − 1) · N < p^k;
8              return(j, c, k);
9          end
10         k = k + 1;
11     forever
12 end.
```

We will see shortly, under what conditions there will be multiple shortest path routes for a given source and destination pair. Note in line 7 of algorithm *SHORTEST-PATH*, c can be simply computed as, $c = \lfloor \frac{p^k - j}{N} \rfloor + 1$. When $c = 1$, there is a unique path of minimum length from node s to node d. If we denote the r^{th} $(1 \leq r \leq c)$, minimum length route by $\mathcal{I}^r =< i_1^r, i_2^r, \ldots, i_k^r >$, then $\mathcal{I}^r =< i_1^r, i_2^r, \ldots, i_k^r >= j + (r-1) \cdot N$, expressed as a base p number, k digits wide. We denote the value $j + (r-1) \cdot N$ by j_r and shall refer to them as *alternate route polynomial values.*

Example 1: Referring to Fig. 4a), if $s = 4$ and $d = 3$, then the following is a trace of algorithm *SHORTEST-PATH* on this source-destination pair.

$$\begin{array}{lllll} \text{Iteration } 1 : k = 1 & \Rightarrow & \text{gap}(4,3,1) = 5 & \Rightarrow & (5 < 2) = \textit{false} \\ \text{Iteration } 2 : k = 2 & \Rightarrow & \text{gap}(4,3,2) = 7 & \Rightarrow & (7 < 4) = \textit{false} \\ \text{Iteration } 3 : k = 3 & \Rightarrow & \text{gap}(4,3,3) = 1 & \Rightarrow & (1 < 8) = \textit{true} \end{array}$$

Thus, $(j = 1, c = 1, k = 3)$ will be returned at the end of the third iteration. Then, we have $\mathcal{I}^1 =< i_1^1, i_2^1, i_3^1 >=< 0, 0, 1 >$. From Fig. 4a), we see that starting at node 4, if we take link 0, we arrive at node 8. Taking link 0 from node 8, we arrive at node 6 and finally, taking link 1 from node 6, we reach the destination 3.

Example 2: With $N = 8$ and $p = 4$, let's compute the route from node 0 to node 4.

$$\begin{array}{lllll} \text{Iteration } 1 : k = 1 & \Rightarrow & \text{gap}(0,4,1) = 4 & \Rightarrow & (4 < 4) = \textit{false} \\ \text{Iteration } 2 : k = 2 & \Rightarrow & \text{gap}(0,4,2) = 4 & \Rightarrow & (4 < 16) = \textit{true} \end{array}$$

In the above example, the algorithm will return (j = 4, c = 2, k = 2) since there are two minimum length paths from node 0 to node 4. We have $\mathcal{I}^1 =< i_1^1, i_2^1 >=< 1, 0 >$ and $\mathcal{I}^1 =< i_1^2, i_2^2 >=< 3, 0 >$. The first path is $0 \to 1 \to 4$ and the second path is $0 \to 3 \to 4$, corresponding to the route polynomial values of 4 and 8 respectively.

Next, we prove some properties of the generalized De-Bruijn network with respect to the algorithm *SHORTEST-PATH*.

Lemma 1 *Algorithm SHORTEST-PATH is deterministic, i.e., it is guaranteed to find a shortest path from s to d, $(0 \leq s, d < N)$.*

Proof: We have $0 \leq d < N$. Thus, from equation 8, we observe $0 \leq j < N$. Since k increases after each unsuccessful iteration (an unsuccessful iteration is one where $j \geq p^k$), at some point, p^k becomes $\geq N$. This makes the condition $j < p^k$ true. Thus a path will always be found from s to d. k in equation 6 represents the number of hops in a path from s to d. Since the algorithm returns as soon as the first k that satisfies equation 6 is found, the path returned is indeed the shortest path (in number of hops). □

Lemma 2 *The diameter of the generalized De-Bruijn network (with the above algorithm), is* $\lceil \log_p N \rceil$.

