Stability Analyses of Longest-Queue-First Link Scheduling in MC-MR Wireless Networks

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ABSTRACT

Longest-queue-first (**LQF**) link scheduling is a greedy link scheduling in multihop wireless networks. Its stability performance in single-channel single-radio (SC-SR) wireless networks has been well studied recently. However, its stability performance in multi-channel multi-radio (MC-MR) wireless networks is largely under-explored. In this paper, we present a stability subregion with closed form of the **LQF** scheduling in MC-MR wireless networks, which is within a constant factor of the network stability region. We also obtain constant lower bounds on the efficiency ratio of the **LQF** scheduling in MC-MR wireless networks under the 802.11 interference model or the protocol interference model.

Categories and Subject Descriptors

C.2.1 [**Computer-Communication Networks**]: Network Architecture and Design—*wireless communication*

Keywords

Stability, multi-channel multi-radio, link scheduling

1. INTRODUCTION

With the rapid technology advances, many off-the-shelf wireless transceivers (i.e., radios) are capable of operating on multiple channels. For example, the IEEE 802.11 b/g standard and IEEE 802.11a standard provide 3 and 12 channels respectively, and MICA2 sensor motes support more than 50 channels. The rapidly diminishing prices of the radios has also made it feasible to equip a wireless node with multiple radios. Providing each node with one or more multi-channel radios offers a promising avenue for enhancing the network capacity by simultaneously exploiting multiple non-overlapping channels through different radio interfaces and mitigating interferences through proper channel assignment. In this paper, we take a queuing-theoretic study of a well-known greedy link scheduling, called *Longest-Queue-First* (**LQF**) link scheduling, in multi-channel multi-radio (MC-MR) wireless networks under the 802.11 interference model or the protocol interference model.

We assume that time is slotted. For each $t \in \mathbb{N}$, the t-th time slot is the time interval $(t-1, t]$. Any packet arriving in a slot is assumed to arrive at the end of the slot, and may only be transmitted in the subsequent slots. In addition, the packet arrivals are assumed to be mutually independent and temporally i.i.d. processes with arrival rate vector α. In each time-slot, the **LQF** scheduling first sorts the communication links in the decreasing order of their queue lengths (ties can be broken arbitrarily) and then schedule their transmissions along this order in the following greedy manner: each link transmits as many packets as possible from its queue using the radios at its two endpoints which have not been used by any preceding links and the channels which have not been used by any preceding *conflicting* links. Let $X(t)$ (respectively, $Y(t)$) denote the vector of cumulative number of packets arriving (respectively, transmitted) in the first t time slots, and $Z(t)$ denote the vector of number of packets queued at the very end of time slot t. Then,

$$Z(t) = Z(0) + X(t) - Y(t).$$

The network is said to be *stable* if the Markov chain $(Z(t))$ is positive recurrent. The *stability region* of the **LQF** scheduling, denoted by Λ, is the set of arrival rate vectors α such that the network is stable. Let P be the maximum stability region of the network, the set of arrival rate vectors such

MobiHoc'12, June 11–14, 2012, Hilton Head Island, SC, USA.

that there exists a scheduling policy stabilizing the network. The *efficiency ratio* of the **LQF** scheduling is defined to be

$$\sup \{\sigma \in \mathbb{R}_+ : \sigma P \subseteq \Lambda\}.$$

The first main contribution of this paper is a stability subregion of the **LQF** scheduling with closed form. Such stability subregion is shown to be within a constant factor of the strict capacity subregion. In addition, it can be checked in polynomial time whether a given vector of packet arrival rates lies in this stability subregion. This computational tractability is particularly favorable for cross-layer optimization, where one needs to allocate the link rates efficiently while still ensuring the network stability under the **LQF** scheduling. The second main contribution of this paper is the discovery of constant lower bounds on the efficiency ratio of the **LQF** scheduling. Specifically, the efficiency ratio of the **LQF** scheduling is at least 1/8 under the 802.11 interference model with uniform interference radii, at least 1/20 under the 802.11 interference model with arbitrary interference radii, and at least

$$1/\left(2\left(\left\lceil \pi/\arcsin\frac{\varphi-1}{2\varphi}\right\rceil+1\right)\right)$$

under the protocol interference model in which the interference radius of each node is at least φ times its communication radius for some $\varphi > 1$.

The efficiency ratio of the **LQF** scheduling in single-channel single-radio (SC-SR) wireless networks is now well understood. Joo et al. [5, 6] made a remarkable contribution to fully characterizing the throughput efficiency ratio of **LQF**. Built upon the prior works by Dimakis and Walrand [4] which presented sufficient conditions for **LQF** to achieve 100% throughput, they proved that the throughput efficiency ratio of **LQF** is exactly the *local pooling factor* (LPF) of the conflict graph of the communication links. The LPF is a pure graph-theoretic parameter. Thus, the works by Joo et al. [5, 6] built an elegant bridge between a queuing-theoretic parameter and a graph-theoretic parameter. Under the 802.11 interference model with uniform interference radii, the LPF is shown to be at least 1/6 in [6]. Sparked by the works in [6], Leconte el al. [7] and Li el al. [8] presented some properties of LPF. Leconte el al. [7] derived tighter lower bounds on LPF in networks of size at most 28 under the 802.11 interference model with uniform interference radii. Li el al. [8] gave an alternative definition of LPF and also introduced a refined notion of LPF. Recently, Wan et al. [16] proved that the LPF is at least 1/16 under the 802.11 interference model with arbitrary interference radii, and at least

$$1/\left(2\left(\left\lceil \pi/\arcsin\frac{\varphi-1}{2\varphi}\right\rceil-1\right)\right)$$

under the protocol interference model in which the interference radius of the sender of each link is at least φ times the link length for some $\varphi > 1$. However, it remains computationally intractable to decide whether a given vector of packet arrival rates meets the so-called local-pooling condition. The efficiency ratios of other link scheduling algorithms were studied in [2, 10, 13, 18, 19].

In contrast, the stability of the **LQF** scheduling in MC-MR wireless networks has been under-studied. Lin and Rasool [9] derived a lower bound 1/10 on the efficiency ratio of the **LQF** scheduling under the 802.11 interference model with uniform interference/communication radii, which is weaker than the 1/8 lower bound derived in this paper under the same setting. The technical approach in [9] is quite different from the approach followed in this paper. In fact, the lower bound 1/10 can be derived in a simpler manner by using the fact that the **LQF** scheduling is actually a 10-approximation algorithm for **Maximum-Weight Independent Set** under the 802.11 interference model with uniform interference/communication radii. Brzezinski et al. [1] considered the variant of the **LQF** scheduling with (temporarily) static channel assignment and the only interference assumed was the primary interference. Even in such restricted setting, no analytical bounds on the efficiency ratio were provided in [1]. The variant of the **LQF** scheduling with (temporarily) dynamic channel assignment, which is subject of this paper, was left as a subject of future research in [1].

