Maximum Independent Set of Links under Physical Interference Model*

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Abstract. This paper addresses the following optimization problem in a plane multihop wireless networks under the physical interference model: From a given a set of communication links whose senders transmit at a fixed uniform power level, select a maximum set of independent links. This problem is known to be NP-hard. The existing approximation algorithms which were claimed to have constant approximation bounds are either valid only in the absence of background noise or simply incorrect in the presence of background noise. In this paper, we develop a new approximation algorithm with constant approximation bound regardless of the value of the background noise. In addition, our approximation bound valid in general is significantly smaller than all the known bounds which are only valid under certain special assumptions.

1 Introduction

This paper studies how to select a maximum number of interference-free links from a given set of communication links subject to the physical interference constraint. We adopt the following model of the multihop wireless networks. All the networking nodes V lie in plane and transmit at a fixed power P. The Euclidean distance between any pair of nodes is denoted by $\|uv\|$. By proper scaling, we assume that the Euclidean distance of a closest pair of the nodes in V is one. The path loss model is then determined by a positive reference loss parameter $\eta < 1$, and the path-loss exponent κ, which is a constant greater than 2 but less than 6 typically. Specifically, when a node u transmits a signal at power P, the power of this signal captured by another node v is $\eta P \|uv\|^{-\kappa}$. The signal quality perceived by a receiver is measured by the *signal to interference and noise ratio* (*SINR*), which is the quotient between the power of the wanted signal and the total power of unwanted signals and the ambient–both internal and external–noise. In order to correctly interpret the wanted signal, the SINR must be no less than certain threshold σ. Formally, consider a link (u, v) and a set W of nodes other than u and transmitting simultaneously with u. Let ξ be the noise power. The SINR of the link (u, v) with respect to W is given by

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$$\frac{\eta P \|uv\|^{-\kappa}}{\xi + \sum_{w \in W} \eta P \|wv\|^{-\kappa}}.$$

If W is empty (i.e., the interference is absent), the SINR reduces to *signal to noise ratio* (*SNR*) given by $\eta P \|uv\|^{-\kappa} / \xi$. Let $R = \left(\frac{\eta P}{\sigma \xi}\right)^{1/\kappa}$. Then, a pair of nodes u and v can communicate with each other in the absence of interference if and only if $\|uv\| \leq R$. The value R is thus referred to as the *maximum transmission radius*.

A set $I = \{(u_i, v_i) : 1 \leq i \leq k\}$ of links are said to be *independent* if

1. all links in I are disjoint, i.e., the $2k$ nodes u_i for $1 \leq i \leq k$ and v_i for $1 \leq i \leq k$ are distinct;
2. for each $1 \leq i \leq k$, the SINR of the link (u_i, v_i) with respect to $\{u_j : 1 \leq j \leq k, j \neq i\}$ is at least σ.

The problem **Maximum Independent Set of Links** (**MISL**) seeks a largest number of independent links from a given set A of links. This optimization problem is NP-hard [3]. The first non-trivial approximation algorithm was proposed in [3], with approximation bound $O\left(\log \frac{\max_{a \in A} \|a\|}{\min_{a \in A} \|a\|}\right)$. We remark that the constant hidden inside this bound is very large, and this bound can be arbitrarily large in general. Recently, Goussevskaia et al. [4] made the first effort on developing a constant approximation. However, as observed in [6], the claimed constant approximation bound and its proof (Lemma 4.5 in [4]) are valid only when the noise $\xi = 0$. Indeed, we can construct counter-example to show that the approximation bound can be arbitrarily large in general. Xu and Tang [6] then made an attempt to fix this flaw. The partition the links into two groups: the short group consists of all links of length at most $R/3^{1/\kappa}$, and the long group consists of the rest links. They select the first set of independent links from the short group by simply applying the algorithm developed in [4], and also select the second set of independent links from the long group by a new algorithm. The larger one of these two sets is their output. Unfortunately, their new algorithm for selecting independent links from the long group is incorrect as the selected links may be not independent (Lemma 2 in [6] and its proof are false). Therefore, despite of these efforts, the existence of a constant approximation algorithm for **MISL** remains open.

In this paper, we develop an approximation algorithm for **MISL** which not only has a constant approximation bound regardless of the value of the noise and the lengths of the communication links, but also has a significantly smaller approximation bound. For example, consider the scenario with $\kappa = 4$ and $\sigma = 16$. The approximation bounds obtained in [4] (in the absence of noise) and in [6] (restricted to short links) are huge constants at least 138135 and 137890 respectively. In contrast, our approximation bound is at most 272, which is more than 500 times smaller than these two known bounds and is independent of the noise value and link lengths.

