Asymptotic Critical Transmission Radius for Greedy Forward Routing in Wireless Ad Hoc Networks

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ABSTRACT

Greedy forward routing (abbreviated by GFR) in wireless ad hoc networks is a localized geographic routing in which each node discards a packet if none of its neighbors is closer to the destination of the packet than itself, or otherwise forwards the packet to the neighbor closest to the destination of the packet. If all nodes have the same transmission radii, the critical transmission radius for GFR is the smallest transmission radius which ensures that packets can be delivered between any source-destination pairs. In this paper, we study the asymptotic critical transmission radius for GFR in randomly deployed wireless ad hoc networks. We assume that the network nodes are represented by a Poisson point process of density n over a convex compact region of unit area with bounded curvature. Let $\beta_0 = 1/\left(\frac{2}{3} - \frac{\sqrt{3}}{2\pi}\right) \approx 1.6^2$. We show that $\sqrt{\frac{\beta_0 \ln n}{\pi n}}$ is asymptotically almost surely (abbreviated by a.a.s.) the threshold of the critical transmission radius for GFR. In other words, for any $\beta > \beta_0$, if the transmission radius is $\sqrt{\frac{\beta \ln n}{\pi n}}$, it is a.a.s. packets can be delivered between any source-destination pairs; for any $\beta < \beta_0$, if the transmission radius is $\sqrt{\frac{\beta \ln n}{\pi n}}$, it is a.a.s. packets can't be delivered between some source-destination pair.

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Keywords

Wireless ad hoc networks, greedy forward routing, random deployment

1. INTRODUCTION

A wireless ad hoc network is a collection of wireless devices distributed over a geographic region. Each ad hoc device is equipped with an omnidirectional antenna. A communication session is established either through a single-hop radio transmission if the communication party is close enough, or through relaying by intermediate devices otherwise. The selection of the intermediate relaying nodes is determined by the routing algorithm. Greedy forward routing (abbreviated by GFR) is one of the localized geogrphic routing algorithms proposed in the literature.

In GFR, each node discards a packet if none of its neighbors is closer to the destination of the packet than itself, or otherwise forwards the packet to the neighbor closest to the destination of the packet. Therefore, each node only need to maintain the locations of its one-hop neighbors and each packet should contain the location of the destination node. Thus, it can be implemented in a localized and memoryless manner. There are some variants of GFR. For example, in [6] and [7], the shortest projected distance to the destination on the straight line joining the current node and the destination node is considered as the greedy metrics. In [6], packets are allowed to be sent backward if there is no forwarding neighbor. In [7], only the nodes whose Voronoi cells intersect with the source-destination line segment are eligible.

Due to the existence of local minima where none of neighbors is closer to the destination than the current node, a packet may be discarded before it reaches its destination. To ensure that every packet can reach its destination, all nodes should have sufficiently large transmission radii to avoid the existence of local minima. Let $B(x, r)$ denote the disk of radius r centered at x. If V is a set of network nodes, represented by a point set, in the plane, let

$$\rho(V) = \max_{\substack{(u,v)\in V^2 \\ u\neq v}} \min_{w\in B(v,\|u-v\|)} \|w-u\|.$$

In the definition, (u, v) is a source-destination pair and w is a node that is closer to v than u. If the transmission radius is not less than $\|w-u\|$, w might be the one to relay packets from u to v. Therefore, for each (u, v), the minimum of $\|w-u\|$ over all nodes on $B(v, \|u-v\|)$ guarantees there exists one node that can route packets from u to v, and the maximum of the minimum over all (u, v) pairs guarantees the existence of relay nodes between any source-destination pair. Clearly, if the transmission radius is at least $\rho(V)$, packets can be delivered between any source-destination pairs. On the other hand, if the transmission radius is less than $\rho(V)$, there must exist some source-destination pair, e.g. the (u, v) that gives the value $\rho(V)$, such that packets can't be delivered. Therefore, $\rho(V)$ is called the *critical transmission radius* for GFR that guarantees the delivery of packets between any source-destination pair of nodes among V.

The analytic work of GFR can date back to 1984 by Takagi and Kleinrock [6] (1984). They studied the optimal transmission radius to maximize the expected progress of packets based on most forward and least backward routing strategy in which every node delivers each packet to the neighbor (not including itself) with the shortest projected distance to the destination on the straight line joining the current node. However, the deliverability of packets is not considered. In the last two decades, there is no significant progress. Recently, Xing et al. [7] (2004) show that in a fully covered homogeneous wireless sensor network, if the transmission radius is larger than 2 times of the sensing radius, the deliverability can be gauranteed between any source-destination pair by greedy forwarding schemes in which a packet is sent to the neighbor either with the shortest Euclidean distance to the destination [2] [4] or with the shortest projected distance to the destination on the straight line joining the current node and the destination node [6] and by bounded Voronoi greedy forwarding scheme in which only those nodes whose Voronoi cells intersect with the line segment between the source and destination are eligible to relay the packet. In this paper, we consider the deliverability by given the asymptotics of $\rho(V)$ where V is Poisson point process. We assume that the deployment region $\mathbb{D}$ is convex compact region whose boundary has bounded curvature. By proper scaling, $\mathbb{D}$ is assumed to have unit area. We use $\mathcal{P}_n$ to denote a Poisson point process of density n over $\mathbb{D}$. Let $\beta_0 = 1/\left(\frac{2}{3} - \frac{\sqrt{3}}{2\pi}\right) \approx 1.6^2$. We show that $\rho(\mathcal{P}_n)$ is asymptotically almost surely at most $\sqrt{\frac{\beta \ln n}{\pi n}}$ for any $\beta > \beta_0$ and at least $\sqrt{\frac{\beta \ln n}{\pi n}}$ for any $\beta < \beta_0$.

In what follows, $\|x\|$ is the Euclidean norm of a point $x \in \mathbb{R}^2$ and $\|x-y\|$ is the Euclidean distance between two points $x, y \in \mathbb{R}^2$. $|A|$ is shorthand for 2-dimensional Lebesgue measure (or area) of a measurable set $A \subset \mathbb{R}^2$. All integrals considered will be Lebesgue integrals. The diameter of a set $A \subset \mathbb{R}^2$ is denoted by $diam(A)$. The topological boundary of a set $A \subset \mathbb{R}^2$ is denoted by ∂A. For any two points $u, v \in \mathbb{R}^2$, the lune of u and v, denoted by L_{uv}, is the set $B(u, \|u-v\|) \cap B(v, \|u-v\|)$. $Po(\lambda)$ represents a Poisson RV with mean λ. An event is said to be *asymptotic almost sure* (abbreviated by a.a.s.) if it occurs with a probability converges to one as $n \to \infty$. The symbols $O, \Theta, \Omega, o, \sim$ always refer to the limit $n \to \infty$. To avoid trivialities, we tacitly assume n to be sufficiently large if necessary. For simplicity of notation, the dependence of sets and random variables on n will be frequently suppressed.

The remaining of this paper is organized as follows. In section 2, we present several useful geometric results. In Section 3, we derive the a.a.s. bounds on the minimum of a collection of Poisson RVs. In section 4, we derive a.a.s. bounds on $\rho(\mathcal{P}_n)$. We summarize this paper in Section 5.

2. GEOMETRIC PRELIMINARIES

If $\|u-v\| = 1/\sqrt{\pi}$, a straightforward calculation yields that $|L_{uv}| = \frac{2}{3} - \frac{\sqrt{3}}{2\pi} = \frac{1}{\beta_0}$. Let $\mathbb{D}$ denote a convex compact set whose boundary has bounded curvature. We use R to denote the minimum of the radius of curvature over $\partial\mathbb{D}$. We have the following lemma.

LEMMA 1. *For any $u, v \in \mathbb{D}$, if $\|u-v\| \leq R$ then*

$$|L_{uv} \cap \mathbb{D}| \geq |L_{uv}|/2.$$

PROOF. Clearly, $|L_{uv} \cap \mathbb{D}| / |L_{uv}|$ achieves the minimum when both u and v are in $\partial\mathbb{D}$. Thus, it is sufficient to show the lemma for $u, v \in \partial\mathbb{D}$. Suppose that $u, v \in \partial\mathbb{D}$. Since $\|u-v\| \leq R$, both $B(u, \|u-v\|)$ and $B(u, \|u-v\|)$ are divided into two parts by $\partial\mathbb{D}$. Let u' denote the intersection point of $\partial B(u, \|u-v\|)$ and $\partial\mathbb{D}$ rather than v, and v' denote the intersection point of $\partial B(v, \|u-v\|)$ and $\partial\mathbb{D}$ rather than u. (See Fig. 1.) Then, the two sectors $\measuredangle u'uv$ and $\measuredangle uvv'$ are both contained in $\mathbb{D}$. Note that the lune L_{uv} is divided into two halves by the segment uv. One of them is contained in $\measuredangle u'uv \cap \measuredangle uvv'$ and thus is contained in $\mathbb{D}$. This implies that $|L_{uv} \cap \mathbb{D}| \geq |L_{uv}|/2$. □

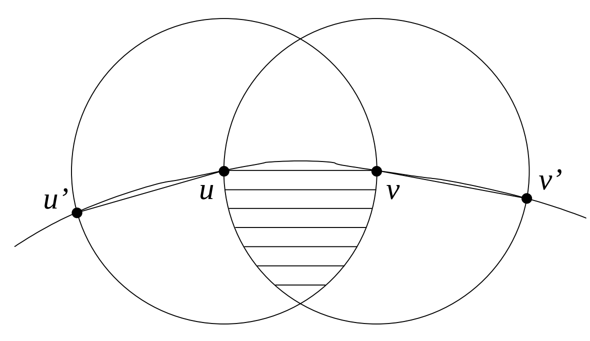

Figure 1: u and v are in $\partial\mathbb{D}$. One of the two half lunas divided by the segment uv is contained in $\mathbb{D}$.