Proof: In the worst case, the algorithm cannot return the path until $p^k \geq N$. Thus the longest path in the network will contain $\lceil log_p N \rceil$ hops. Since the algorithm always returns the shortest path between a given pair of nodes, the diameter of the network is $\lceil log_p N \rceil$. □

Lemma 3 *The minimum length path from a given source to a given destination is unique if, k, the number of hops in the minimum length path, is less than the diameter of the network.*

Proof: From Lemma 2, we know the diameter of the generalized De-Bruijn network is $\lceil \log_p N \rceil$. Thus if k is less than the diameter we know that $p^k < N$. We want to find the maximum c, ($c \geq 1$), such that $j + (c-1) \cdot N < p^k$. With $p^k < N$ and $0 \leq j < N$, the only positive value of c that satisfies the above inequality is 1, meaning that there is a unique path from the given source to the given destination. □

Lemma 4 *If algorithm SHORTEST-PATH returns $k = D$, where D is the diameter of the network, then*

$$c = \begin{cases} \lceil p^k/N \rceil & \text{for } 0 \leq j < \delta \text{ or } \delta = 0 \\ \lceil p^k/N \rceil - 1 & \text{for } j \geq \delta > 0 \end{cases}$$

with $\delta = p^k \bmod N$ and j and c are returned by SHORTEST-PATH.

Proof: In other words, this lemma says that if the shortest distance separating two nodes is equal to the diameter of the network, then there are $\lceil p^k/N \rceil$ or $\lceil p^k/N \rceil - 1$ different paths from the given source to the given destination depending on the value of j returned by *SHORTEST-PATH*. □

From Lemma 2, we have $D = \lceil \log_p N \rceil$. With $k = D$, two cases are possible : 1). $p^k = N$ and 2) $p^k > N$, since $D = \lceil \log_p N \rceil$. When $p^k = N$, the only value of c satisfying the equation, $j + (c-1) \cdot N < p^k$, is 1. Thus there is a unique path from s to d in this case.

For the case $p^k > N$, we refer to Fig. 5. We define two paths to be different as long as at least one edge or one node is distinct between them. In Fig. 5, j denotes

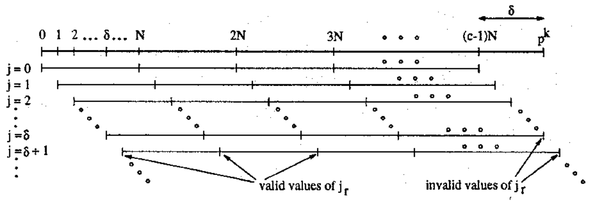

Figure 5: Valid values of j when $p^k > N$

the route polynomial value returned by the algorithm *SHORTEST-PATH* for a given s and d, as before. If we divide the interval $[0, p^k]$, into intervals of length N starting at the value of j returned by the algorithm, we get the picture shown in Fig. 5. We note, $j = j_1$, from our earlier definition of alternate route polynomial values.

With δ defined as $p^k \bmod N$, we observe from the figure that, when $j < \delta$ or when $\delta = 0$, $j_2 = j + N$, $j_3 = j + 2N, \ldots, j_c = j + (c-1) \cdot N$, are all less than p^k, where $c = \lceil p^k/N \rceil$. Thus $\lceil p^k/N \rceil$ different values for the route polynomial are obtained. Note that the two conditions $j < \delta$ and $\delta = 0$ are mutually exclusive. When $j \geq \delta$, we observe that $j_c = j + (c-1) \cdot N \geq p^k$ when $c = \lceil p^k/N \rceil$ while $j_c = j + (c-1) \cdot N < p^K$ when $c = \lceil p^k/N \rceil - 1$. Thus the algorithm will return $c = \lceil p^k/N \rceil - 1$. The c values of j returned by *SHORTEST-PATH* are mutually distinct in at least one position in their k-digit representation with base-p digits thus giving c different paths between the given node pairs.

To summarize, the number of paths when $k = D$, is given by either $\lceil p^k/N \rceil$ or $\lceil p^k/N \rceil - 1$, depending upon the value of j relative to $p^k \bmod N$.

Lemma 5 *If the minimum length path from a given source to a given destination has exactly $k = D$ hops, where D is the diameter of the network, then there are at most p alternate paths of minimum length between the same pair of nodes.*

Proof: From Lemma 4, we know that the number of alternate minimum length paths (when $k = D$) is at most $\lceil p^k/N \rceil$.