The remainder of this paper is organized as follows. Section 2 introduces some basic results from functional analysis and probability theory. Section 3 defines the stability region of a MC-MR wireless network. Section 4 presents a stability subregion of the **LQF** scheduling. Section 5 derives the lower bounds on the efficiency ratio of the **LQF** scheduling. Finally, we conclude this paper in Section 6.

2. PRELIMINARIES

Let I be an interval in the real line $\mathbb{R}$. A function $f : I \to \mathbb{R}$ is *absolutely continuous* on I if for every $\varepsilon > 0$, there is a $\delta > 0$ such that whenever a finite sequence of pairwise disjoint sub-intervals $[s_k, t_k]$ of I satisfies

$$\sum_k |s_k - t_k| < \delta,$$

then

$$\sum_k |f(s_k) - f(t_k)| < \varepsilon.$$

If f is absolutely continuous, then f has a derivative f' almost everywhere; the points at which f is differential are called the *regular points* of f. The following property of absolutely continuous functions is implicitly used in [3].

Lemma 2.1. *Let f be an absolutely continuous non-negative function on $\mathbb{R}_+$ and κ be a positive constant. Suppose that for every almost every regular point t, $f'(t) \leq -\kappa$ whenever $f(t) > 0$. Then, f is non-increasing, and once it reaches zero it stays zero forever. Moreover, $f(t) = 0$ for all $t \geq f(0)/\kappa$.*

A function f on $\mathbb{R}_+$ is said to be *Lipschitz continuous with Lipschitz constant C* (or simply *C-Lipschitz continuous*) for some constant $C > 0$ if for any $s, t \in \mathbb{R}_+$,

$$|f(s) - f(t)| \leq C |s - t| .$$

Lipschitz continuous functions are absolutely continuous. The following lemma may be hidden in a textbook, and we give a short proof for the sake of completeness.

LEMMA 2.2. *Suppose that $f_1, f_2, \cdots, f_k$ are k C-Lipschitz continuous functions on $\mathbb{R}_+$. Then, $\max_{1 \leq i \leq t} f_k(t)$ is also C-Lipschitz continuous.*

PROOF. For any $t \in \mathbb{R}_+$, let

$$g(t) = \max_{1 \leq i \leq t} f_i(t)$$

and

$$K(t) = \{1 \leq j \leq k : f_j(t) = g(t)\} .$$

Consider any distinct $s > t \geq 0$. If there is any $i \in K(s) \cap K(t)$, then

$$|g(s) - g(t)| = |f_i(s) - f_i(t)| \leq C(s - t) .$$

Now suppose that $K(s) \cap K(t) = \emptyset$. Choose $i \in K(s)$ and $j \in K(t)$. Then,

$$f_i(s) > f_j(s), f_j(t) > f_i(t) .$$

Then, there exists $r \in (t, s)$ such that $f_i(r) = f_j(r)$. So,

$$\begin{aligned}
&|g(s) - g(t)| \\
&= |f_i(s) - f_j(t)| \\
&\leq |f_i(s) - f_i(r)| + |f_j(r) - f_j(t)| \\
&\leq C(s - r) + C(r - t) \\
&= C(s - t) .
\end{aligned}$$

Therefore, g is also C-Lipschitz continuous. □

A function f which takes values in k-dimensional Euclidean space is said to be absolutely (respectively, Lipschitz) continuous if each of its component is absolutely (respectively, Lipschitz) continuous. For any vector x in an Euclidean Space, $\|x\|_\infty$ and $\|x\|_1$ denote the maximum norm (also called uniform norm) and the Manhattan norm of x respectively.

Let (f_n) be a sequence of functions on $\mathbb{R}_+$ and let f be a continuous function on $\mathbb{R}_+$. We say that $f_n \to f$ *uniformly on compact sets*, or simply $f_n \to f$ u.o.c., if for each $t > 0$,

$$\sup_{0 \leq s \leq t} |f_n(s) - f(s)| \to 0.$$

The following lemma was stated in Lemma 4.1 of [3].

LEMMA 2.3. *Let (f_n) be a sequence of nondecreasing real-valued functions on $\mathbb{R}_+$, and f be a continuous function on $\mathbb{R}_+$. Assume that (f_n) converges pointwise to f. Then the convergence is u.o.c.*

The following theorem on the convergence of random variables is stated in Theorem 2.2.3 of [12].

THEOREM 2.4. *Suppose that a sequence of random variables (ξ_n) converge to a random variable ξ in probability.*

1. *If ξ_n is uniformly integrable, then $\mathbf{E}[|\xi|] < \infty$ and $\lim_{n\to\infty} \mathbf{E}[\xi_n] = \mathbf{E}[\xi]$.*
2. *If $\xi_n \geq 0$, $\mathbf{E}[\xi] < \infty$, and $\lim_{n\to\infty} \mathbf{E}[\xi_n] = \mathbf{E}[\xi]$, then ξ_n is uniformly integrable.*

3. NETWORK STABILITY REGION

Consider an instance of MC-MR multihop wireless network with a set V of networking nodes and a set A of node-level communication links. Each node v has $\tau(v)$ radios, and there are λ non-overlapping channels. In the fine-grained network representation [15] of the MC-MR wireless network, each communication link is encoded by an ordered *quintuple* specifying the transmitting node, the receiver node, the radio at the transmitting node, the radio at the receiving node, and the channel. Specifically, for each node-level link (u, v) in A, we make $\lambda \cdot \tau_u \cdot \tau_v$ replications (u, v, i, j, k) for $1 \leq i \leq \tau_u$, $1 \leq j \leq \tau_v$, and $1 \leq k \leq \lambda$. A replication (u, v, i, j, k) always utilizes the i-th radio at u and the j-th radio at v over the k-th channel. For each subset B of A, we use $B^{\tau,\lambda}$ to denote the set of all replications of the links in B. In particular, $A^{\tau,\lambda}$ is the set of all replicated links of the links in A. A subset I of $A^{\tau,\lambda}$ can transmit at the same time if and only if (1) all replication links in I are radio-disjoint, in other words, no pair share a common radio, and (2) for each channel k, all the replication links in I transmitting over channel k are conflict-free. Let $\mathcal{I}^{\tau,\lambda}$ denote the collection of the subsets of $A^{\tau,\lambda}$ which can transmit successfully at the same time. For each $I \in \mathcal{I}^{\tau,\lambda}$, its *service rate* is the vector $d \in \mathbb{R}_+^A$ given by

$$d_a = \left| I_j \cap \{a\}^{\tau,\lambda} \right|$$

for each $a \in A$.