Lemma 2. *Assume $c = 0.039$, $R > 0$, and $a_1, b_1, a_2, b_2 \in \mathbb{R}^2$. Let $z_1 = \frac{1}{2}(a_1 + b_1)$, $r_1 = \|a_1 - b_1\|$, $z_2 = \frac{1}{2}(a_2 + b_2)$, and $r_2 = \|a_2 - b_2\|$. If $r_1, r_2 \in \left[\frac{1}{2}R, R\right]$, $\|z_1 - z_2\| \leq \sqrt{3}R$, $a_1, b_1 \notin L_{a_2 b_2}$, and $a_2, b_2 \notin L_{a_1 b_1}$, then*

$$|L_{a_1 b_1} \cup L_{a_2 b_2}| - |L_{a_1 b_1}| \geq cR \|z_1 - z_2\|.$$

Proof. The proof is given in Appendix. □

For any convex compact set $C \subset \mathbb{R}^2$, we use C_{-r} to denote the set of points of C that are away from ∂C by at least r.

Lemma 3. *Suppose that $C \subset \mathbb{R}^2$ is a convex compact set with diameter at most d. Then,*

$$|C_{-r}| \geq |C| - \pi d r.$$

Proof. First, we assume C is a polygon. To get the lower bound of $|C_{-r}|$, we draw a rectangle by each edge of C with width r toward the inner of C. Since $C \setminus C_{-r}$ is fully covered by these rectangles, we have $|C_{-r}| \geq |C| - peri(C)\, r$, where $peri(C)$ is the perimeter of C. According to the isodiametric inequality [5] [3], the disk with diameter d has the longest perimeter πd over all convex compact sets with diameter d. Thus $peri(C) < \pi d$, which implies that $|C_{-r}| \geq |C| - \pi d r$.

If C is a convex compact set, the lemma can be proved using the fact that C can be approximated by a sequence of polygons contained in C. □

An ε-tessellation is a technique that divides the plane by vertical and horizontal lines into a grid in which each grid cell has width ε. Without loss of generality, we assume the origin is a corner of cells. In a tesselation, a polyquadrate is a collection of cells intersecting with a convex compact set. For example, in Fig. 2, the shaded cells form a polyquadrate induced by a polygon. The horizontal span of a polyquadrate is the horizontal distance measured in the number of cells from the left to the right. The vertical span of a polyquadrate is defined similarly but in the vertical direction. If the span of a polygon is s and the width of each cell is l, the span of the corresponding polyquadrate is at most $\lceil s/l \rceil + 1$.

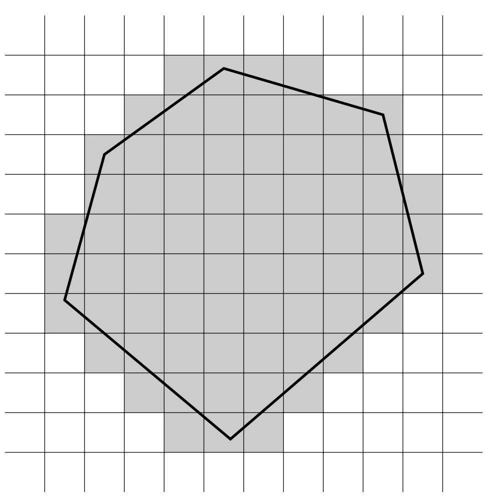

Figure 2: The cells intersecting with the polygon form a polyquadrate.

Lemma 4. *If S consists of m cells and τ is a positive integer constant, the number of polyquadrates with span at most τ and intersecting with S is $\Theta(m)$.*

Proof. For a specified cell, since τ is a constant, the number of polyquadrates that contain the cell and have span at most τ is also a constant (depending on τ). For each cell in S, the number of polyquadrates that contain the cell and have span at most τ is $\Theta(1)$. Therefore, since there are m cells in S, the total number of polyquadrates with span at most τ and intersecting with S is $\Theta(m)$. □

At the end of this section, we introduce a technique to obtain the Jacobian determinant in the change of variables that will be implicitly used in Subsection 4(B). Assume a tree topology is fixed over $x_1, x_2, \cdots, x_k \in \mathbb{R}^2$. Without loss of generality, we may assume (x_{k-1}, x_k) is one of edges. Let $z_{k-1} = \frac{1}{2}(x_{k-1} + x_k)$, $r = \frac{1}{2}\|x_k - x_{k-1}\|$, and θ be the slope of $x_{k-1}x_k$. For $1 \leq i \leq k-2$, we use $p(x_i)$ to denote x_i's parent in the tree rooted at x_k, and let $z_i = \frac{1}{2}(x_i + p(x_i))$. Let I_2 denote a 2×2 identity matrix and $\mathbf{0}$ denote a 2×2 zero matrix. Then, the Jacobian determinant for changing variables $x_1, \cdots, x_{k-1}, x_k$ by $z_1, \cdots, z_{k-1}, (r, \theta)$ is

$$\begin{aligned}
&\left|\frac{\partial(x_1, \cdots, x_{k-1}, x_k)}{\partial(z_1, \cdots, z_{k-1}, r, \theta)}\right| \\
&= \left|\frac{\partial(x_1 + p(x_1), \cdots, x_{k-1} + p(x_{k-1}), x_k)}{\partial(z_1, \cdots, z_{k-1}, r, \theta)}\right| \\
&= 4^{k-1}\left|\frac{\partial\left(\frac{x_1 + p(x_1)}{2}, \cdots, \frac{x_{k-1} + p(x_{k-1})}{2}, x_k\right)}{\partial(z_1, \cdots, z_{k-1}, r, \theta)}\right| \\
&= 4^{k-1}\left|\frac{\partial(z_1, \cdots, z_{k-1}, x_k - z_{k-1})}{\partial(z_1, \cdots, z_{k-1}, r, \theta)}\right| \\
&= 4^{k-1}\left|\begin{matrix} I_2 & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \cdots & I_2 & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \begin{matrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{matrix} \end{matrix}\right| \\
&= 4^{k-1} r.
\end{aligned}$$

In the first equality, each non-root variable is added by its parent variable. The equality stands since the Jacobian determinant is equal to 1 as we add one variable to another. We remark that in most cases, the function in the integral is independent of the variable θ. Thus, 2π will be the outcome of the integral over θ in such cases.

3. MINIMUM OF A COLLECTION OF POISSON RVS

Let ϕ be the function over $(0,\infty)$ defined by $\phi(\mu) = 1-\mu+\mu\ln\mu$. A straightforward calculation yields $\phi'(\mu) = \ln\mu$ and $\phi''(\mu) = 1/\mu$. Thus, ϕ is strictly convex and has the unique minium zero at $\mu = 1$. (See Fig. 3.) Let $\phi^{-1} : (0,1] \to [0,1)$

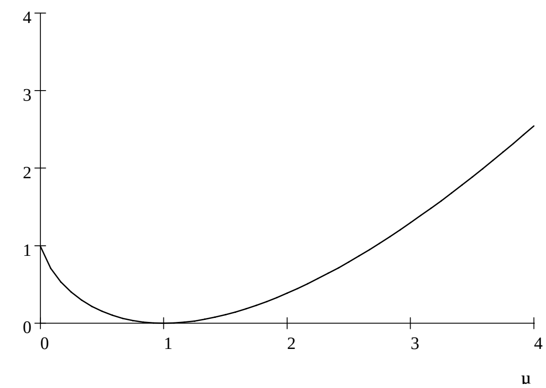

Figure 3: $\phi(\mu) = 1 + \mu\ln\mu - \mu$.

be the inverse of the restriction of ϕ to $(0,1]$. We define a

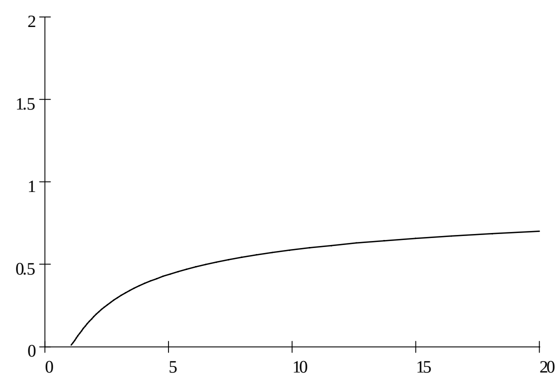

Figure 4: The x-axis is β, and the y-axis is $\phi^{-1}(1/\beta)$.

functions $\mathcal{L}$ over $(0,\infty)$ by

$$\mathcal{L}(\beta) = \begin{cases} \beta\phi^{-1}(1/\beta) & \text{if } \beta \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

The curve of $\beta \to \phi^{-1}(1/\beta)$ is illustrated in Fig. 4, and the curve of $\mathcal{L}$ is illustrated in Fig. 5. $\mathcal{L}$ is a monotonic increasing function of β.

We first present an estimation of the lower-tail distribution of a Poisson RV.

LEMMA 5. *For any $\mu \in (0,1)$, as $\lambda \to \infty$,*

$$\Pr(Po(\lambda) \leq \mu\lambda) \sim \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\mu}(1-\mu)} \frac{1}{\sqrt{\lambda}} e^{-\lambda\phi(\mu)}.$$

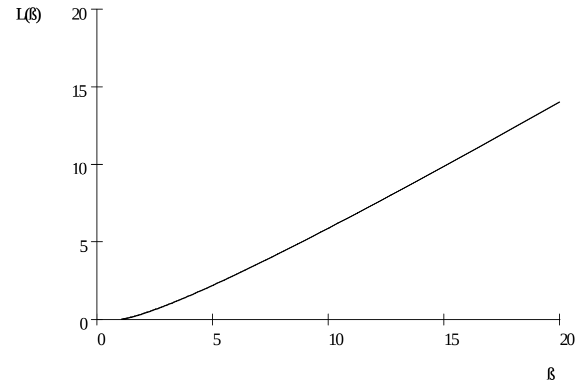

Figure 5: The curve is $\mathcal{L}(\beta)$.