The remaining of this paper is organized as follows. Section 2 introduces a new concept of relative interference and its fundamental properties. Section 3 presents the design and analysis of our approximation algorithm. Finally, Section 4 discusses some generalizations and future works.

2 Relative Interference

Consider a link $a = (u, v)$ and a set W of "interfering" nodes other than u. In the presence of the interference from W, the SINR of a is

$$\frac{\eta P \|uv\|^{-\kappa}}{\xi + \sum_{w \in W} \eta P \|wv\|^{-\kappa}} = \frac{1}{\frac{\xi}{\eta P} \|uv\|^{\kappa} + \sum_{w \in W} (\|uv\| / \|wv\|)^{\kappa}}$$
$$= \frac{1}{\frac{1}{\sigma} (\|uv\| / R)^{\kappa} + \sum_{w \in W} (\|uv\| / \|wv\|)^{\kappa}}.$$

It's straightforward to verify that such SINR is at least σ if and only if

$$\sigma \frac{\sum_{w \in W} (\|uv\| / \|wv\|)^{\kappa}}{1 - (\|uv\| / R)^{\kappa}} \leq 1.$$

Motivated by this charaterization, we define the *relative interference* of node w other than u to the link a as

$$RI(w; a) = \sigma \frac{(\|uv\| / \|wv\|)^{\kappa}}{1 - (\|uv\| / R)^{\kappa}},$$

and define the *relative interference* of a set W of nodes other than u to a as

$$RI(W; a) = \sum_{w \in W} RI(w; a).$$

Then, in the presence of the interference from W, the link (u, v) succeeds if and only if the relative interference of W to the link (u, v) is at most one. We would like to remark that the relative interference defined in this paper is different from the one defined in [4]. In [4], the relative interference of node w other than u to the link a was defined to be $(\|uv\| / \|wv\|)^{\kappa}$, and the relative interference of a set W of nodes other than u to a was defined to be $\sum_{w \in W} (\|uv\| / \|wv\|)^{\kappa}$.

Now, we present a some simple sufficient condition for $RI(W; a)$ to be less than some given value $\phi \in (0, 1]$. Our condition will utilize the Riemann zeta function in the following form: $\zeta(x) = \sum_{j=1}^{\infty} j^{-x}$. Note $\zeta(1) = \infty$, and for any $x > 1$, $\zeta(x) < x/(x-1)$. The values of $\zeta(x)$ for small values of x are

$$\zeta(1.5) = 2.612, \zeta(2) = \frac{\pi^2}{6}, \zeta(2.5) = 1.341, \zeta(3) = 1.202,$$
$$\zeta(3.5) = 1.127, \zeta(4) = \frac{\pi^4}{90}, \zeta(5) = 1.041, \zeta(6) = \frac{\pi^6}{945} = 1.017.$$

Lemma 1. *Consider a link $a = (u, v)$ and a set W of nodes other than u. Suppose that $0 < \phi \leq 1$ and*

$$\rho = 1 + \left(\frac{\sigma \left(16\zeta\left(\kappa - 1\right) + 8\zeta\left(\kappa\right) - 6\right)}{\phi \left(1 - \left(\left\|a\right\| / R\right)^{\kappa}\right)} \right)^{1/\kappa}.$$

If all the nodes in $W \cup \{u\}$ have mutual distances at least $\rho \|uv\|$, then $RI\left(W; a\right) < \phi$.

Proof. For each $j \geq 1$, define

$$W_j = \{w \in W : j\rho r \leq \|uw\| < (j+1)\rho r\},$$

Then, $W_1, W_2, \cdots$ form a partition of W. By a classic result due to Bateman and Erdös [2], $|W_1| \leq 18$. For $j \geq 2$, using the folklore area argument we have $|W_j| \leq 8(2j+1)$. In addition, for each $w \in W_j$ with $j \geq 1$, we have

$$\|wv\| \geq \|wu\| - \|uv\| \geq j\rho r - r \geq j(\rho - 1) r.$$

So,

$$\sigma \sum_{w \in W} \left(\frac{\|uv\|}{\|wv\|} \right)^{\kappa} = \sigma \sum_{j=1}^{\infty} \sum_{w \in W_j} \left(\frac{\|uv\|}{\|wv\|} \right)^{\kappa} \leq \sigma \sum_{j=1}^{\infty} \frac{|W_j|}{\left(j\left(\rho - 1\right)\right)^{\kappa}}$$