With $k = D$, we have, $p^k \geq N$. Thus $p^{k-1} < N$. Multiplying both sides of the latter inequality by p, we have $p^k < N \cdot p$. Which gives $\lceil p^k/N \rceil \leq p$. The number of alternate minimum length paths is thus bounded above by p. □

Lemma 6 *If there are multiple minimum length paths between a given pair of nodes, then these paths are all edge disjoint.*

Proof: We show, by contradiction, that alternate paths (of length D), have no edge in common.

Let there be two paths P^1 and P^2 between nodes s and d and let one edge e be common to the two paths.

This is shown in Fig. 6. Let the common edge e be between nodes x and y. If we denote path i from node a to node b by P_{ab}^i, then we have $P_{sd}^1 = P_{sx}^1.e.P_{yd}^1$ where . denotes concatenation. Similarly $P_{sd}^2 = P_{sx}^2.e.P_{yd}^2$. □

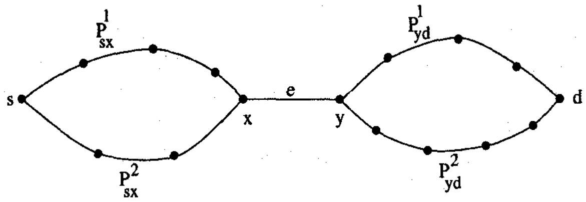

Figure 6: Multiple paths of minimum length are all edge disjoint

Let $L(P)$ denote the length of a path P. Then we have $L(P^1) = L(P^2) = D$, from Lemma 4. Note from Fig. 6 that $L(P_{sx}^1) < D)$ and $L(P_{sx}^2) < D$. This results in a contradiction of Lemma 3, which shows there is a unique path between two nodes when the length of the path is less than the diameter. Likewise, we have $L(P_{yd}^1) < D$ and $L(P_{yd}^2) < D$ which is again a contradiction.

Thus we conclude that paths of length $k = D$, will be distinct in all their edges.

Lemma 7 *If there are multiple minimum length paths between a given pair of nodes, then these paths are all node disjoint, except at the source and destination nodes.*

Proof: A proof similar to the one for Lemma 5, can be constructed. □

One of the observations in the De-Bruijn graph is that some of the links are self links (i.e., connects a node to itself). This means in the virtual topology the self links will be absent since assignment of the same wavelength to a node's transmitter and receiver will result in a redundant virtual link. In the (p, k) De-Bruijn graph, there are exactly p self links. For example, in Fig. 3, the (3,2) and (2,3) De-Bruijn graphs depicted have, exactly, 3 and 2 self links respectively.

In the generalized De-Bruijn graph, however, it is not trivial to formulate a closed form expression for the number of self links in the graph. We state this as a lemma below.

Lemma 8 *The number of self links in the (p, N) Generalized De-Bruijn graph is*

$$p + \gcd(N, p-1) - 1, \tag{9}$$

where, gcd $\equiv$ *the greatest common divisor.*

Proof: Denoting the address of a node by a, the number of self links can be determined by computing the number of distinct solutions to the equation $a \equiv (a \cdot p + i) \bmod N$, where $0 \leq a \leq N-1$ & $0 \leq i \leq p-1$. Note that $a \equiv (a \cdot p + i) \bmod N \Leftrightarrow a \cdot (p-1) + i = k \cdot N$ for some k.

We first determine the range of all such k. Since $0 \leq a \leq N-1$ and $0 \leq i \leq p-1$, $0 \leq a \cdot (p-1) + i \leq (p-1) \cdot N$. Therefore $0 \leq k \leq p-1$. The solution is of the form $a = \frac{k \cdot N - i}{p-1}$. We note that when $k = 0$ or $k = p-1$, there is a unique solution. For $1 \leq k \leq p-2$ there are two cases to consider: i) when $k \cdot N \not\equiv 0 \pmod{p-1}$, then again, there is a unique solution, and, ii) when $k \cdot N \equiv 0 \pmod{p-1}$, then there are two solutions, namely, $a = \frac{k \cdot N}{p-1}$ and $\frac{k \cdot N}{p-1} - 1$. The number of such cases (i.e., when there are two solutions) is $\gcd(N, p-1) - 1$. The total number of solutions therefore, is $p + \gcd(N, p-1) - 1$. □

4 TWDM Media Access Protocol for the Generalized De-Bruijn Network

4.1 Basic Properties

As mentioned in section 1, *Time and Wavelength Division Multiplexing*, (or TWDM) is a preferable method of providing media access for stations connected in a lightwave network, either through a shared optical transmission medium like a bus or a shared optical device like the passive star. We first establish some basic properties and the underlying assumptions for deploying a TWDM media access scheme on a lightwave network.