A set

$$\Pi = \left\{ (I_j, \ell_j) \in \mathcal{I}^{\tau,\lambda} \times \mathbb{R}_+ : 1 \leq j \leq m \right\}$$

is called a *(fractional) link schedule* of some $d \in \mathbb{R}_+^A$ if

$$d_a = \sum_{j=1}^{m} \ell_j \left| I_j \cap \{a\}^{\tau,\lambda} \right|$$

for each $a \in A$. The two values m and $\sum_{j=1}^{m} \ell_j$ are referred to as the *size* and *length* (or *latency*) of Π respectively. For any $d \in \mathbb{R}_+^A$, the *minimum latency* $\chi^*(d)$ of d is defined as the minimum length of all fractional link schedules of d. The stability region of the MC-MR wireless network is

$$P = \left\{ d \in \mathbb{R}_+^A : \chi^*(d) < 1 \right\} .$$

For any subset S of $A^{\tau,\lambda}$, a subset I of S is said to be a *maximal independent* of S if $I \in \mathcal{I}^{\tau,\lambda}$ and for any link $e \in S \setminus I$, $I \cup \{e\} \notin \mathcal{I}^{\tau,\lambda}$. For any $B \subseteq A$, a set $I \in \mathcal{I}^{\tau,\lambda}$ is said to be *B-maximal* if $I \cap B^{\tau,\lambda}$ is a maximal independent set of $B^{\tau,\lambda}$. We use $\mathcal{M}_B^{\tau,\lambda}$ to denote the collection of B-maximal independent sets of $B^{\tau,\lambda}$, M_B to denote the set of service rates of the sets in $\mathcal{M}_B^{\tau,\lambda}$, and Φ_B to denote the convex hull of M_B.

4. STABILITY SUBREGION

Consider an instance of MC-MR wireless network specified in Section 3. Two links in A are said to have a *conflict* if they cannot transmit at the same time over the same channel. Furthermore, a conflicting pair of distinct links in A are said to have *primary* conflict if there share one common end, and *secondary* conflict otherwise. For the sake of convenience, each link is said to have a *self-conflict* with itself. The *concise conflict graph* [20] of the MC-MR wireless network is the edge-weighted graph G on A in which there is an edge between each conflicting pair of links (a, b) whose weight denoted by $c(a, b)$, is defined as follows:

- If $b = a$ (i.e., self-conflict), then
$$c_{a,b} = 1 - \left(1 - \frac{1}{\tau_u}\right)\left(1 - \frac{1}{\tau_v}\right)\left(1 - \frac{1}{\lambda}\right)$$
where u and v are the two endpoints of a.
- If a and b have a common endpoint u (i.e., a and b have a primary conflict), then
$$c_{a,b} = 1 - \left(1 - \frac{1}{\tau_u}\right)\left(1 - \frac{1}{\lambda}\right).$$
- If a and b have the secondary conflict, then
$$c_{a,b} = \frac{1}{\lambda}.$$

Note that $c(a, b) = c(b, a)$. Let $\mathcal{I}$ denote the collection of the independent sets in G. In other words, $\mathcal{I}$ is the collection of the subsets of A which can transmit successfully at the same time over the same channel. Note that G can be regarded as a generalization of the conventional conflict graph of the underlying SC-SR wireless network by adding a self-loop at each link and assigning each edge a weight specified by the function c. Thus, $\mathcal{I}$ is essentially the collection of the independent sets of links in the underlying SC-SR wireless network.

For any link $a \in A$, $N_G(a)$ denotes the set of neighbors of a in G. Since G has a self-loop at each vertex, a is a neighbor to itself, and hence $a \in N_G(a)$. Thus, $N_G(a)$ consists of all links in A (including itself) having conflict with a. For any link a, any subset B of links, and any $d \in \mathbb{R}_+^A$, define
$$\Gamma(B, a; d) = \sum_{b \in N_G(a) \cap B} c_{a,b} d_b.$$
By Lemma 2.3 in [20], for any $d \in M_B$ and any $a \in B$,
$$\Gamma(B, a; d) \geq 1.$$
Thus, for any $d \in \Phi_B$ and any $a \in B$,
$$\Gamma(B, a; d) \geq 1$$
as well.

Consider a link ordering $\prec$ of A. For any link $a \in A$, $N_G^{\prec}(a)$ denotes the set of neighbors of a in G preceding a in the ordering $\prec$ plus a itself. For any $d \in \mathbb{R}_+^A$, the value
$$\max_{a \in A} \Gamma\left(N_G^{\prec}(a), a; d\right)$$
is referred to as *d-weighted inductivity* of $\prec$ and is denoted by $\Delta^{\prec}(d)$. The smallest d-weighted inductivity of all possible link orderings, denoted by $\Delta^*(d)$, is called the *d-weighted inductivity* of the network. It was shown in [20] that
$$\Delta^*(d) = \max_{\emptyset \neq B \subseteq A} \min_{b \in B} \Gamma(B, b; d).$$
and $\Delta^*(d)$ is achieved by a special ordering, called *smallest-last ordering*, which is produced successively as follows: Initialize B to A. For $i = |A|$ down to 1, let a_i be a link minimizing $\Gamma(B, b; d)$ among all links b in B, and delete a_i from B. Then the ordering $\langle a_1, a_2, \cdots, a_{|A|}\rangle$ is a smallest-last ordering.

Let
$$Q^* = \left\{d \in \mathbb{R}_+^A : \Delta^*(d) < 1\right\}.$$
The following theorem shows that Q^* is a stability subregion of **LQF** scheduling.

THEOREM 4.1. $Q^* \subseteq \Lambda$.

We shall prove Theorem 4.1 by applying the **Malyshev-Menshikov Criterion** [11] for ergodicity of discrete-time countable-state Markov chains. For any $n \in \mathbb{N}$, we denote by $Z^{(n)}(t)$ (respectively, $X^{(n)}(t)$, $Y^{(n)}(t)$) the vector of queue length (cumulative number of arriving packets, cumulative number of transmitted packets) in a system at the end of time-slot t with its initial total queue length $\left\|Z^{(n)}(0)\right\|_1 = n$. Let
$$T = \left\lceil \frac{1 - \left(1 - \frac{1}{\|\tau\|_\infty}\right)^2 \left(1 - \frac{1}{\lambda}\right)}{1 - \Delta^*(\alpha)} \right\rceil.$$
By the **Malyshev-Menshikov Criterion** [11], Theorem 4.1 follows immediately from the theorem below.