PROOF. First, for any $\mu \in (0,1)$, we show that the lower tail distribution of a Poisson RV can be given by

$$\Pr(Po(\lambda) \leq \mu\lambda) \sim \frac{1}{1-\mu}\Pr(Po(\lambda) = \mu\lambda).$$

Since

$$\frac{\Pr(Po(\lambda) = k-1)}{\Pr(Po(\lambda) = k)} = \frac{\frac{\lambda^{k-1}}{(k-1)!}e^{-\lambda}}{\frac{\lambda^k}{k!}e^{-\lambda}} = \frac{k}{\lambda},$$

we have

$$\Pr(Po(\lambda) \leq \mu\lambda) = \sum_{k=\mu\lambda}^{0} \Pr(Po(\lambda) = k)$$

$$= \sum_{k=0}^{\mu\lambda} \frac{\binom{\mu\lambda}{k}}{\lambda^k} \Pr(Po(\lambda) = \mu\lambda)$$

$$\sim \sum_{k=0}^{\mu\lambda} \frac{(\mu\lambda)^k}{\lambda^k} \Pr(Po(\lambda) = \mu\lambda)$$

$$\sim \frac{1}{1-\mu}\Pr(Po(\lambda) = \mu\lambda).$$

By Sterling's formula, we have

$$\Pr(Po(\lambda) \leq \mu\lambda) \sim \frac{1}{1-\mu}\frac{\lambda^{\mu\lambda}}{(\mu\lambda)!}e^{-\lambda}$$

$$\sim \frac{1}{1-\mu}\frac{\lambda^{\mu\lambda}}{\sqrt{2\pi\mu\lambda}(\mu\lambda)^{\mu\lambda}e^{-\mu\lambda}}e^{-\lambda}$$

$$= \frac{1}{1-\mu}\frac{1}{\sqrt{2\pi\mu\lambda}\mu^{\mu\lambda}}e^{-\lambda+\mu\lambda}$$

$$= \frac{1}{1-\mu}\frac{1}{\sqrt{2\pi\mu\lambda}}e^{-\lambda+\mu\lambda-\mu\lambda\ln\mu}$$

$$= \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{\mu}(1-\mu)}\frac{1}{\sqrt{\lambda}}e^{-\lambda(1-\mu+\mu\ln\mu)}$$

$$= \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{\mu}(1-\mu)}\frac{1}{\sqrt{\lambda}}e^{-\lambda\phi(\mu)}.$$

Thus, the lemma is proved. □

The next lemma gives an a.a.s. lower bound for the minimum of a collection of Poisson RVs.

LEMMA 6. *Assume that* $\lim \frac{\lambda_n}{\ln n} = \beta$ *for some* $\beta > 1$. *Let* $Y_1, Y_2, \cdots, Y_{I_n}$ *be* I_n *Poisson RVs with means at least* λ_n.

1. *If* $I_n = o\left(n\sqrt{\ln n}\right)$, *then for any* $1 < \beta' < \beta$, $\min_{i=1}^{I_n} Y_i > \mathcal{L}(\beta') \ln n$ *a.a.s..*
2. *If* $I_n = O\left(\sqrt{\frac{n}{\ln n}}\right)$, *then for any* $1 < \beta' < \beta$, $\min_{i=1}^{I_n} Y_i > \frac{1}{2}\mathcal{L}(2\beta') \ln n$ *a.a.s..*

PROOF. We first assume that $Y_1, Y_2, \cdots, Y_{I_n}$ all have means λ_n. Let Y be a Poisson RV with mean λ_n. We claim that for any $\mu > 0$,

$$\Pr\left[\min_{i=1}^{I_n} Y_i \le \mu\lambda_n\right] \le I_n \Pr\left[Y \le \mu\lambda_n\right].$$

Let X_i be the indicator of the event $Y_i \le \mu\lambda_n$. Then X_i is a Bernoulli RV with probability $\Pr[Y \le \mu\lambda_n]$. Let $X = X_1 + \cdots + X_{I_n}$. Then, $\min_{i=1}^{I_n} Y_i \le \mu\lambda_n$ if and only if $X \ge 1$. By Markov's inequality,

$$\Pr\left[\min_{i=1}^{I_n} Y_i \le \mu\lambda_n\right] = \Pr[X \ge 1]$$
$$\le E[X] = \sum_{i=1}^{I_n} E[X_i] = I_n \Pr[Y \le \mu\lambda_n].$$

Now, assume that $I_n = o\left(n\sqrt{\ln n}\right)$. Since $\mathcal{L}(\beta') < \mathcal{L}(\beta)$, we have $\mathcal{L}(\beta')/\beta < \phi^{-1}(1/\beta)$. We choose a constant $\mu \in \left(\mathcal{L}(\beta')/\beta, \phi^{-1}(1/\beta)\right)$. Then, $\mu \in (0,1), \mu\beta > \mathcal{L}(\beta')$ and $\beta\phi(\mu) > 1$. Thus, for sufficiently large n, $\mu\lambda_n \ge \mathcal{L}(\beta') \ln n$, which implies that

$$\Pr\left[\min_{i=1}^{I_n} Y_i \le \mathcal{L}(\beta') \ln n\right] \le \Pr\left[\min_{i=1}^{I_n} Y_i \le \mu\lambda_n\right]$$
$$\le I_n \Pr[Y \le \mu\lambda_n].$$

By Lemma 5,

$$\Pr\left[\min_{i=1}^{I_n} Y_i \le \mathcal{L}(\beta') \ln n\right]$$
$$\lesssim \frac{1}{\sqrt{2\pi\beta}} \frac{1}{\sqrt{\mu}(1-\mu)} \frac{I_n}{n\sqrt{\ln n}} n^{1-(\lambda_n/\ln n)\phi(\mu)}.$$

Since

$$1 - (\lambda_n/\ln n)\phi(\mu) \to 1 - \beta\phi(\mu) < 0,$$

we have

$$\Pr\left[\min_{i=1}^{I_n} Y_i \le \mathcal{L}(\beta') \ln n\right] = o(1).$$

Hence $\min_{i=1}^{I_n} Y_i > \mathcal{L}(\beta') \ln n$ a.a.s..

Next, assume that $I_n = O\left(\sqrt{\frac{n}{\ln n}}\right)$. Since $\mathcal{L}(2\beta') < \mathcal{L}(2\beta)$, we have $\mathcal{L}(2\beta')/(2\beta) < \phi^{-1}(1/(2\beta))$. We choose a constant $\mu \in \left(\mathcal{L}(2\beta')/(2\beta), \phi^{-1}(1/(2\beta))\right)$. Thus, $\mu \in (0,1), \mu\beta > \frac{1}{2}\mathcal{L}(2\beta')$ and $\beta\phi(\mu) > 1/2$. Thus, for sufficiently large n, $\mu\lambda_n \ge \frac{1}{2}\mathcal{L}(2\beta') \ln n$, which implies that

$$\Pr\left[\min_{i=1}^{I_n} Y_i \le \frac{1}{2}\mathcal{L}(2\beta') \ln n\right] \le \Pr\left[\min_{i=1}^{I_n} Y_i \le \mu\lambda_n\right]$$
$$\le I_n \Pr[Y \le \mu\lambda_n].$$

By Lemma 5,

$$\Pr\left[\min_{i=1}^{I_n} Y_i \le \frac{1}{2}\mathcal{L}(2\beta') \ln n\right]$$
$$\lesssim \frac{1}{\sqrt{2\pi\beta}} \frac{1}{\sqrt{\mu}(1-\mu)} \frac{I_n}{\sqrt{n\ln n}} n^{1/2-(\lambda_n/\ln n)\phi(\mu)}.$$

Since

$$1/2 - (\lambda_n/\ln n)\phi(\mu) \to 1/2 - \beta\phi(\mu) < 0,$$

we have

$$\Pr\left[\min_{i=1}^{I_n} Y_i \le \frac{1}{2}\mathcal{L}(2\beta') \ln n\right] = o(1).$$

Hence $\min_{i=1}^{I_n} Y_i > \frac{1}{2}\mathcal{L}(2\beta') \ln n$ a.a.s..

Finally, we consider that general case that $Y_1, Y_2, \cdots, Y_{I_n}$ have means $\lambda_{n,1}, \lambda_{n,2}, \cdots, \lambda_{n,I_n}$ respectively with $\lambda_{n,i} \ge \lambda_n$ for each $1 \le i \le I_n$. Let $Y'_1, Y'_2, \cdots, Y'_{I_n}$ be I_n Poisson RVs with means λ_n. For each $1 \le i \le I_n$, let Y''_i be a Poisson RV with mean $\lambda_{n,i} - \lambda_n$ which is independent with Y'_i. Then by the superposition property of Poisson RVs, $Y_i = Y'_i + Y''_i$. Therefore, $\min_{i=1}^{I_n} Y_i \ge \min_{i=1}^{I_n} Y'_i > \mu\lambda_n$. By the above argument, the lemma also holds in this general case. □

At the end of this section, we state the Palm theory [1] on the Poisson process that will be used in Subsection 4(B).

THEOREM 7. *Let* $n > 0$. *Suppose* $k \in N$, *and* $h(\mathcal{Y}, \mathcal{X})$ *is a bounded measurable function defined on all pairs of the form* $(\mathcal{Y}, \mathcal{X})$ *with* $\mathcal{X} \subset \mathbb{R}^2$ *being a finite subset and* $\mathcal{Y}$ *being a subset of* $\mathcal{X}$, *satisfying* $h(\mathcal{Y}, \mathcal{X}) = 0$ *except when* $\mathcal{Y}$ *has* k *elements. Then*

$$\mathbf{E}\left[\sum_{\mathcal{Y} \subseteq \mathcal{P}_n} h(\mathcal{Y}, \mathcal{P}_n)\right] = \frac{n^k}{k!} \mathbf{E}\left[h(\mathcal{X}_k, \mathcal{X}_k \cup \mathcal{P}_n)\right]$$

where the sum on the left-hand side is over all subsets $\mathcal{Y}$ *of the random Poisson point set* $\mathcal{P}_n$, *and on the righthand side the set* $\mathcal{X}_k$ *is a binomial process with* k *nodes, independent of* $\mathcal{P}_n$.

4. GREEDY FORWARD ROUTING

The main result of this paper is given in the following theorem.