A TWDM system contains stations or user nodes which have their transmitters tuned for transmission at a fixed wavelength and receivers tuned for reception at another wavelength. The time domain is divided into time slots of equal duration with the slots long enough to contain a fixed sized packet. The time slots are logically arranged into repeating cycles, with each station transmitting once or several times within a cycle at predetermined time slots and wavelengths. This can be described by means of a table such as the one shown in Fig. 7. We call this table a *transmission schedule.* In Fig. 7, we show a transmission schedule with 3 wavelengths, for a (3,2) De-Bruijn network shown previously, in Fig. 3a). In a transmission schedule, the columns denote time slots and the rows represent wavelengths. An entry $k \rightarrow l, m, n$ in row i and column j of the transmission schedule means that station k has the right to transmit at wavelength i in time slot j and this transmission can be simultaneously received by stations l, m and n directly (i.e., stations l, m and n have their respective receivers all tuned to wavelength i). We call the number of time slots in a transmission schedule the *cycle length.*

In Fig. 8 we show a 9 node, (3,2) De-Bruijn graph embedded as a virtual topology on an optical passive

	1	2	3
0	$0 \rightarrow 0,1,2$	$3 \rightarrow 0,1,2$	$6 \rightarrow 0,1,2$
1	$1 \rightarrow 3,4,5$	$4 \rightarrow 3,4,5$	$7 \rightarrow 3,4,5$
2	$2 \rightarrow 6,7,8$	$5 \rightarrow 6,7,8$	$8 \rightarrow 6,7,8$

Figure 7: Transmission Schedule for a (3,2)-De-Bruijn Network with 3 wavelengths

star. The transmission schedule depicted in Fig. 8 corresponds to this embedding. Fig. 8 shows a split view of the system, with all the transmitters on the left and all the corresponding receivers on the right. The numbers on the lines connecting a node's transmitter or receiver to the passive star denotes the wavelength used. Such a virtual topology embedding presents three constraints :

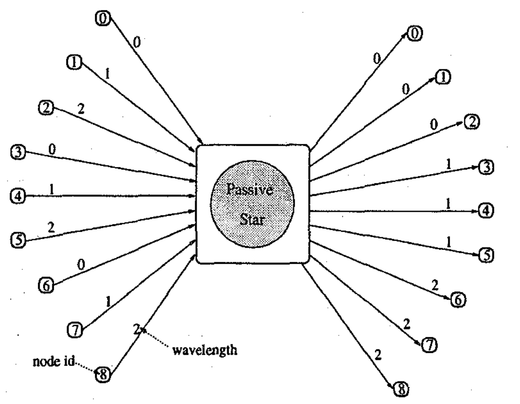

Figure 8: A (3,2) De-Bruijn network embedded as a virtual topology on an optical star

1. If a node transmits on a particular wavelength, then all nodes to which there is directed edge from this node must receive on the same wavelength. e.g. there are three directed edges from node 3, $3 \rightarrow 0$, $3 \rightarrow 1$ and $3 \rightarrow 2$. Thus nodes 0,1 and 2 must all receive on the wavelength node 3 transmits on.

2. If a node receives on a particular wavelength, then all nodes which have a directed edge to it must transmit on the same wavelength. e.g. there are three directed edges to node 1, $0 \rightarrow 1$, $3 \rightarrow 1$ and $6 \rightarrow 1$. Thus nodes 0,3 and 6 must all transmit on the wavelength node 1 receives on.

3. If multiple nodes transmit on the same wavelength, then they must transmit in different slots to avoid collisions. e.g. nodes 0, 3 and 6 must all transmit in different time slots.