THEOREM 4.2. *For any* $\alpha \in Q^*$,
$$\lim_{n \to \infty} \mathbf{E}\left[\left\|\frac{Z^{(n)}(nT)}{n}\right\|_1\right] = 0.$$

The proof of Theorem 4.2 utilizes Theorem 2.4. By the strong law of large numbers,
$$\left\|\frac{X^{(n)}(nT)}{n}\right\|_1 = T\left\|\frac{X^{(n)}(nT)}{nT}\right\|_1 \to T\|\alpha\|_1$$
almost surely, and
$$\mathbf{E}\left[\left\|\frac{X^{(n)}(nT)}{n}\right\|_1\right] = T \cdot \mathbf{E}\left[\left\|\frac{X^{(n)}(nT)}{nT}\right\|_1\right] = T\|\alpha\|_1.$$
By Theorem 2.4, the sequence $\left(\left\|\frac{X^{(n)}(nT)}{n}\right\|_1\right)$ is uniformly integrable. Since
$$\left\|\frac{Z^{(n)}(nT)}{n}\right\|_1 \leq \left\|\frac{X^{(n)}(nT)}{n}\right\|_1,$$

the sequence $\left(\left\|\frac{Z^{(n)}(nT)}{n}\right\|_1\right)$ is also uniformly integrable. Again, by Theorem 2.4, Theorem 4.2 would hold if

$$\left\|\frac{Z^{(n)}(nT)}{n}\right\|_1 \to 0$$

in probability. We will actually prove a stronger result that

$$\left\|\frac{Z^{(n)}(nT)}{n}\right\|_1 \to 0$$

almost surely. Consider a sample path (i.e. realization) ω of

$$\left(Z^{(n)}(0) : n \in \mathbb{N}\right) \cup \left(X^{(n)}(t) : n, t \in \mathbb{N}\right).$$

It is is said to be *well-behaved* if

$$\lim_{t\to\infty} \frac{X^{(n)}(t,\omega)}{t} = \alpha.$$

By the strong law of large numbers, every sample path is almost surely well-behaved. We will prove that for any well-behaved sample path ω,

$$\left\|\frac{Z^{(n)}(nT,\omega)}{n}\right\|_1 \to 0,$$

from which we can conclude that

$$\left\|\frac{Z^{(n)}(nT)}{n}\right\|_1 \to 0$$

almost surely, and hence Theorem 4.2 holds.

Fix a well-behaved sample path ω. Denote $X^{(n)}(t,\omega)$ (respectively, $Y^{(n)}(t,\omega)$, $Z^{(n)}(t,\omega)$) by $x^{(n)}(t)$ (respectively, $y^{(n)}(t)$, $z^{(n)}(t)$). Then, all of them are deterministic. In order to show that

$$\left\|\frac{z^{(n)}(nT)}{n}\right\|_1 \to 0,$$

it is sufficient to show that for any infinite increasing sequence S of positive integers, there is an infinite subsequence S' of S along which

$$\left\|\frac{z^{(n)}(nT)}{n}\right\|_1 \to 0.$$

So, we further fix an infinite increasing sequence S of positive integers. For convenience, we define $x^{(n)}(0)$ and $y^{(n)}(0)$ to be the vector of zeros. We extend $x^{(n)}(t)$ (respectively, $y^{(n)}(t)$, $z^{(n)}(t)$) to all non-negative real numbers by *linear* interpolation. Then, for any $t \geq 0$ and any $n \in \mathbb{N}$,

$$\frac{z^{(n)}(nt)}{n} = \frac{z^{(n)}(0)}{n} + \frac{x^{(n)}(nt)}{n} - \frac{y^{(n)}(nt)}{n}.$$

LEMMA 4.3. *For any $t \geq 0$, $\lim_n \frac{x^{(n)}(nt)}{n} = \alpha t$.*

PROOF. The lemma holds trivially when $t = 0$. Since ω is well-behaved for any positive integer t,

$$\lim_n \frac{x^{(n)}(nt)}{n} = t \lim_n \frac{x^{(n)}(nt)}{nt} = \alpha t.$$

Now consider any non-integer $t > 0$. Since

$$\frac{x^{(n)}(\lfloor nt\rfloor)}{\lfloor nt\rfloor}\frac{\lfloor nt\rfloor}{nt} \leq \frac{x^{(n)}(nt)}{nt} \leq \frac{x^{(n)}(\lceil nt\rceil)}{\lceil nt\rceil}\frac{\lceil nt\rceil}{nt}$$

we have

$$\lim_n \frac{x^{(n)}(nt)}{nt} = \alpha.$$

Thus,

$$\lim_n \frac{x^{(n)}(nt)}{n} = \alpha t.$$

So, the lemma holds. □

For each $t \in \mathbb{N}$, let $I^{(n)}(t) \in \mathcal{I}^{\tau,\lambda}$ be the set of replicated links which are scheduled to transmit in the t-th time slot, and $d^{(n)}(t)$ be the service rate of $I^{(n)}(t)$. The average service rate in a time interval $[t_1, t_2]$ is defined to be

$$\overline{y}^{(n)}(t_1,t_2) = \frac{y^{(n)}(t_2) - y^{(n)}(t_1)}{t_2 - t_1}.$$

It has the following properties.

LEMMA 4.4. *Consider any $0 \leq t_1 < t_2$.*

1. *For any $s = \varepsilon t_1 + (1-\varepsilon) t_2$ for some $\varepsilon \in [0,1]$,*
$$\overline{y}^{(n)}(t_1,t_2) = (1-\varepsilon)\,\overline{y}^{(n)}(t_1,s) + \varepsilon\overline{y}^{(n)}(s,t_2),$$
2. *$\overline{y}^{(n)}(t_1,t_2)$ is a convex combination of*
$$\left\{d^{(n)}(t) : \lfloor t_1\rfloor + 1 \leq t \leq \lceil t_2\rceil, t \in \mathbb{N}\right\}.$$
3. $\left\|\overline{y}^{(n)}(t_1,t_2)\right\|_\infty \leq \|\tau\|_\infty$.

PROOF. (1). The first part of the lemma holds trivially if $\varepsilon = 0$ or 1. So we assume that $\varepsilon \in (0,1)$. Then,

$$\begin{aligned}
&\overline{y}^{(n)}(t_1,t_2)\\
&= \frac{y^{(n)}(t_2) - y^{(n)}(t_1)}{t_2 - t_1}\\
&= \frac{y^{(n)}(t_2) - y^{(n)}(s)}{t_2 - t_1} + \frac{y^{(n)}(s) - y^{(n)}(t_1)}{t_2 - t_1}\\
&= \frac{t_2 - s}{t_2 - t_1}\frac{y^{(n)}(t_2) - y^{(n)}(s)}{t_2 - s} + \frac{s - t_1}{t_2 - t_1}\frac{y^{(n)}(s) - y^{(n)}(t_1)}{s - t_1}\\
&= \varepsilon\overline{y}^{(n)}(s,t_2) + (1-\varepsilon)\,\overline{y}^{(n)}(t_1,s),
\end{aligned}$$

and hence the first part of the lemma holds as well.

(2). If $\lfloor t_1\rfloor + 1 = \lceil t_2\rceil$, then

$$\overline{y}^{(n)}(t_1,t_2) = d^{(n)}(\lceil t_2\rceil)$$

and hence the second part of the lemma holds trivially. So, we assume that Note that $\lfloor t_1\rfloor + 1 < \lceil t_2\rceil$. Note that for any $t \in \mathbb{R}_+$,

$$\begin{aligned}
\overline{y}^{(n)}(\lceil t\rceil - 1, t) &= d^{(n)}(\lceil t\rceil),\\
y^{(n)}(t, \lfloor t\rfloor + 1) &= d^{(n)}(\lfloor t\rfloor + 1).
\end{aligned}$$

By the first part of the lemma, $\overline{y}^{(n)}(t_1,t_2)$ is a convex combination of

$$\begin{aligned}
&y^{(n)}(t_1, \lfloor t_1\rfloor + 1);\\
&y^{(n)}(s, s+1), s \in \mathbb{N} \text{ and } \lfloor t_1\rfloor + 1 \leq s \leq \lceil t_2\rceil - 1;\\
&y^{(n)}(\lceil t_2\rceil - 1, t_2).
\end{aligned}$$

Thus, the second part of the lemma holds.