THEOREM 8. *Suppose that* $n\pi r_n^2 = (\beta + o(1)) \ln n$ *for some* $\beta > 0$.

1. *If* $\beta > \beta_0$, *then* $\rho(\mathcal{P}_n) \le r_n$ *is a.a.s..*
2. *If* $\beta < \beta_0$, *then* $\rho(\mathcal{P}_n) > r_n$ *is a.a.s..*

4.1 Upper Bounds for the Critical Transmission Radius

This subsection is dedicated to the proof of Theorem 8(1). We need a technique tool called minimal scan statistics for the proof. For any finite point set $V \subset \mathbb{D}$ and any $r > 0$, define

$$\mathcal{S}(V, r) = \min_{u,v \in \mathbb{D}, \|u-v\|=r} |V \cap L_{uv}|.$$

We claim that the event $\mathcal{S}(\mathcal{P}_n, r_n) > 0$ implies the event $\rho(\mathcal{P}_n) \le r_n$. Note that $\rho(\mathcal{P}_n) \le r_n$ if and only if for any pair of nodes u and v with $\|u - v\| > r_n$, there is at least one node inside $B(u, r_n) \cap B(v, \|u - v\|)$. Assume to the contrary that $\rho(\mathcal{P}_n) > r_n$. Then there are a pair of nodes u and v such that $\|u - v\| > r_n$ and no one node of $\mathcal{P}_n$ is inside $B(u, r_n) \cap B(v, \|u - v\|)$. Let w be the intersection point of the segment uv and the circle $\partial B(u, \|u - v\|)$. (See Fig. 6.) Then $\|u - w\| = r_n$, and $B(w, \|u - w\|) \subset B(v, \|u - v\|)$.

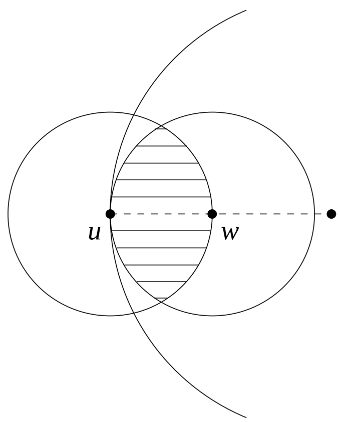

Figure 6: w is the intersection point of the segment uv and the circle $B(u, r)$. The shaded area is $B(u, r) \cap B(w, r)$ which is contained in $B(u, r) \cap B(v, \|u - v\|)$.

Hence, $L_{uw} \subset B(u, r_n) \cap B(v, \|u - v\|)$. This implies that L_{uw} contains no nodes of $\mathcal{P}_n$. Thus, $\mathcal{S}(\mathcal{P}_n, r) = 0$, which is a contradiction. Therefore, our claim is true.

Based on the previous claim, to prove $\rho(\mathcal{P}_n) \le r_n$ is a.a.s., it is enough to show that $\mathcal{S}(\mathcal{P}_n, r_n) > 0$ a.a.s.. Below, we shall give a stronger result that provides an a.a.s. lower bound for $\mathcal{S}(\mathcal{P}_n, r_n)$ with $r_n = \sqrt{\frac{(\beta+o(1))\ln n}{\pi n}}$ and implies $\mathcal{S}(\mathcal{P}_n, r_n) > 0$ is a.a.s. if $\beta > \beta_0$.

LEMMA 9. *Suppose that $n\pi r_n^2 = (\beta + o(1)) \ln n$ for some $\beta > \beta_0$. Then for any constant $\beta_1 \in (\beta_0, \beta)$, it is a.a.s. that*

$$\mathcal{S}(\mathcal{P}_n, r_n) \ge \frac{1}{2}\mathcal{L}\left(\frac{\beta_1}{\beta_0}\right) \ln n.$$

PROOF. Choose a constant $\beta_2 \in (\beta_1, \beta)$ and let $\varepsilon = \frac{1}{6\sqrt{2}\beta_0}\left(1 - \frac{\beta_2}{\beta}\right)$. Let $d = \sqrt{3} r_n$. Consider an εd-tessellation. Let I_n denote the number of polyquadrates in $\mathbb{D}$ with span at most $\frac{1}{\varepsilon}$ and area at least $\frac{\beta_2}{\beta_0}\frac{\pi r_n^2}{\beta}$, and Y_i be the number of nodes on the i-th polyquadrate. Then Y_i is a Poisson RV with rate at least $\left(\frac{\beta_2}{\beta_0} + o(1)\right) \ln n$. By Lemma 4,

$$I_n = O\left(\left(\frac{1}{\varepsilon d}\right)^2\right) = O\left(\frac{n}{\ln n}\right).$$

By Lemma 6, it is a.a.s. that

$$\frac{\min_{i=1}^{I_n} Y_i}{\ln n} \ge \mathcal{L}\left(\frac{\beta_1}{\beta_0}\right).$$

Now, let I_n' denote the number of polyquadrates in $\mathbb{D} \setminus \mathbb{D}_{-d}$ with span at most $\frac{1}{\varepsilon}$ and area at least $\frac{1}{2}\frac{\beta_2}{\beta_0}\frac{\pi r_n^2}{\beta}$, and Y_i' be the number of nodes on the i-th polyquadrate. Then Y_i' is a Poisson RV with rate at least $\frac{1}{2}\left(\frac{\beta_2}{\beta_0} + o(1)\right) \ln n$. By Lemma 4,

$$I_n' = O\left(\frac{1}{\varepsilon d}\right) = O\left(\sqrt{\frac{n}{\ln n}}\right).$$

By Lemma 6, it is a.a.s. that

$$\frac{\min_{i=1}^{I_n'} Y_i'}{\ln n} \ge \frac{1}{2}\mathcal{L}\left(\frac{\beta_1}{\beta_0}\right).$$

Therefore, it is a.a.s. that

$$\frac{\min\left(\min_{i=1}^{I_n} Y_i, \min_{i=1}^{I_n'} Y_i'\right)}{\ln n} \ge \frac{1}{2}\mathcal{L}\left(\frac{\beta_1}{\beta_0}\right).$$

Thus, the lemma follows if we can show that

$$\mathcal{S}(\mathcal{P}_n, r_n) \ge \min\left(\min_{i=1}^{I_n} Y_i, \min_{i=1}^{I_n'} Y_i'\right).$$

To prove this inequality, it is sufficient to show that for any lune L of two points in $\mathbb{D}$ which are separated by a distance of r_n, it either contains a polyquadrate in $\mathbb{D}$ with span at most $\frac{1}{\varepsilon}$ and area at least $\frac{\beta_2}{\beta_0}\frac{\pi r_n^2}{\beta}$, or contains a polyquadrate in $\mathbb{D} \setminus \mathbb{D}_{-d}$ with span at most $\frac{1}{\varepsilon}$ and area at least $\frac{1}{2}\frac{\beta_2}{\beta_0}\frac{\pi r_n^2}{\beta}$. We shall prove this in two cases.

Case 1: L is contained in $\mathbb{D}$. Let P denote the polyquadrate induced by $L_{-\sqrt{2}\varepsilon d}$. Then, $P \subseteq L \subseteq \mathbb{D}$, and the span of P is at most $\left\lceil \frac{d-2\sqrt{2}\varepsilon d}{\varepsilon d} \right\rceil + 1 \le \frac{1}{\varepsilon}$. By Lemma 3 and using the fact that $|L| = \pi r_n^2/\beta_0 = \pi d^2/(3\beta_0)$, we have

$$\begin{aligned}|P| &\ge \left|L_{-\sqrt{2}\varepsilon d}\right| \ge |L| - \pi d\left(\sqrt{2}\varepsilon d\right) \\ &= |L| - \sqrt{2}\varepsilon\pi d^2 = |L|\left(1 - 3\sqrt{2}\beta_0\varepsilon\right) \\ &> |L|\left(1 - 6\sqrt{2}\beta_0\varepsilon\right) \\ &= \frac{\beta_2}{\beta}|L| = \frac{\beta_2}{\beta_0}\frac{\pi r_n^2}{\beta}.\end{aligned}$$

Case 2: L is not contained in $\mathbb{D}$. Then L must be disjoint with $\mathbb{D}_{-d}$. Let $L' = L \cap \mathbb{D}$ and let P' denote the polyquadrate induced by $L'_{-\sqrt{2}\varepsilon d}$. Then $P' \subseteq L' \subseteq \mathbb{D} \setminus \mathbb{D}_{-d}$ and the the span of P is also at most $\frac{1}{\varepsilon}$. By Lemma 3 and Lemma 1, we have

$$\begin{aligned}|P'| &\geq |L'_{-\sqrt{2}\varepsilon d}| \geq |L'| - \pi d\left(\sqrt{2}\varepsilon d\right)\\ &\geq \frac{1}{2}|L| - \sqrt{2}\pi\varepsilon d^2\\ &= \frac{1}{2}|L|\left(1-6\sqrt{2}\beta_0\varepsilon\right)\\ &= \frac{1}{2}\frac{\beta_2}{\beta}|L| = \frac{1}{2}\frac{\beta_2}{\beta_0}\frac{\pi r_n^2}{\beta}.\end{aligned}$$

Thus, the lemma is proved. □

4.2 Lower Bounds for the Critical Transmission Radius

This subsection is dedicated to the proof of Theorem 8(2). Assume β_1 and β_2 are positive constants, and R_1 and R_2 are given by $n\pi R_1^2 = \beta_1 \ln n$ and $n\pi R_2^2 = \beta_2 \ln n$, respectively. Choose β_1, β_2 such that $\max\left(\frac{1}{4}\beta_0, \beta\right) < \beta_1 < \beta_2 < \beta_0$ and $\frac{\pi^2}{c^2}\left(1-\frac{R_1}{R_2}\right) < 1$. Here c is given by Lemma 2. We have $\frac{1}{2}R_2 \leq R_1 \leq R_2$. Divide $\mathbb{D}$ by $\left(4\sqrt{\frac{\ln n}{n\pi}}\right)$-tessellation. Let I_n denote the number of cells fully contained in $\mathbb{D}$. Here we have $I_n = O\left(\frac{n}{\ln n}\right)$. For each cell fully contained in $\mathbb{D}$, we draw a disk with radius $\frac{1}{2}\sqrt{\frac{\ln n}{n\pi}}$ at the center of the cell. For $1 \leq i \leq I_n$, let E_i be the event that there exist two nodes $X, Y \in \mathcal{P}_n$ such that their midpoint is on the i-th disk and distance is between R_1 and R_2, and there is no other node on the lune L_{XY}. Then,

$$\Pr[\rho(\mathcal{P}_n) > r_n] \geq \Pr[\text{at least one } E_i \text{ occurs}].$$