Employing a TWDM transmission schedule such as the one shown in Fig. 7 entails a uniform and symmetric embedding of the virtual topology on the optical passive star. For a regular graph this means that all wavelengths are shared by the same number of transmitters and the same number of receivers receive on all wavelengths. For example, in Fig. 7, all wavelengths are shared by 3 transmitters and also by 3 receivers. In general, sets of x transmitters and sets of y receivers share each wavelength where x and y may be different depending on the topology and the number of transmitters and receivers at each node.

For ease of discussion we treat the generalized De-Bruijn network as a regular bipartite graph $\mathcal{G}(N \times N)$ with the transmitters on the left hand side and the receivers on the right hand side (Fig. 4). A set of transmitters and receivers form a *component* if there is a path between any two of them assuming the edges in this bipartite graph are bidirectional. In other words, forgetting the unidirectional nature of the virtual link between a transmitter and a receiver, a component is a *connected component* in graph theoretic terminology. For example, Fig. 9 shows two components of the (2,10) generalized De-Bruijn network in Fig. 4a. One of the components is constituted by the transmitters of nodes 1 and 6 and the receivers of nodes 2 and 3, while the other one by the transmitters of nodes 3 and 8 and the receivers of nodes 6 and 7. The two components are shown with solid and dashed lines respectively in Fig. 9. There are a total of five components in the graph. Henceforth we will employ this bipartite representation of the generalized De-Bruijn graph to establish its properties with respect to the support of a TWDM protocol.

The notion of a component is central to assignment of wavelengths and supporting a TWDM access scheme. In each component, a wavelength can be assigned starting at any transmitter (receiver). Then all receivers (transmitters) connected to this transmitter (receiver) are forced to receive (transmit) at this wavelength. Continuing in this manner, we end up with all transmitters and receivers within a component assigned to the same wavelength.

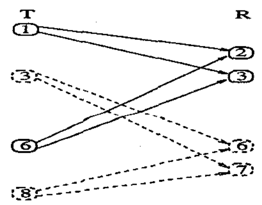

Figure 9: Two components of a (2,10) generalized De-Bruijn graph (there are a total of five components in this graph)

Observation 1 *All transmitters and receivers constituting a component are assigned to the same wavelength. In other words, within a component only one wavelength can be employed.*

Observation 1 implies that the maximum number of wavelengths that can be employed, W_{max}, is equal to the number of components in the graph, C. Assuming all components contain the same number of transmitters (i.e., $\frac{N \cdot T}{W_{max}}$), since transmitters in the same component have to share one wavelength, the transmission cycle length will be at least $\frac{N \cdot T}{W_{max}}$. We denote this by α_{min}. All the other components can proceed parallely in the TWDM transmission cycle. With this scheme, each row in the TWDM transmission schedule represents a component. Clearly, the more the number of components, the greater the number of wavelengths exploited by the TWDM scheme.

If less than W_{max} wavelengths are available, then S components can share a wavelength. In that case the transmission cycle length and the number of wavelengths required are :

$$\alpha = S \cdot \alpha_{min} \text{ and } W = \frac{W_{max}}{S} = \frac{N \cdot T}{S \cdot \alpha_{min}},\ S \leq C \qquad (10)$$

4.2 TWDM with $T = R = 1$

Here we consider a generalized De-Bruijn graph with $T = R = 1$, i.e., each node is equipped with a single fixed wavelength transmitter and a single fixed wavelength. There are two cases to consider : i) $N \bmod p = 0$, and, ii) $N \bmod p \neq 0$.

$N \bmod p = 0$ **:** In this case, the number of nodes, $N = p \cdot C$ where C is the number of components, each component having p transmitters and receivers. Thus, upto $W_{max} = C$ distinct wavelengths can be exploited. Fig. 10a shows a (2,10) generalized De-Bruijn graph with five components and five assigned wavelengths.