(3). The third part of the lemma follows from the second part of the lemma and the fact that

$$\left\| d^{(n)}(t) \right\|_\infty \leq \|\tau\|_\infty$$

for any $t \in \mathbb{N}$. $\square$

Note that for any $0 \leq t_1 < t_2$,

$$\begin{aligned} &\frac{\frac{y^{(n)}(nt_2)}{n} - \frac{y^{(n)}(nt_1)}{n}}{t_2 - t_1} \\ &= \frac{y^{(n)}(nt_2) - y^{(n)}(nt_1)}{nt_2 - nt_1} \\ &= \overline{y}^{(n)}(nt_1, nt_2). \end{aligned}$$

By the third part of Lemma 4.4, $\frac{y^{(n)}(nt)}{n}$ is $\|\tau\|_\infty$-Lipschitz continuous, and hence is equicontinuous. By the Arzela-Ascoli theorem, there is an infinite subsequence S_1 of S along which $\frac{y^{(n)}(nt)}{n}$ converges to some function $\beta(t)$. In addition, $\beta(t)$ is also $\|\tau\|_\infty$-Lipschitz continuous. Since

$$\left\| \frac{z^{(n)}(0)}{n} \right\|_1 = 1$$

for any $n \in \mathbb{N}$, there is an infinite subsequence S' of S_1 along which $\frac{z^{(n)}(0)}{n}$ converges. Therefore, along the sequence S', $\frac{z^{(n)}(0)}{n}$, $\frac{x^{(n)}(nt)}{n}$ and $\frac{y^{(n)}(nt)}{n}$ all converge. Since both $\frac{x^{(n)}(nt)}{n}$ and $\frac{y^{(n)}(nt)}{n}$ are increasing function of t for each n, they converge u.o.c along S' to αt and $\beta(t)$ respectively by Lemma 2.3. As

$$\frac{z^{(n)}(nt)}{n} = \frac{z^{(n)}(0)}{n} + \frac{x^{(n)}(nt)}{n} - \frac{y^{(n)}(nt)}{n},$$

$\frac{z^{(n)}(nt)}{n}$ also converges u.o.c. along S' to some function $\gamma(t)$. Since

$$\left\| \frac{z^{(n)}(0)}{n} \right\|_1 = 1,$$

$\|\gamma(0)\|_1 = 1$. In addition,

$$\gamma(t) = \gamma(0) + \alpha t - \beta(t).$$

Clearly, $\gamma(t)$ is also Lipschitz continuous with Lipschitz constant $\|\alpha\|_\infty + \|\tau\|_\infty$, and so is $\|\gamma(t)\|_\infty$ by Lemma 2.2.

A time $t \in \mathbb{R}_+$ is said to be a *regular point* if each component of $\gamma(t)$ and $\|\gamma(t)\|_\infty$ are differentiable at t. Since both $\gamma(t)$ and $\|\gamma(t)\|_\infty$ are Lipschitz continuous, almost every time $t \in \mathbb{R}_+$ is a regular point. Since

$$\beta(t) = \gamma(t) - \gamma(0) - \alpha t$$

$\beta(t)$ is also differentiable at any regular point t, and

$$\gamma'(t) = \alpha - \beta'(t).$$

In the next, we derive the properties of $\|\gamma(t)\|'_\infty$ and $\beta'(t)$ at any regular point $t > 0$ with $\|\gamma(t)\|_\infty > 0$. For any regular point t, denote

$$A_0(t) = \left\{ a \in A \ : \gamma_a(t) = \max_{b \in A} \ \gamma_b(t) \right\},$$
$$A_1(t) = \left\{ a \in A_0(t) : \gamma'_a(t) = \max_{b \in A_0(t)} \ \gamma'_b(t) \right\}.$$

Then, we have the following lemma.

Lemma 4.5. *Consider any regular point $t > 0$ with $\|\gamma(t)\|_\infty > 0$.*

1. *For any $a \in A_1(t)$, $\gamma'_a(t) = \|\gamma(t)\|'_\infty$.*
2. *$\beta'(t) \in \Phi_{A_1(t)}$.*

The proof of Lemma 4.5 is quite involved, and so is relegated to Appendix. We apply Lemma 4.5 to show that for any regular point $t > 0$ with $\|\gamma(t)\|_\infty > 0$,

$$\|\gamma(t)\|'_\infty \leq \frac{\Delta^*(\alpha) - 1}{1 - \left(1 - \frac{1}{\|\tau\|_\infty}\right)^2 \left(1 - \frac{1}{\lambda}\right)}.$$

Let a be the link in $A_1(t)$ with minimum $\Gamma(A_1(t), a; \alpha)$. Then,

$$\Gamma(A_1(t), a; \alpha) \leq \Delta^*(\alpha).$$

By the second part of Lemma 4.5, for each link $a \in A_1(t)$,

$$\Gamma\left(A_1(t), a; \beta'(t)\right) \geq 1$$

Thus,

$$\begin{aligned} &\Gamma\left(A_1(t), a; \gamma'(t)\right) \\ &= \Gamma\left(A_1(t), a; \alpha - \beta'(t)\right) \\ &= \Gamma(A_1(t), a; \alpha) - \Gamma\left(A_1(t), a; \beta'(t)\right) \\ &\leq \Delta^*(\alpha) - 1. \end{aligned}$$

On the other hand, by the first part of Lemma 4.5,

$$\begin{aligned} &\Gamma\left(A_1(t), a; \gamma'(t)\right) \\ &= \textstyle\sum_{b \in N_G(a) \cap A_1(t)} c_{a,b} \gamma'_b(t) \\ &= \textstyle\sum_{b \in N_G(a) \cap A_1(t)} c_{a,b} \|\gamma(t)\|'_\infty \\ &= \|\gamma(t)\|'_\infty \textstyle\sum_{b \in N_G(a) \cap A_1(t)} c_{a,b} \\ &\geq \|\gamma(t)\|'_\infty \, c_{a,a} \\ &\geq \|\gamma(t)\|'_\infty \left(1 - \left(1 - \frac{1}{\|\tau\|_\infty}\right)^2 \left(1 - \frac{1}{\lambda}\right)\right). \end{aligned}$$

Therefore,

$$\|\gamma(t)\|'_\infty \leq \frac{\Delta^*(\alpha) - 1}{1 - \left(1 - \frac{1}{\|\tau\|_\infty}\right)^2 \left(1 - \frac{1}{\lambda}\right)}.$$