We have $E_1, \cdots, E_{I_i}$ are identical. Let o_i denote the center of the i-th disk, and u, v be two points such that their midpoint is on the i-th disk and distance is between R_1 and R_2. (See Fig. 7.) For any point $w \in L_{uv}$, we have

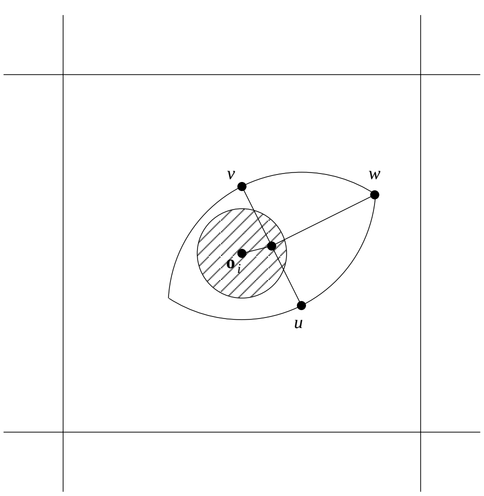

Figure 7: The lune is fully contained in the cell.

$$\begin{aligned}\|w - o_i\| &\leq \left\|w - \frac{1}{2}(u+v)\right\| + \left\|o_i - \frac{1}{2}(u+v)\right\|\\ &\leq \frac{\sqrt{3}}{2}R + \frac{1}{2}\sqrt{\frac{\ln n}{n\pi}} \approx 1.885\sqrt{\frac{\ln n}{n\pi}}\\ &< 2\sqrt{\frac{\ln n}{n\pi}}.\end{aligned}$$

Obviously, u, v and L_{uv} are contained in the i-th cell. Therefore, $E_1, \cdots, E_{I_i}$ are independent. Then,

$$\begin{aligned}\Pr[\text{none of } E_i \text{ occurs}] &= (1-\Pr[E_1])^{I_n}\\ &= e^{I_n \ln(1-\Pr[E_1])}\\ &\leq e^{-I_n \Pr(E_1)}.\end{aligned}$$

If $I_n \Pr(E_1) \to \infty$, we may have

$$\Pr[\rho(\mathcal{P}_n) > r_n] \to 1,$$

and Theorem 8(2) follows. In the following, we will prove that $I_n \Pr(E_1) \to \infty$.

We introduce several relevant events and derive their probabilities. Let A denote the first disk. Assume V is a point set and $Y \subset V$. Let $h_1(Y, V)$ denote a function such that $h_1(Y = \{x_1, x_2\}, V) = 1$ only if $\frac{1}{2}(x_1 + x_2) \in A$, $R_1 \leq \|x_1 - x_2\| \leq R_2$, and there is no other node of V in the lune area $L_{x_1x_2}$; otherwise, $h_1(Y, V) = 0$. Then, E_1 is the event that there exist two nodes $X, Y \in \mathcal{P}_n$ such that $h_1(\{X, Y\}, \mathcal{P}_n) = 1$. In the remaining of this subsection, we use X_1, X_2, X_3 and X_4 to denote independent random points with uniform distribution over $\mathbb{D}$ and independent of $\mathcal{P}_n$. Let F_1 be the event that

$$h_1(\{X_1, X_2\}, \{X_1, X_2\} \cup \mathcal{P}_n) = 1,$$

F_2 be the event that

$$\left(\begin{array}{l} h_1(\{X_1, X_2\}, \{X_1, X_2, X_3\} \cup \mathcal{P}_n)\\ \cdot h_1(\{X_1, X_3\}, \{X_1, X_2, X_3\} \cup \mathcal{P}_n)\end{array}\right) = 1,$$

and F_3 be the event that

$$\left(\begin{array}{l} h_1(\{X_1, X_2\}, \{X_1, X_2, X_3, X_4\} \cup \mathcal{P}_n)\\ \cdot h_1(\{X_3, X_4\}, \{X_1, X_2, X_3, X_4\} \cup \mathcal{P}_n)\end{array}\right) = 1.$$

We claim that

$$\Pr[E_1] \geq \frac{n^2}{2!}\Pr[F_1] - \frac{n^3}{2}\Pr[F_2] - \frac{n^4}{8}\Pr[F_3]. \quad (1)$$

We shall prove this claim by the Palm theory and Boole's inequalities. For clarity, we use X_1', X_2', X_3' and X_4' to denote elements of $\mathcal{P}_n$. For any $\{x_1, x_2, x_3\} \subseteq V$, let

$$\begin{aligned}&h_2(\{x_1, x_2, x_3\}, V)\\ &= h_1(\{x_1, x_2\}, V) \cdot h_1(\{x_1, x_3\}, V)\\ &+ h_1(\{x_2, x_1\}, V) \cdot h_1(\{x_2, x_3\}, V)\\ &+ h_1(\{x_3, x_1\}, V) \cdot h_1(\{x_3, x_2\}, V).\end{aligned}$$

For any $\{x_1, x_2, x_3, x_4\} \subseteq V$, let

$$\begin{aligned}&h_3(\{x_1, x_2, x_3, x_4\}, V)\\ &= h_1(\{x_1, x_2\}, V) \cdot h_1(\{x_3, x_4\}, V)\\ &+ h_1(\{x_1, x_3\}, V) \cdot h_1(\{x_2, x_4\}, V)\\ &+ h_1(\{x_1, x_4\}, V) \cdot h_1(\{x_2, x_3\}, V).\end{aligned}$$

Let $F_1'(\{X_1', X_2'\})$ be the event that

$$h_1\left(\left\{X_1', X_2'\right\}, \mathcal{P}_n\right) = 1,$$

$F_2'(\{X_1', X_2', X_3'\})$ be the event that

$$h_2\left(\left\{X_1', X_2', X_3'\right\}, \mathcal{P}_n\right) = 1,$$

and $F_3'(\{X_1', X_2', X_3', X_4'\})$ be the event that

$$h_3\left(\left\{X_1', X_2', X_3', X_4'\right\}, \mathcal{P}_n\right) = 1.$$

According to the Palm theory (Theorem 7), we have

$$\begin{aligned}
&\sum_{\{X_1', X_2'\} \subseteq \mathcal{P}_n} \Pr\left[F_1'\left(\left\{X_1', X_2'\right\}\right)\right] \\
&= \mathbf{E}\left[\sum_{\{X_1', X_2'\} \subseteq \mathcal{P}_n} h_1\left(\left\{X_1', X_2'\right\}, \mathcal{P}_n\right)\right] \\
&= \frac{n^2}{2!}\mathbf{E}\left[h_1\left(\{X_1, X_2\}, \{X_1, X_2\} \cup \mathcal{P}_n\right)\right] \\
&= \frac{n^2}{2}\Pr\left[F_1\right]; \qquad (2)
\end{aligned}$$

$$\begin{aligned}
&\sum_{\{X_1', X_2', X_3'\} \subseteq \mathcal{P}_n} \Pr\left[F_2'\left(\left\{X_1', X_2', X_3'\right\}\right)\right] \\
&= \mathbf{E}\left[\sum_{\{X_1', X_2', X_3'\} \subseteq \mathcal{P}_n} h_2\left(\left\{X_1', X_2', X_3'\right\}, \mathcal{P}_n\right)\right] \\
&= \frac{n^3}{3!}\mathbf{E}\left[h_2\left(\{X_1, X_2, X_3\}, \{X_1, X_2, X_3\} \cup \mathcal{P}_n\right)\right] \\
&= 3\frac{n^3}{3!}\Pr\left[F_2\right] = \frac{n^3}{2}\Pr\left[F_2\right]; \qquad (3)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{\{X_1', X_2', X_3', X_4'\} \subseteq \mathcal{P}_n} \Pr\left[F_3'\left(\left\{X_1', X_2', X_3', X_4'\right\}\right)\right] \\
&= \mathbf{E}\left[\sum_{\{X_1', X_2', X_3', X_4'\} \subseteq \mathcal{P}_n} h_3\left(\left\{X_1', X_2', X_3', X_4'\right\}, \mathcal{P}_n\right)\right] \\
&= \frac{n^3}{3!}\mathbf{E}\left[h_3\left(\{X_1, X_2, X_3, X_4\}, \{X_1, X_2, X_3, X_4\} \cup \mathcal{P}_n\right)\right] \\
&= 3\frac{n^4}{4!}\Pr\left[F_2\right] = \frac{n^4}{8}\Pr\left[F_3\right]. \qquad (4)
\end{aligned}$$

Applying Boole's inequalities and Eq. (2), (3), and (4), we have

$$\begin{aligned}
\Pr\left[E_1\right] \geq &\sum_{\{X_1', X_2'\} \subseteq \mathcal{P}_n} \Pr\left[F_1'\left(\left\{X_1', X_2'\right\}\right)\right] \\
&- \sum_{\{X_1', X_2', X_3'\} \subseteq \mathcal{P}_n} \Pr\left[F_2'\left(\left\{X_1', X_2', X_3'\right\}\right)\right] \\
&- \sum_{\{X_1', X_2', X_3', X_4'\} \subseteq \mathcal{P}_n} \Pr\left[F_3'\left(\left\{X_1', X_2', X_3', X_4'\right\}\right)\right] \\
&= \frac{n^2}{2}\Pr\left[F_1\right] - \frac{n^3}{2}\Pr\left[F_2\right] - \frac{n^4}{8}\Pr\left[F_3\right].
\end{aligned}$$

Hence, our claim is true.