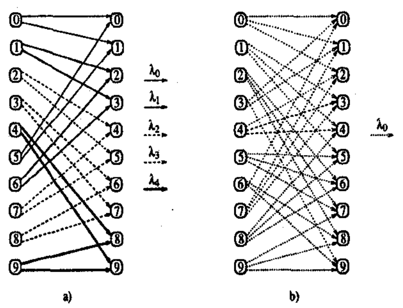

Figure 10: Wavelength assignment examples for: a) a (2,10) generalized De-Bruijn graph, and, b) a (3,10) generalized De-Bruijn graph

$N \bmod p \neq 0$ **:** This situation results in the generalized De-Bruijn graph having a single component and thus only one wavelength can be exploited. Thus this case is somewhat restrictive in that the TWDM media access scheme reduces to a pure TDM scheme. Fig. 10b shows a (3,10) generalized De-Bruijn graph with one assigned wavelength.

This is summarized in the lemma below:

Lemma 9 *The maximum number of wavelengths that can be used with $T = R = 1$ is $\frac{N}{p}$, when $N \bmod p = 0$, and 1, when $N \bmod p \neq 0$.*

Proof: We first prove the $N \bmod p = 0$ case. Let $\mathcal{S} = \{0, 1, \ldots, N-1\}$ denote the set of all node indices in the graph. We prove this lemma by construction. We divide the receivers on the right hand side of the bipartite graph into sets $\mathcal{R}_i = \{i \cdot p + k \mid 0 \leq k \leq p-1\}$, for $0 \leq i \leq \frac{N}{p} - 1$. The set of receivers on the right hand side of the bipartite graph, connected to a transmitter t, $0 \leq t < N$, on the left side, is exactly $\mathcal{R}_{t \bmod \frac{N}{p}}$. Therefore we divide all transmitters on the left hand into sets $\mathcal{L}_i = \{k \cdot \frac{N}{p} + i \mid 0 \leq k \leq p-1\}$, for $0 \leq i \leq \frac{N}{p} - 1$. We have $\mathcal{S} = \cup_{i=0}^{\frac{N}{p}-1} \mathcal{L}_i = \cup_{i=0}^{\frac{N}{p}-1} \mathcal{R}_i$. Also $\mathcal{L}_i \cap \mathcal{L}_j = \phi$, and $\mathcal{R}_i \cap \mathcal{R}_j = \phi$, for $i \neq j$.

It is easy to see that $\forall i$, $0 \leq i \leq \frac{N}{p} - 1$, $\exists x \in \mathcal{L}_i$, and $\exists y \in \mathcal{R}_i$, such that $x \to y$ is an edge in $\mathcal{G}(N \times N)$. Furthermore it can be seen that $\forall i$, if $x \in \mathcal{L}_i$ then there is no $y \in \mathcal{R}_i$, $i \neq j$, such that $i \to j$ is an edge in $\mathcal{G}(N \times N)$. The converse is also true. This means that the transmitters in $\mathcal{L}_i$ and the receivers in $\mathcal{R}_i$ form a connected component. Thus we have a total of $\frac{N}{p}$ components in the graph. The number of wavelengths that can therefore be employed is observation 1.

Next we prove the $N \bmod p \neq 0$ case. For each transmitter t on the left side, the set of receivers connected to it on the right hand side is $\mathcal{Z}(t) = \{r \mid r = (t \cdot p + i) \bmod N,\ 0 \leq i < p\}$, for $0 \leq t < N$. We note that if $\mathcal{Z}(i) \cap \mathcal{Z}(j) \neq \phi$, $i \neq j$, then transmitters i and j belong to the same component since they are connected to at least one common receiver. Also note that if $\mathcal{Z}(i) \cap \mathcal{Z}(j) \neq \phi$ and $\mathcal{Z}(i) \cap \mathcal{Z}(k) \neq \phi$, $i \neq j \neq k$, then transmitters i, j and k all belong to the same component. Let $m = \lfloor \frac{N}{p} \rfloor$ and $n = N \bmod p$. m denotes the number of transmitters on the left connected to receivers on the right in a non-overlapping manner, i.e., the number of transmitters that each connect to a distinct set of receivers. There are two cases to consider:
Case 1. $m = 1$: In this case we observe $\mathcal{S} = \mathcal{Z}(0) \cup \mathcal{Z}(1)$ and $\mathcal{Z}(0) \cap \mathcal{Z}(1) \neq \phi$. This implies that the bipartite graph is connected and that there is only one component.