The property established in the previous paragraph together with Lemma 2.1 yields that $\|\gamma(t)\|_\infty = 0$ for

$$t \geq \frac{\|\gamma(0)\|_\infty}{\frac{1 - \Delta^*(\alpha)}{1 - \left(1 - \frac{1}{\|\tau\|_\infty}\right)^2 \left(1 - \frac{1}{\lambda}\right)}}.$$

Since

$$T=\left\lceil \frac{1-\left(1-\frac{1}{\|\tau\|_\infty}\right)^2\left(1-\frac{1}{\lambda}\right)}{1-\Delta^*(\alpha)} \right\rceil$$
$$\geq \frac{1-\left(1-\frac{1}{\|\tau\|_\infty}\right)^2\left(1-\frac{1}{\lambda}\right)}{1-\Delta^*(\alpha)}$$
$$=\frac{1}{\frac{1-\Delta^*(\alpha)}{1-\left(1-\frac{1}{\|\tau\|_\infty}\right)^2\left(1-\frac{1}{\lambda}\right)}}$$
$$=\frac{\|\gamma(0)\|_1}{\frac{1-\Delta^*(\alpha)}{1-\left(1-\frac{1}{\|\tau\|_\infty}\right)^2\left(1-\frac{1}{\lambda}\right)}}$$
$$\geq \frac{\|\gamma(0)\|_\infty}{\frac{1-\Delta^*(\alpha)}{1-\left(1-\frac{1}{\|\tau\|_\infty}\right)^2\left(1-\frac{1}{\lambda}\right)}},$$

we have $\|\gamma(t)\|_\infty = 0$ whenever $t \geq T$. Consequently, $\|\gamma(t)\|_1 = 0$ whenever $t \geq T$. Therefore,

$$\lim_n \left\| \frac{z^{(n)}(nT)}{n} \right\|_1 = \|\gamma(T)\|_1 = 0.$$

This completes the proof of Theorem 4.2.

5. THE EFFICIENCY RATIO

In this section, we derive the lower bounds on the efficiency ratio of the **LQF** scheduling.

Given a link ordering $\prec$ of A, its *backward local independence number* (BLIN) is defined to be

$$\max_{a\in A} \max\left\{|I| : I \subseteq N_G^{\preceq}(a), I \in \mathcal{I}\right\}.$$

An *orientation* of G is a digraph obtained from G by imposing an orientation on each edge of G. Note that in each orientation D of G also has a self-loop at each vertex, and consequently, $a \in N_D^{in}(a) \cap N_D^{out}(a)$ for each $a \in A$. Given an orientation D of G, its *inward local independence number* (ILIN) is defined to be

$$\max_{a\in A} \max\left\{|I| : I \subseteq N_D^{in}(a), I \in \mathcal{I}\right\}.$$

Since BLIN and ILIN only depend on the topology of G rather than the edge weight function c, the following properties which hold in the convectional conflict graph of the underlying SC-SR wireless network also hold in the G:

- Under the 802.11 interference model with uniform interference radii, the lexicographic ordering of A has BLIN at most 6 [6].
- Under the 802.11 interference model with arbitrary interference radii, there is an orientation D of G with ILIN at most 8 [17].
- Under the protocol interference model in which the interference radius of the sender of each link is at least φ times the link length for some $\varphi > 1$, there is an orientation D of G with ILIN $\left\lceil \pi/\arcsin\frac{\varphi-1}{2\varphi} \right\rceil - 1$ [14].

Theorem 5.1. *The following two statements are true:*

1. *If there is a link ordering of A with BLIN μ, then $Q^* \supseteq \frac{1}{\mu+2}P$.*
2. *If there is an orientation of G with ILIN μ, then $Q^* \supseteq \frac{1}{2(\mu+2)}P$.*

Proof. (1). Consider any $d \in P$. By the first part of Corollary 2.8 in [20],

$$\Delta^*(d) \leq (\mu+2)\chi^*(d) < \mu+2.$$

Thus, $d \in (\mu+2)Q^*$. Hence, $P \subseteq (\mu+2)Q^*$, which implies that $Q^* \supseteq \frac{1}{\mu+2}P$.

(2). Consider any $d \in P$. By the second part of Corollary 2.8 in [20],

$$\Delta^*(d) \leq 2(\mu+2)\chi^*(d) < 2(\mu+2).$$

Thus, $d \in 2(\mu+2)Q^*$. Hence, $P \subseteq 2(\mu+2)Q^*$, which implies that $Q^* \supseteq \frac{1}{2(\mu+2)}P$. □

Theorem 4.1 and Theorem 5.1 immediately implies the following lower bounds on the efficiency ratio of the **LQF** scheduling.

- 1/8 under the 802.11 interference model with uniform interference radii, the efficiency ratio of the **LQF** scheduling is at least .
- 1/20 under the 802.11 interference model with arbitrary interference radii, the efficiency ratio of the **LQF** scheduling is at least .
- $1/\left(2\left(\left\lceil \pi/\arcsin\frac{\varphi-1}{2\varphi}\right\rceil+1\right)\right)$ under the protocol interference model in which the interference radius of the sender of each link is at least φ times the link length for some $\varphi > 1$.

6. DISCUSSIONS

Most of the recent works on the stability of **LQF** scheduling established the stability by using Theorem 4.2 of [3], which states that in the context of multiclass queuing networks the stability of fluid-limit systems imply the stability of the original system under certain conditions. One crucial condition is that the queuing service discipline is *working-conserving* (middle of pp. 65 in [3]): a server is idle only when there is no customer waiting for the service. Apparently, the **LQF** scheduling in wireless networks is not working-conserving, as a link with non-empty queue may be idle due to the interference from other nearby links. So, the direct applicability of Theorem 4.2 of [3] to wireless link scheduling is questionable. Instead, we have applied the **Malyshev-Menshikov Criterion** [11] to establish the stability of **LQF** scheduling. An attractive feature of this technical approach is that we push the deterministic (sample-path) arguments as far as possible while trying to avoid the heavy machinery of stochastic processes. The advantages of this approach are clear: the sample-path arguments are simple and intuitive; thus they provide a clear insight into the issues at hand. Sample-path analysis also helps pinpoint what and when stochastic conditions are needed to guarantee the stability. For example, the packet arrival process in this paper is only required to be mutually independent and temporally i.i.d., while the packet arrival processes in [4, 5, 6] have to meet additional conditions.

The local pooling factor (LPF) of a MC-MR wireless network can be defined as follows. For any non-empty subset B of A, let

$$\sigma_B = \min \left\{ c \in \mathbb{R}^+ : \exists x, y \in \Phi_B \text{ s.t. } x \leq cy \right\}.$$

Then, the LPF is the parameter

$$\sigma^* = \min_{\emptyset \neq B \subseteq A} \sigma_B.$$

Using the same (and easier) argument as in Section 4, we can show that $\Lambda \supseteq \sigma^* P$. Thus, the LPF σ^* is also a lower bound on the efficiency ratio. We can also derive the same lower bounds on σ^*. Specifically, σ^* is at least 1/8 under the 802.11 interference model with uniform interference radii, at least 1/20 under the 802.11 interference model with arbitrary interference radii, and at least

$$1/\left(2\left(\left\lceil \pi/\arcsin\frac{\varphi-1}{2\varphi}\right\rceil+1\right)\right)$$

under the protocol interference model in which the interference radius of the sender of each link is at least φ times the link length for some $\varphi > 1$.