In the next, we derive the probabilities of F_1, F_2, and F_3. Let S_1 denote the set

$$\left\{(x_1, x_2) \;\middle|\; \begin{array}{c} \frac{1}{2}\left(x_1 + x_2\right) \in A, \\ R_1 \leq \left\|x_1 - x_2\right\| \leq R_2 \end{array}\right\}.$$

We have

$$\begin{aligned}
&\Pr\left[F_1\right] \\
&= \int\int_{S_1} \Pr\left[F_1 \mid X_1 = x_1, X_2 = x_2\right] dx_1 dx_2 \\
&= \int\int_{S_1} e^{-n\left|L_{x_1 x_2}\right|} dx_1 dx_2 \\
&= \int\int_{S_1} e^{-n\frac{1}{\beta_0}\pi\left\|x_1 - x_2\right\|^2} dx_1 dx_2.
\end{aligned}$$

Let $z = \frac{x_1 + x_2}{2}$ and $r = \frac{1}{2}\left\|x_1 - x_2\right\|$. Then,

$$\begin{aligned}
\Pr\left[F_1\right] &= \int_{z \in A}\int_{r = \frac{R_1}{2}}^{\frac{R_2}{2}} e^{-\frac{4}{\beta_0}n\pi r^2} 8\pi r dr dz \\
&= 4\int_{z \in A}\int_{r = \frac{R_1}{2}}^{\frac{R_2}{2}} e^{-\frac{4}{\beta_0}n\pi r^2} 2\pi r dr dz \\
&= 4\int_{z \in A}\int_{r = \frac{R_1}{2}}^{\frac{R_2}{2}} e^{-\frac{4}{\beta_0}n\pi r^2} d\left(\pi r^2\right) dz \\
&= -\left(\frac{\beta_0}{n} e^{-\frac{4}{\beta_0}n\pi r^2}\Big|_{r = \frac{R_1}{2}}^{\frac{R_2}{2}}\right)|A| \\
&= \frac{\beta_0}{4n^2}\left(n^{-\frac{\beta_1}{\beta_0}} - n^{-\frac{\beta_2}{\beta_0}}\right)\ln n. \qquad (5)
\end{aligned}$$

Let S_2 denote the set

$$\left\{(x_1, x_2, x_3) \;\middle|\; \begin{array}{c} \frac{x_1 + x_2}{2}, \frac{x_1 + x_3}{2} \in A; \\ R_1 \leq \left\|x_1 - x_2\right\| \leq R_2; \\ R_1 \leq \left\|x_1 - x_3\right\| \leq R_2; \\ x_1, x_2 \notin L_{x_1 x_3}; \\ x_1, x_3 \notin L_{x_1 x_2} \end{array}\right\}.$$

Applying Lemma 2, if $(x_1, x_2, x_3) \in S_2$, we have

$$\begin{aligned}
&\Pr\left[F_2 \;\middle|\; \begin{array}{c} X_1 = x_1, X_2 = x_2, \\ X_3 = x_3 \end{array}\right] \\
&\leq e^{-n\left|L_{x_1 x_2} \cup L_{x_1 x_3}\right|} \\
&\leq e^{-n\left(\frac{1}{\beta_0}\pi\left\|x_1 - x_2\right\|^2 + cR_2\left\|\frac{x_1 + x_2}{2} - \frac{x_1 + x_3}{2}\right\|\right)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\Pr\left[F_2\right] \\
&= \int\int\int_{S_2} \Pr\left[F_2 \;\middle|\; \begin{array}{c} X_1 = x_1, X_2 = x_2, \\ X_3 = x_3 \end{array}\right] \\
&\quad \cdot dx_1 dx_2 dx_3 \\
&\leq \int\int\int_{S_2} e^{-n\left(\frac{1}{\beta_0}\pi\left\|x_1 - x_2\right\|^2 + cR_2\left\|\frac{x_1 + x_2}{2} - \frac{x_1 + x_3}{2}\right\|\right)} \\
&\quad \cdot dx_1 dx_2 dx_3.
\end{aligned}$$

Let $z_1 = \frac{x_1 + x_2}{2}$, $r_1 = \frac{1}{2}\left\|x_1 - x_2\right\|$, $z_2 = \frac{x_1 + x_3}{2}$, and $\rho = \left\|z_1 - z_2\right\|$. Then,

$$\begin{aligned}
&\Pr\left[F_2\right] \\
&\leq 16\int_{z_1 \in A}\int_{r_1 = \frac{R_1}{2}}^{\frac{R_2}{2}}\int_{z_2 \in A} e^{-n\left(\frac{4}{\beta_0}\pi r_1^2 + cR_2\left\|z_1 - z_2\right\|\right)} \\
&\quad \cdot 2\pi r_1 dr_1 dz_1 dz_2 \\
&\leq 16\int_{z_1 \in A}\int_{r_1 = \frac{R_1}{2}}^{\frac{R_2}{2}} e^{-\frac{4}{\beta_0}n\pi r_1^2} 2\pi r_1 dr_1 dz_1
\end{aligned}$$

$$\cdot \int_{z_2\in A} e^{-cnR_2\|z_1-z_2\|} dz_2$$

$$\le 16 \int_{z_1\in A}\int_{r_1=\frac{R_1}{2}}^{\frac{R_2}{2}} e^{-\frac{4}{\beta_0}n\pi r_1^2} d\left(\pi r_1^2\right) dz_1$$

$$\cdot \int_{\rho=0}^{\infty} e^{-cnR_2\rho} 2\pi\rho d\rho$$

$$= -\left(\frac{4\beta_0}{n} e^{-\frac{4}{\beta_0}n\pi r^2}\Big|_{r=\frac{R_1}{2}}^{\frac{R_2}{2}}\right)|A| \cdot \frac{2\pi}{(cnR_2)^2}$$

$$= \frac{2\pi\beta_0}{c^2\left(nR_2^2\right)n^3}\left(n^{-\frac{\beta_1}{\beta_0}} - n^{-\frac{\beta_2}{\beta_0}}\right)\ln n. \quad (6)$$

Let S_3 denote the set

$$\left\{ (x_1,x_2,x_3,x_4) \;\middle|\; \begin{array}{c} \frac{x_1+x_2}{2}, \frac{x_3+x_4}{2} \in A; \\ R_1 \le \|x_1-x_2\| \le R_2; \\ R_1 \le \|x_3-x_4\| \le R_2; \\ x_1,x_2 \notin L_{x_3x_4}; \\ x_3,x_4 \notin L_{x_1x_2} \end{array} \right\}.$$

Applying Lemma 2, if $(x_1,x_2,x_3,x_4)\in S_3$, we have

$$\Pr\left[F_3 \;\middle|\; \begin{array}{c} X_1=x_1, X_2=x_2, \\ X_3=x_3, X_4=x_4 \end{array}\right]$$
$$\le e^{-n\left|L_{x_1x_2}\cup L_{x_3x_4}\right|}$$
$$\le e^{-n\left(\frac{1}{\beta_0}\pi\|x_1-x_2\|^2 + cR_2\left\|\frac{x_1+x_2}{2}-\frac{x_3+x_4}{2}\right\|\right)}.$$

Therefore,

$$\Pr[F_3]$$
$$= \int\int\int\int_{S_3} \Pr\left[F_3 \;\middle|\; \begin{array}{c} X_1=x_1, X_2=x_2, \\ X_3=x_3, X_4=x_4 \end{array}\right]$$
$$\cdot dx_1dx_2dx_3dx_4$$
$$\le \int\int\int\int_{S_3} e^{-n\left(\frac{1}{\beta_0}\pi\|x_1-x_2\|^2 + cR_2\left\|\frac{x_1+x_2}{2}-\frac{x_3+x_4}{2}\right\|\right)}$$
$$\cdot dx_1dx_2dx_3dx_4.$$

Let $z_1 = \frac{x_1+x_2}{2}$, $r_1 = \frac{1}{2}\|x_1-x_2\|$, $z_2 = \frac{x_3+x_4}{2}$, $r_2 = \frac{1}{2}\|x_3-x_4\|$, and $\rho = \|z_1-z_2\|$. Then,

$$\Pr[F_3]$$
$$\le \int_{z_1\in A}\int_{r_1=\frac{R_1}{2}}^{\frac{R_2}{2}}\int_{z_2\in A}\int_{r_2=\frac{R_1}{2}}^{\frac{R_2}{2}} e^{-n\left(\frac{4}{\beta_0}\pi r_1^2 + cR_2\|z_1-z_2\|\right)}$$
$$\cdot (8\pi r_1dr_1dz_1)(8\pi r_2dr_2dz_2)$$
$$\le \left(4\int_{z_1\in A}\int_{r_1=\frac{R_1}{2}}^{\frac{R_2}{2}} e^{-\frac{4}{\beta_0}n\pi r_1^2} 2\pi dr_1dz_1\right)$$
$$\cdot\left(8\pi\frac{R_2}{2}\left(\frac{R_2}{2}-\frac{R_1}{2}\right)\int_{z_2\in A} e^{-cnR_2\|z_1-z_2\|}dz_2\right)$$
$$\le \left(4\int_{z_1\in A}\int_{r_1=\frac{R_1}{2}}^{\frac{R_2}{2}} e^{-\frac{4}{\beta_0}n\pi r_1^2} d\left(\pi r_1^2\right)dz_1\right)$$
$$\cdot\left(8\pi\frac{R_2}{2}\left(\frac{R_2}{2}-\frac{R_1}{2}\right)\int_{\rho=0}^{\infty} e^{-cnR_2\rho}2\pi\rho d\rho\right)$$
$$= \left(\frac{\beta_0\ln n}{4n^2}\left(n^{-\frac{\beta_1}{\beta_0}} - n^{-\frac{\beta_2}{\beta_0}}\right)\right)\left(\frac{4\pi^2}{(cnR_2)^2}R_2(R_2-R_1)\right)$$
$$= \frac{\pi^2\beta_0}{c^2n^4}\left(1-\frac{R_1}{R_2}\right)\left(n^{-\frac{\beta_1}{\beta_0}} - n^{-\frac{\beta_2}{\beta_0}}\right)\ln n. \quad (7)$$

Put Eq. (1), (5), (6) and (7) together. We have

$$\Pr[E_1] \ge \left(\frac{\beta_0}{8} - \frac{\pi\beta_0}{c^2\left(nR_2^2\right)} - \frac{\pi^2\beta_0}{8c^2}\left(1-\frac{R_1}{R_2}\right)\right)$$
$$\cdot\left(n^{-\frac{\beta_1}{\beta_0}} - n^{-\frac{\beta_2}{\beta_0}}\right)\ln n$$
$$\sim \frac{\beta_0}{8}\left(1-\frac{\pi^2}{c^2}\left(1-\frac{R_1}{R_2}\right)\right)\left(n^{-\frac{\beta_1}{\beta_0}} - n^{-\frac{\beta_2}{\beta_0}}\right)\ln n.$$

Since $\frac{\pi^2}{c^2}\left(1-\frac{R_1}{R_2}\right) < 1$ and $I_n = \Omega\left(\frac{\ln n}{n}\right)$, we have

$$\Pr[E_1] = \Omega\left(\left(n^{-\frac{\beta_1}{\beta_0}} - n^{-\frac{\beta_2}{\beta_0}}\right)\ln n\right),$$

and

$$I_n \Pr[E_1] = \Omega\left(n^{1-\frac{\beta_1}{\beta_0}}\right) \to \infty.$$

This complete the proof of Theorem 8(2).