Case 2. $m > 1$: We note that $((t+m) \cdot p + i) \bmod N = (t \cdot p + N - r + i) \bmod N = (t \cdot p + i - r) \bmod N$ and $((t+1+m) \cdot p + i) \bmod N = (t \cdot p + p + N - r + i) \bmod N = (t \cdot p + i + p - r) \bmod N$. We have $\mathcal{Z}(t) \cap \mathcal{Z}(t+m) = \{r \mid r = (t \cdot p + i) \bmod N$, for $i = r, \ldots, p-1-r\}$ and $\mathcal{Z}(t) \cap \mathcal{Z}(t+1+m) = \{r \mid r = (t \cdot p+i) \bmod N$, for $i = p-r, \ldots, p-1\}$. Thus transmitters $t, t+m$ and $t+1+m$ are in the same component. In

other words, two neighboring transmitters are in the same component. Continuing in this manner it can be shown that all the transmitters belong to the same component. Thus there is only one component in the graph and only wavelength can be employed by the TWDM transmission scheme. □

We use the construction in lemma 9 to derive another result below which will be useful in the $T, R \geq 1$ case.

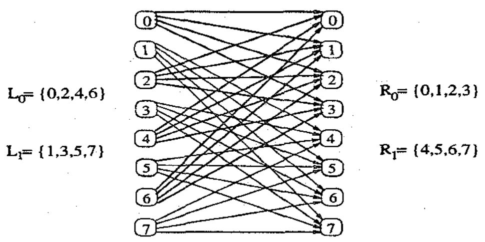

Figure 11: Two (4×4) complete bipartite graphs are the components of a $(4, 8)$ generalized De-Bruijn graph

Lemma 10 *Each component of the generalized De-Bruijn graph $\mathcal{G}(N \times N)$ is a fully connected bipartite graph $\mathcal{G}(p \times p)$, provided $N \bmod p = 0$.*

Proof: In lemma 9 we showed that there are $\frac{N}{p}$ components in $\mathcal{G}(N \times N)$. Component i, constituted by $\mathcal{L}_i$ and $\mathcal{R}_i$ is a bipartite graph $\mathcal{G}_i(p \times p)$. Examining the sets $\mathcal{L}_i$ and $\mathcal{R}_i$, $0 \leq i \leq \frac{N}{p} - 1$, from Lemma 9, we can easily see that $\forall i$, $\forall x \in \mathcal{L}_i$ and $\forall y \in \mathcal{R}_i$, $x \to y$ is an edge in $\mathcal{G}_i(p \times p)$. Thus $\mathcal{G}_i(p \times p)$ is a complete bipartite graph $\forall i$. □

Figure 11 illustrates the result of the above lemmas. Shown in this figure is a (4,8) generalized De-Bruijn graph which has two (4 X 4) bipartite graphs as components.

4.3 TWDM with $T, R \geq 1$

This is a more general case of the one discussed above. Since the degree of each transmitter and receiver is p, we have $T, R \leq p$. We assume both T and R are factors of p for simplicity (i.e., $p \bmod T = 0$ and $p \bmod R = 0$). We characterize the generalized De-Bruijn graph with multiple transmitters and receivers below and discuss the *Consecutive Partition Allocation (CPA)* strategy for wavelength and time slot assignment for nodes in the generalized De-Bruijn graph. We'll call $\mathcal{G}(p, N, T, R)$ *transmission graph* for the generalized De-Bruijn graph.