ACKNOWLEDGEMENTS: This work is supported in part by the National Science Foundation of USA under grants CNS-0831831 and CNS-0916666, and by the National Natural Science Foundation of P. R. China under grant 61128005. The authors would like to thank Dr. Alexandre Proutiere for shepherding the revision of this paper, and the anonymous reviewers for many helpful suggestions.

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Appendix

In this appendix, we prove Lemma 4.5. Fix a regular point $t > 0$ with $\|\gamma(t)\|_\infty > 0$. We make the convention that the maximum of an empty set is zero. Define $\eta(t)$ as follows: if $A_1(t) = A_0(t)$,

$$\eta(t) = \|\gamma(t)\|_\infty - \max_{a \in A \setminus A_0(t)} \gamma_a(t);$$

otherwise,

$$\eta(t) = \max_{a \in A_0(t)} \gamma'_a(t) - \max_{a \in A_0(t) \setminus A_1(t)} \gamma'_a(t).$$

Then, $\eta(t) > 0$.

Lemma 8.1. *There exists $\delta > 0$ such that for any $s \in [t, t+\delta]$,*

$$\min_{a \in A_1(t)} \gamma_a(s) > \max_{a \in A \setminus A_1(t)} \gamma_a(s) + \frac{s-t}{2}\eta(t).$$

Proof. We consider two cases.

Case 1: $A_1(t) = A_0(t)$. By the continuity of γ, there exists $\delta \in (0,1)$ such that for any $s \in [t, t+\delta]$,

$$\begin{aligned}
&\min_{a \in A_1(t)} \gamma_a(s) \\
&= \min_{a \in A_0(t)} \gamma_a(s) \\
&> \max_{a \in A \setminus A_0(t)} \gamma_a(s) + \frac{1}{2}\eta(t) \\
&= \max_{a \in A \setminus A_1(t)} \gamma_a(s) + \frac{1}{2}\eta(t) \\
&\geq \max_{a \in A \setminus A_1(t)} \gamma_a(s) + \frac{s-t}{2}\eta(t).
\end{aligned}$$

Case 2: $A_1(t) \neq A_0(t)$. There exists $\delta > 0$ such that for any $s \in (t, t+\delta]$,

$$\min_{a \in A_0(t)} \gamma_a(s) > \max_{a \in A \setminus A_0(t)} \gamma_a(s),$$
$$\max_{a \in A} \left| \frac{\gamma_a(s) - \gamma_a(t)}{s-t} - \gamma'_a(t) \right| < \frac{\eta(t)}{4}.$$

Fix an $s \in (t, t+\delta]$. Then,

$$\max_{a \in A \setminus A_1(t)} \gamma_a(s) = \max_{a \in A_0(t) \setminus A_1(t)} \gamma_a(s).$$

For any $a \in A_1(t)$ and any $b \in A_0(t) \setminus A_1(t)$, we have

$$\begin{aligned}
&\frac{\gamma_a(s) - \gamma_b(s)}{s-t} \\
&= \frac{\gamma_a(s) - \gamma_a(t)}{s-t} - \frac{\gamma_b(s) - \gamma_b(t)}{s-t} \\
&\geq \gamma'_a(t) - \gamma'_b(t) - \left| \frac{\gamma_a(s) - \gamma_a(t)}{s-t} - \gamma'_a(t) \right| \\
&\quad - \left| \frac{\gamma_b(s) - \gamma_b(t)}{s-t} - \gamma'_b(t) \right| \\
&> \eta(t) - \frac{\eta(t)}{4} - \frac{\eta(t)}{4} \\
&= \frac{\eta(t)}{2},
\end{aligned}$$

which implies

$$\gamma_a(s) > \gamma_b(s) + \frac{s-t}{2}\eta(t).$$

Hence, for any $s \in (t, t+\delta]$,

$$\begin{aligned}
&\min_{k \in A_1(t)} \gamma_a(s) \\
&> \max_{a \in A_0(t) \setminus A_1(t)} \gamma_a(s) + \frac{s-t}{2}\eta(t) \\
&= \max_{a \in A \setminus A_1(t)} \gamma_a(s) + \frac{s-t}{2}\eta(t).
\end{aligned}$$

Note the above inequality holds trivially when $s = t$. Therefore, the lemma holds. □

Now, we give the proof of the first part of Lemma 4.5 using Lemma 8.1. Consider any $a \in A_1(t)$ and any $\varepsilon > 0$. By Lemma 8.1, there exists $\delta > 0$ such that for any $s \in (t, t+\delta]$,

$$\min_{a \in A_1(t)} \gamma_a(s) > \max_{a \in A \setminus A_1(t)} \gamma_a(s),$$
$$\left\| \frac{\gamma(s) - \gamma(t)}{s-t} - \gamma'(t) \right\|_\infty < \varepsilon.$$

Thus, for any $s \in (t, t+\delta]$,

$$\begin{aligned}
&\frac{\|\gamma(s)\|_\infty - \|\gamma(t)\|_\infty}{s-t} - \gamma'_a(t) \\
&= \frac{\max_{b \in A_1(t)} \gamma_b(s) - \|\gamma(t)\|_\infty}{s-t} - \gamma'_a(t) \\
&= \max_{b \in A_1(t)} \frac{\gamma_b(s) - \|\gamma(t)\|_\infty}{s-t} - \gamma'_a(t) \\
&= \max_{b \in A_1(t)} \left(\frac{\gamma_b(s) - \|\gamma(t)\|_\infty}{s-t} - \gamma'_a(t) \right) \\
&= \max_{b \in A_1(t)} \left(\frac{\gamma_b(s) - \gamma_b(t)}{s-t} - \gamma'_b(t) \right),
\end{aligned}$$

which implies

$$\begin{aligned}
&\left| \frac{\|\gamma(s)\|_\infty - \|\gamma(t)\|_\infty}{s-t} - \gamma'_a(t) \right| \\
&\leq \max_{b \in A_1(t)} \left| \frac{\gamma_b(s) - \gamma_b(t)}{s-t} - \gamma'_b(t) \right| \\
&< \varepsilon.
\end{aligned}$$

Therefore,

$$\|\gamma(t)\|'_\infty = \gamma'_a(t),$$

and the first part of Lemma 4.5 holds.