5. CONCLUSION

Greedy forward routing is a localized and memoryless geographic routing. However, it cannot guarantee the delivery of a packet from its source to its destination if the transmission of the nodes are not large enough. The smallest transmission radius which ensures the successful delivery of any packet is referred to as the critical transmission radius. In this paper, we provides tight a.a.s. bounds on the critical transmission radius when the networking nodes are represented by a Poisson point process.

As a future work, one may investigate a number of other parameters related to GFR. These parameters include the average of one-hop progress, the expected number of hops between a source and destination, the ratio of the total length of the path to the Euclidean distance between the source and the destination. It is also interesting to study the asymptotics of other localized geographic routings.

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APPENDIX

PROOF LEMMA 2. Note that $|L_{a_1b_1} \cup L_{a_2b_2}| - |L_{a_1b_1}| = |L_{a_2b_2} \setminus L_{a_1b_1}|$. If $r_2 > r_1$, we have $|L_{a_2b_2} \setminus L_{a_1b_1}| > |L_{a_1b_1} \setminus L_{a_2b_2}|$. Therefore, without loss of generality, we may assume $r_1 \geq r_2$. For a lune L_{uv}, the portion of boundary of L_{uv} contributed by either $\partial B(u, \|u - v\|)$ or $\partial B(v, \|u - v\|)$ are called the sides, and the intersection points of two sides are called vertices. This lemma is proved in the following two cases.

Case 1: Suppose the segment a_2b_2 intersects with at most one side of $L_{a_1b_1}$. First, under this assumption, we claim the minimum of $|L_{a_2b_2} \setminus L_{a_1b_1}|$ occurs if a_2, b_2 are on the boundary of $L_{a_1b_1}$ and $r_1 = r_2 = \frac{1}{2}R$. This claim is based on the following three observations: (1) If r_1 and r_2 are fixed, $|L_{a_2b_2} \setminus L_{a_1b_1}|$ is minimal as a_2, b_2 are on $\partial L_{a_1b_1}$. As illustrated in Fig. 8(a), if $L_{a_2b_2}$ is moved away from $L_{a_1b_1}$, the area surrounded by arcs a_2b_2, a_2c_2 and a_2c_2 is always outside of $L_{a_1b_1}$. (2) For any fixed r_1, if a_2, b_2 are on $\partial L_{a_1b_1}$, $|L_{a_2b_2} \setminus L_{a_1b_1}|$ is minimal as $r_2 = \frac{1}{2}R$. As illustrated in Fig. 8(b), if $\|a_2 - b_2\| \leq \|a_2' - b_2'\|$ and lines a_2b_2 and a_2b_2 are parallel, $L_{a_2b_2} \setminus L_{a_1b_1}$ is contained in $L_{a_2'b_2'} \setminus L_{a_1b_1}$. (3) If a_2, b_2 are on $\partial L_{a_1b_1}$ and $r_2 = \frac{1}{2}R$, $|L_{a_2b_2} \setminus L_{a_1b_1}|$ is minimal as $r_1 = \frac{1}{2}R$. Now, we assume a_2, b_2 are on the boundary of $L_{a_1b_1}$ and $r_1 = r_2 = \frac{1}{2}R$. For convenience, we assume a_2 is coincident with a vertex of $L_{a_1b_1}$, and b_2 is coincident with b_1. Let c_2 denote the vertex of $L_{a_2b_2}$ far from a_1. (See Fig. 8(c).) Since the area surrounded by the arc a_2b_2 and segment a_2b_2 is equal to the area surrounded by the arc b_2c_2 and segment b_2c_2, we have

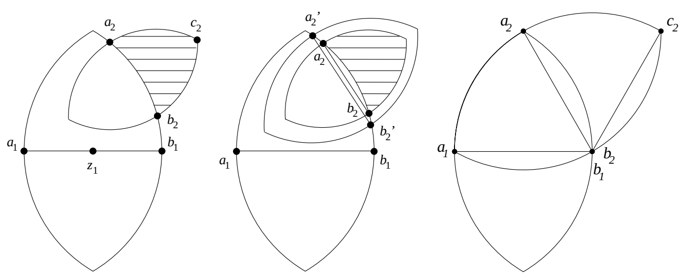

Figure 8: The minimum of $|L_{a_2b_2} \setminus L_{a_1b_1}|$ as the line a_2b_2 intersect with one side of $L_{a_1b_1}$.

$$|L_{a_2b_2} \setminus L_{a_1b_1}| = |\measuredangle a_2b_2c_2| = \frac{1}{6}\pi \left(\frac{1}{2}R\right)^2.$$

Therefore, in this case, for any r_1 and r_2, we have

$$\begin{aligned}|L_{a_2b_2} \setminus L_{a_1b_1}| &\geq \frac{1}{6}\pi \left(\frac{1}{2}R\right)^2 \\ &= \frac{\pi}{24\sqrt{3}} R \left(\sqrt{3}R\right) \\ &\geq \frac{\pi}{24\sqrt{3}} R \|z_1 - z_2\| \\ &\approx 0.075R \|z_1 - z_2\|.\end{aligned}$$

Case 2: Suppose the segment a_2b_2 intersects with both sides of $L_{a_1b_1}$. Let x (respectively, y) denote the intersection point of the line a_2b_2 with $\partial L_{a_1b_1}$ near to a_2 (respectively, b_2). Without loss of generality, we assume y is closer to a_1b_1 than x. Let $0 \leq \theta' \leq \pi/2$ denote the angle between rays b_1a_1 and b_2a_2. (If $\theta' = 0$, it means b_1a_1 and b_2a_2 are parallel.) Let c_1 denote the vertex of $L_{a_1b_1}$ contained in $L_{a_2b_2}$, and c_2 denote the vertex of $L_{a_2b_2}$ near to c_1, and H denote the region of the half lune of $L_{a_2b_2}$ divided by a_2b_2 and containing c_2. In the remaining discussion, we only focus on the area of $H \setminus L_{a_1b_1}$. Assume b_2 is on the boundary of $L_{a_1b_1}$. (If b_2 is not on $\partial L_{a_1b_1}$, we may shift $L_{a_2b_2}$ along the line a_2b_2 until b_2 is on $\partial L_{a_1b_1}$. During shifting, $|H \setminus L_{a_1b_1}|$ has the same value.) Let a_2' denote the point such that lines a_1b_1 and $a_2'b_2$ are parallel, the segment $a_2'b_2$ crosses over $L_{a_1b_1}$, and $\|a_2' - b_2\| = r_1$. Let e denote the perpendicular projection of z_1 onto the line $a_2'b_2$, h denote the perpendicular projection of b_2 onto the line a_1b_1, z_2' denote the the intersection point of the segment $a_2'b_2$ and circle $\partial B(b_2, \|z_2 - b_2\|)$, z_2'' denote the midpoint of a_2' and b_2, and d denote the intersection point of the ray b_2c_1 and $\partial L_{a_2b_2}$. (See Fig. 9.)

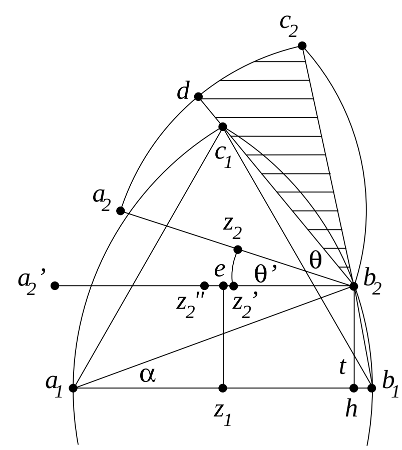

Figure 9: The intersection of two lunes.