At each node we partition the outgoing links into T groups, called *t-groups*, with one transmitter assigned to each group. The outgoing links from each node are numbered $0, 1, \ldots, p-1$. With this partitioning, the first transmitter is assigned to links $0, 1, \ldots, \frac{p}{T} - 1$, the second transmitter to links $\frac{p}{T}, \ldots, \frac{2 \cdot p}{T} - 1$, and so on. We then label a transmitter by the two tuple (*t-node-id,t-group-id*), where *t-node-id* is the index of the node the transmitter is located in and *t-group-id* is the group index of the transmitter. Similarly we partition the p incoming links at each node into R groups, called *r-groups* with one receiver designated to each r-group. Thus the first receiver at each node is assigned links $0, 1, \ldots, \frac{p}{R} - 1$, the second gets assigned links $\frac{p}{R}, \ldots, \frac{2 \cdot p}{R} - 1$ and so on. A receiver is also labeled by the tuple (*r-node-id,r-group-id*), where *r-node-id* is the index of the node the receiver is in and *r-group-id* is the group index of the receiver. The vertex set of $\mathcal{G}(p, N, T, R)$ is the union of the transmitter set

$$\{(a, t) \mid 0 \leq a < N, 0 \leq t < T\}$$

and the receiver set

$$\{(b, r) \mid 0 \leq b < N, 0 \leq r < T\}.$$

For any edge $a \to b$ in the De-Bruijn graph, if this is the i-th outgoing link of node a and the j-th incoming link of node b, then the corresponding link in the graph $\mathcal{G}(p, N, T, R)$ is

$$(a, \lfloor \frac{i}{\frac{p}{T}} \rfloor) \to (b, \lfloor \frac{i}{\frac{p}{T}} \rfloor).$$

Next we examine the maximum number of wavelengths, W_{max}, usable in $\mathcal{G}(p, N, T, R)$. The result for the $N \bmod p = 0$ case is proved below. The $N \bmod p \neq 0$ case is non-trivial to prove. For detailed proof of that case, see [11].

Lemma 11 *For $N \bmod p = 0$, $\mathcal{G}(p, N, T, R)$ can employ $\frac{N \cdot T \cdot R}{p}$ wavelengths.*

Proof: We know through lemma 9 and lemma 10 that there are $\frac{N}{p}$ components in the graph when N mod $p = 0$ and that each component is a complete bipartite digraph. We now examine one $(p \times p)$ component of the De-Bruijn graph in isolation. Let the nodes in the component be labeled $0, 1, \ldots, p-1$. Then, for any node a in the graph, its i-th outgoing link is $a \to i$. Similarly the j-th incoming link for a node b would be $j \to b$.

We prove this lemma by construction too. We observe that the set of receivers a transmitter (a, t) in the component connects to is

$$\{(t \cdot \frac{p}{T} + i, \lfloor \frac{a}{\frac{p}{R}} \rfloor) \mid 0 \leq i < \frac{p}{T}\},$$

and the set of transmitters a receiver (b, r) connects from is

$$\{(r \cdot \frac{p}{R} + j, \lfloor \frac{b}{\frac{p}{T}} \rfloor) \mid 0 \leq j < \frac{p}{R}\}.$$

For any $0 \leq t < T$ and $0 \leq r < R$, let

$$A_{t,r} = \{(r\frac{p}{R} + i, t) \mid 0 \leq i < \frac{p}{R}\},$$

$$B_{t,r} = \{(t\frac{p}{T} + j, r) \mid 0 \leq j < \frac{p}{T}\}.$$

It's easy to verify that

- the set of receivers that each transmitter in $A_{t,r}$ connects to in the transmission graph is exactly $B_{t,r}$;
- the set of transmitters that each receiver in $B_{t,r}$ connects from is exactly $A_{t,r}$.

Therefore, the transmitters in the set $A_{t,r}$ and the receivers in the set $B_{t,r}$ constitute a subcomponent. Notice that each of these subcomponents is also a complete bipartite graph $(\frac{p}{R} \times \frac{p}{T})$. Thus each complete bipartite component $(p \times p)$ of the original generalized De-Bruijn graph $\mathcal{G}(N \times N)$ has $T \cdot R$ subcomponents. Since there are $\frac{N}{p}$ components in the original graph, it means that the maximum number of wavelengths that can be employed is $\frac{N \cdot T \cdot R}{p}$. $\square$

In a related paper, we have extended this result to a fully connected graph (or equivalently, a single hop network) [15]. Both the case of a single hop network with self-loops and without, are considered in [15].

5 Conclusions

In this paper we have shown that the generalized De-Bruijn graph can be a suitable virtual topology for designing lightwave networks using optical passive stars. We have developed several topological properties of the generalized De-Bruijn graph in this paper and shown how TWDM protocols can be employed on it. A key contribution of this paper is to present the Consecutive Partition Allocation strategy for identifying components in the graph which leads to an easy wavelength and time slot assignment strategy on this virtual topology. We have extended the CPA results to several other topologies. Work is in progress to develop methods for scaling multihop lightwave networks on different topologies as well as supporting them on multiple passive stars.

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