We move on to prove the second part of Lemma 4.5. Consider any $\varepsilon \in (0,1)$. By Lemma 8.1, there exists $\delta > 0$ such that for any $s \in (t, t+\delta]$,

$$\min_{a\in A_1(t)} \gamma_a(s) > \max_{b\in A\setminus A_1(t)} \gamma_b(s) + \frac{s-t}{2}\eta(t),$$

and

$$\left\| \frac{\beta(s)-\beta(t)}{s-t} - \beta'(t) \right\|_\infty \le \varepsilon.$$

Let $\varepsilon_1 = \varepsilon / \|\tau\|_\infty$. By the u.o.c. convergence, there exists a sufficiently large $n \in S'$ such that

$$n \ge \max\left\{ \frac{2}{\varepsilon_1\delta}, \frac{8}{\varepsilon_1\delta\cdot\eta(t)}\|\tau\|_\infty \right\}$$

and for any $s \in [t,\ t+\delta]$,

$$\left\| \frac{y^{(n)}(ns)}{n} - \beta(s) \right\|_\infty < \frac{\varepsilon}{2}\delta$$

$$\left\| \frac{z^{(n)}(ns)}{n} - \gamma(s) \right\|_\infty < \frac{\varepsilon_1}{16}\delta\eta(t).$$

We make the following two claims.

Claim 8.2. $\left\| \overline{y}^{(n)}(n(t+\varepsilon_1\delta), n(t+\delta)) - \beta'(t) \right\|_\infty \le 3\varepsilon.$

Proof. By the third part of Lemma 4.4,

$$\begin{aligned}&\left\| \overline{y}^{(n)}(n(t+\varepsilon_1\delta), n(t+\delta)) - \overline{y}^{(n)}(nt, n(t+\delta)) \right\|_\infty\\ &\le \varepsilon_1\|\tau\|_\infty\\ &= \varepsilon.\end{aligned}$$

Thus, it is sufficient to show that

$$\left\| \overline{y}^{(n)}(nt, n(t+\delta)) - \beta'(t) \right\|_\infty \le 2\varepsilon,$$

which will be proved subsequently. Since

$$\begin{aligned}&\left\| \overline{y}^{(n)}(nt, n(t+\delta)) - \frac{\beta(t+\delta)-\beta(t)}{\delta} \right\|_\infty\\ =&\left\| \frac{y^{(n)}(n(t+\delta)) - y^{(n)}(nt)}{n\delta} - \frac{\beta(t+\delta)-\beta(t)}{\delta} \right\|_\infty\\ \le&\frac{1}{\delta}\left\| \frac{y^{(n)}(n(t+\delta))}{n} - \beta(t+\delta) \right\|_\infty +\\ &\frac{1}{\delta}\left\| \frac{y^{(n)}(nt)}{n} - \beta(t) \right\|_\infty\\ <&\frac{1}{\delta}\cdot\frac{\varepsilon}{2}\delta + \frac{1}{\delta}\cdot\frac{\varepsilon}{2}\delta\\ =&\varepsilon.\end{aligned}$$

we have

$$\begin{aligned}&\left\| \overline{y}^{(n)}(nt, n(t+\delta)) - \beta'(t) \right\|_\infty\\ \le&\left\| \overline{y}^{(n)}(nt, n(t+\delta)) - \frac{\beta(t+\delta)-\beta(t)}{\delta} \right\|_\infty\\ &+\left\| \frac{\beta(t+\delta)-\beta(t)}{\delta} - \beta'(t) \right\|_\infty\\ <&2\varepsilon.\end{aligned}$$

Therefore, our claim holds. □

Claim 8.3. $\overline{y}^{(n)}(n(t+\varepsilon_1\delta), n(t+\delta)) \in \Phi_{A_1(t)}$.

Proof. By the second part of Lemma 4.4, $\overline{y}^{(n)}(n(t+\varepsilon_1\delta), n(t+\delta))$ is a convex combination of

$$\left\{ d^{(n)}(j) : \lfloor n(t+\varepsilon_1\delta)\rfloor + 1 \le j \le \lceil n(t+\delta)\rceil \right\}.$$

Thus, it is sufficient to show that $d^{(n)}(j) \in M_{A_1(t)}$ for any integer j between $\lfloor n(t+\varepsilon_1\delta)\rfloor + 1$ and $\lceil n(t+\delta)\rceil$.

We first show that for any $s \in [t+\varepsilon_1\delta/2, nt+\delta]$,

$$\min_{a\in A_1(t)} z_a^{(n)}(ns) > \max_{b\in A\setminus A_1(t)} z_b^{(n)}(ns) + \|\tau\|_\infty.$$

Consider any link $a \in A_1(t)$, and any link $b \in A\setminus A_1(t)$. Then,

$$\gamma_a(s) - \gamma_b(s) > \frac{s-t}{2}\eta(t) \ge \frac{\varepsilon_1\delta}{4}\eta(t),$$

and

$$\begin{aligned}&\left| \frac{z_a^{(n)}(ns) - z_b^{(n)}(ns)}{n} - (\gamma_a(s) - \gamma_b(s)) \right|\\ \le&\left| \frac{z_a^{(n)}(ns)}{n} - \gamma_a(s) \right| + \left| \frac{z_b^{n}(ns)}{n} - \gamma_b(s) \right|\\ <&\frac{\varepsilon_1\delta}{8}\eta(t).\end{aligned}$$

Therefore,

$$\begin{aligned}&\frac{z_a^{(n)}(ns) - z_b^{(n)}(ns)}{n}\\ &> \gamma_a(s) - \gamma_b(s) - \frac{\varepsilon_1\delta}{8}\eta(t)\\ &> \frac{\varepsilon_1\delta}{8}\eta(t),\end{aligned}$$

which implies

$$\begin{aligned}z_a^{(n)}(ns) &> z_b^{(n)}(ns) + n\frac{\varepsilon_1\delta}{8}\eta(t)\\ &\ge z_b^{(n)}(ns) + \|\tau\|_\infty.\end{aligned}$$

So, the desired inequality holds.

Consider any integer j such that $j-1 \in [n(t+\varepsilon_1\delta/2), n(t+\delta)]$. Then, at the end of the $(j-1)$-th times-slot, the queue length of each link in $A_1(t)$ is at least $\|\tau\|_\infty$ and is larger than any link not in $A_1(t)$. Thus, $I^{(n)}(j)$ is $A_1(t)$-maximal, which implies that $d^{(n)}(j) \in \Phi_{A_1(t)}$. Since

$$\begin{aligned}&\lfloor n(t+\varepsilon_1\delta)\rfloor - n(t+\varepsilon_1\delta/2)\\ &> n(t+\varepsilon_1\delta) - 1 - n(t+\varepsilon_1\delta/2)\\ &= n\varepsilon_1\delta/2 - 1 \ge 0\end{aligned}$$

and

$$\lceil n(t+\delta)\rceil - 1 < n(t+\delta),$$

for each integer j between $\lfloor n(t+\varepsilon_1\delta)\rfloor + 1$ and $\lceil n(t+\delta)\rceil$, we have $d^{(n)}(j) \in \Phi_{A_1(t)}$. □

Since $\Phi_{A_1(t)}$ is compact, the above two claims together with the fact that ε can be chosen arbitrarily small imply the correctness of the second part of Lemma 4.5.