First, we consider the lower bound of the area of $|L_{a_2b_2} \setminus L_{a_1b_1}|$. Let θ denote the angle of $\angle a_2b_2d$, α denote the angle of $\angle b_1a_1b_2$. Since $\angle c_1a_1b_1 = \pi/3$, we have

$$\begin{aligned}\angle c_1a_1b_2 &= \angle c_1a_1b_1 - \angle b_2a_1b_1 \\ &= \frac{\pi}{3} - \alpha.\end{aligned}$$

Since $\|a_1 - c_1\| = \|a_1 - b_2\|$, we have

$$\begin{aligned}\angle a_1b_2c_1 &= \frac{1}{2}(\pi - \angle c_1a_1b_2) \\ &= \frac{1}{2}\left(\pi - \left(\frac{\pi}{3} - \alpha\right)\right) \\ &= \frac{\pi}{3} + \frac{1}{2}\alpha.\end{aligned}$$

Since lines $a_2'b_2$ and a_2b_2 are parallel, we have

$$\begin{aligned}\angle a_1b_2c_1 &= \angle a_2b_2d + \angle a_2'b_2a_2 + \angle a_1b_2a_2 \\ &= \angle a_2b_2d + \angle a_2'b_2a_2 + \angle b_1a_1b_2\end{aligned}$$

$$= \theta + \theta' + \alpha.$$

Therefore,

$$\theta = \frac{\pi}{3} - \frac{1}{2}\alpha - \theta'.$$

Since $\angle a_2 b_2 c_2 = 3/\pi$, we have

$$\angle c_2 b_2 d = \frac{\pi}{3} - \theta = \frac{1}{2}\alpha + \theta'.$$

Since $\|c_1 - b_2\| \leq \|c_2 - b_2\|$ and $r_2 \leq r_1$, $|L_{a_2 b_2} \setminus L_{a_1 b_1}|$ is not less than the area of the sector $\measuredangle c_2 b_2 d$. Therefore,

$$|L_{a_2 b_2} \setminus L_{a_1 b_1}| \geq |\measuredangle c_2 b_2 d| = \frac{1}{2} r_2^2 \left(\frac{1}{2}\alpha + \theta' \right). \quad (8)$$

In the next, we are going to show that without loss of generality, we may assume b_2 is on the boundary $L_{a_1 b_1}$. This is can be verified by shifting argument as follows:

If b_2 is not on the boundary of $L_{a_1 b_1}$, we may shift $L_{a_2 b_2}$ along the line $a_2 b_2$ but don't let a_2 and b_2 cross the boundary of $L_{a_1 b_1}$. During shifting, $|H \setminus L_{a_1 b_1}|$ has the same value. So we only need to find out the maximum of $\|z_1 - z_2\|$. The maximum of $\|z_1 - z_2\|$ occurs either as a_2 is shifted to x or as b_2 is shifted to y. We claim that the maximum of $\|z_1 - z_2\|$ occurs as b_2 is at y, i.e. b_2 is on $\partial L_{a_1 b_1}$. Let p be the perpendicular point from z_1 to the line $a_2 b_2$. Since $0 \leq \theta' \leq \pi/2$, we have $\|p - x\| \geq \|p - y\|$. Besides, if b_2 is shifted to y, $\|z_2 - a_2\| \geq \|p - a_2\|$. Since $\|z_1 - z_2\|^2 = \|z_1 - p\|^2 + \|z_2 - p\|^2$ and $\|z_1 - p\|$ is constant during shifting, $\|z_1 - z_2\|$ is maximal if and only if $\|z_2 - p\|$ is maximal. Let a', z', and b' respectively denote the location of a_2, z_2, and b_2 as b_2 is at y; and a'', z'', and b'' respectively denote the location of a_2, z_2, and b_2 as a_2 is at x. According to the position of y, p, and z'', there are six variations. Let $[u, v, w]$ denote the relative position of y, p, and z'' if we record them in the direction from x to y.

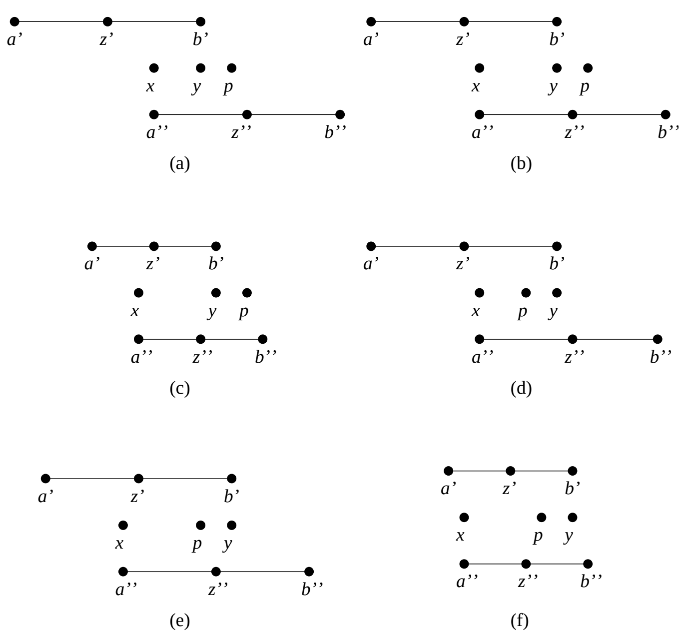

Figure 10: Shift a lune along its waist.

(i) $[y, p, z'']$: (See Fig. 10(a).) Then,

$$\|z' - p\| = \|z' - b'\| + \|b' - p\|,$$
$$\|z'' - p\| = \|z'' - a''\| - \|a'' - p\|.$$

Since $\|z' - b'\| = \|z'' - a''\|$, we have $\|z' - p\| \geq \|z'' - p\|$.

(ii)$[y, z'', p]$: (See Fig. 10(b).) Then,

$$\|z' - p\| = \|z' - b'\| + \|b' - p\|,$$
$$\|z'' - p\| = \|p - y\| - \|y - z''\|.$$

Since $\|b' - p\| = \|p - y\|$, we have $\|z' - p\| \geq \|z'' - p\|$.

(iii) $[z'', y, p]$: (See Fig. 10(c).) Then,

$$\begin{aligned} \|z' - p\| &= \|z' - b'\| + \|b' - p\| \\ &= \|z' - y\| + \|y - p\|, \\ \|z'' - p\| &= \|z'' - y\| + \|y - p\|. \end{aligned}$$

Since $\|z' - y\| \geq \|z'' - y\|$, we have $\|z' - p\| \geq \|z'' - p\|$.

(iv) $[p, y, z'']$: (See Fig. 10(d).) Then,

$$\begin{aligned} \|z' - p\| &= \|z' - b'\| - \|b' - p\| \\ &= \|z' - b'\| - \|y - p\|, \\ \|z'' - p\| &= \|z'' - a''\| - \|a'' - p\| \\ &= \|z'' - a''\| - \|x - p\|. \end{aligned}$$

Since $\|z' - b'\| = \|z'' - a''\|$ and $\|y - p\| \leq \|x - p\|$, we have $\|z' - p\| \geq \|z'' - p\|$.

(v) $[p, z'', y]$: (See Fig. 10(e).) Since the same equations used in (iv) still works, we have $\|z' - p\| \geq \|z'' - p\|$.

(vi) $[z'', p, y]$: (See Fig. 10(f).) Then,

$$\begin{aligned} \|z' - p\| &= \|z' - b'\| - \|b' - p\| \\ &= \|z' - y\| - \|y - p\|, \\ \|z'' - p\| &= \|z'' - y\| - \|y - p\|. \end{aligned}$$

Since $\|z' - y\| \geq \|z'' - y\|$, we have $\|z' - p\| \geq \|z'' - p\|$.

Therefore, we may assume b_2 is on the boundary $L_{a_1 b_1}$. Now, we consider the distance between e and z_2'. Let t denote the distance between b_2 and the line $a_1 b_1$, and β denote the angle of $\angle a_1 b_1 b_2$. Here $\frac{\pi}{3} \leq \beta \leq \frac{\pi}{2}$. If $r_2 = r_1$, we have

$$\|z_2' - e\| = \|z_2'' - e\|, \text{ and}$$
$$\|z_2'' - e\| = \|h - b_1\| = t \cot \beta \leq \frac{1}{\sqrt{3}} t.$$

If $r_1 = \|c_1 - b_2\|$, let z_2''' denote the position of z_2'. Since z_2''' and z_2'' are to the different side of e,

$$\begin{aligned} \|z_2''' - e\| &= \|z_2'' - z_2'''\| - \|z_2'' - e\| \\ &= \frac{1}{2} \left(\|a_2' - b_2\| - \|c_1 - b_2\| \right) - \|z_2'' - e\| \\ &= \frac{1}{2} \left(\|c_1 - b_1\| - \|c_1 - b_2\| \right) - \|h - b_1\| \\ &\leq \frac{1}{2} \|b_2 - b_1\| - \|h - b_1\| \\ &= \frac{1}{2} t \csc \beta - t \cot \beta \leq \frac{1}{2} t \csc \beta \leq \frac{1}{\sqrt{3}} t. \end{aligned}$$

Since r_1 is the largest value for r_2 and $\|c_1 - b_2\|$ is the smallest possible value for r_1, we have

$$\|z_2' - e\| \leq \max \left(\|z_2'' - e\|, \|z_2''' - e\| \right) \leq \frac{1}{\sqrt{3}} t. \quad (9)$$

Thus,

$$\begin{aligned}\|z_1 - z_2\| &\le \|z_1 - e\| + \|e - z_2'\| + \|z_2' - z_2\| \\ &\le \frac{1+\sqrt{3}}{\sqrt{3}} t + r_2\theta'.\end{aligned}$$

From Eq. (8) and (9), we have

$$\begin{aligned}&|L_{a_2b_2} \setminus L_{a_1b_1}| \ge \frac{1}{2} r_2^2 \left(\frac{1}{2}\alpha + \theta'\right) \\ &\ge \frac{1}{4} r_2^2 \sin\alpha + \frac{1}{2} r_2^2 \theta' = \frac{1}{4} r_2^2 \left(\frac{t}{r_1}\right) + \frac{1}{2} r_2^2 \theta' \\ &= \frac{1}{4}\left(\frac{r_2}{r_1}\right) r_2 t + \frac{1}{2} r_2^2 \theta' \ge \frac{1}{16} R t + \frac{1}{4} R \left(r_2 \theta'\right) \\ &\ge \frac{\sqrt{3}}{16\left(1+\sqrt{3}\right)} R \left(\frac{1+\sqrt{3}}{\sqrt{3}} t\right) + \frac{1}{4} R \left(r_2\theta'\right) \\ &\ge \frac{\sqrt{3}}{16\left(1+\sqrt{3}\right)} R \left(\frac{1+\sqrt{3}}{\sqrt{3}} t + r_2\theta'\right) \\ &\ge \frac{\sqrt{3}}{16\left(1+\sqrt{3}\right)} R \|z_1 - z_2\| \\ &\approx 0.039 R \|z_1 - z_2\|.\end{aligned}$$

Thus, the proof is complete. $\square$