$$= \sigma \left(\rho - 1\right)^{-\kappa} \left(|W_1| + \sum_{j=2}^{\infty} \frac{|W_j|}{j^{\kappa}} \right) < \sigma \left(\rho - 1\right)^{-\kappa} \left(18 + 8 \sum_{j=2}^{\infty} \frac{2j+1}{j^{\kappa}} \right)$$

$$\leq \sigma \left(\rho - 1\right)^{-\kappa} \left(16\zeta\left(\kappa - 1\right) + 8\zeta\left(\kappa\right) - 6\right) = \phi \left(1 - \left(\frac{\|uv\|}{R} \right)^{\kappa} \right).$$

Thus, the lemma follows.

Next, we present a necessary condition for $RI\left(W; a\right) \leq 1$.

Lemma 2. *Consider a link $a = (u, v)$ and a set W of nodes other than u whose distances from u is at most $\rho \|uv\|$. If $RI\left(W; a\right) \leq 1$, then*

$$|W| \leq \frac{\left(\rho + 1\right)^{\kappa}}{\sigma} \left(1 - \left(\frac{\|uv\|}{R} \right)^{\kappa} \right).$$

Proof. Assume to the contrary that

$$|W| > \frac{\left(\rho + 1\right)^{\kappa}}{\sigma} \left(1 - \left(\frac{\|uv\|}{R} \right)^{\kappa} \right).$$

For any $w \in W$,

$$\|wv\| \leq \|wu\| + \|uv\| \leq \left(\rho + 1\right) \|uv\|.$$

Thus,

$$\sigma \sum_{w \in W} \left(\frac{\|uv\|}{\|wv\|} \right)^{\kappa} \geq \frac{\sigma |W|}{\left(\rho + 1\right)^{\kappa}} > \left(1 - \left(\frac{\|uv\|}{R} \right)^{\kappa} \right).$$

Hence, $RI\left(W; a\right) > 1$, which is a contradiction. So, the lemma holds.

3 Algorithm Design and Analysis

Our approximation algorithm for **MISL** is outlined in Table 1. The algorithm is associated with a parameter $\phi \in (0,1]$, whose value will be determined later on. Let A be the set of given communication links of length at most R. Three variables are maintained by the algorithm. S is the sequence–sorted in the increasing order of length–of links which haven't been selected or discarded, I stores the set of selected independent links, and U stores the senders of the links in I. The algorithm is iterative. In each iteration, the first link a in S is moved from S to I, and its sender is added to U. Then, some links from S are discarded in two steps. The first step discards all links in S whose senders are close to the sender of a. A parameter ρ is used to guide such discarding in the first step. The value of ρ is specified in the algorithm. It decreases with ϕ and but increases with the length of a. After the computation of ρ, all links in S whose senders lie in the disk of radius $\rho \|a\|$ centered at the sender of a are removed from S. The second step removes all links remaining in S to which the relative interference of U exceed $1-\phi$. Such iteration is then repeated until S is empty.

Table 1. The description of the approximation algorithm for **MISL**

Approximation Algorithm for MISL:
$S \leftarrow$ sequence of links in A in the increasing order of length;
$I \leftarrow \emptyset$, $U \leftarrow \emptyset$;
While $S \neq \emptyset$
$a = (u,v) \leftarrow$ the first link in S;
$S \leftarrow S \setminus \{a\}$, $I \leftarrow I \cup \{a\}$, $U \leftarrow U \cup \{u\}$;
$\rho \leftarrow 1 + \left(\frac{\sigma(16\zeta(\kappa-1)+8\zeta(\kappa)-6)}{\phi(1-(\|uv\|/R)^\kappa)}\right)^{1/\kappa}$;
$S' \leftarrow \{(u',v') \in S : \|u'u\| < \rho \|uv\|\}$;
$S \leftarrow S \setminus S'$;
$S'' \leftarrow \{a'' \in S : RI(U,a'') > 1-\phi\}$;
$S \leftarrow S \setminus S''$;
Output I.

Next, we prove the correctness of the algorithm and derive its approximation bound. We introduce the following notations. Suppose that the algorithm runs in k iterations, and $a_i = (u_i, v_i)$ is the link selected in the i-th iteration for each $1 \leq i \leq k$. Then, the output I consists of $a_1, a_2, \cdots, a_k$. Let

$$U_0 = \emptyset,$$

$$\rho_0 = 1 + \left(\frac{\sigma\left(16\zeta\left(\kappa-1\right)+8\zeta\left(\kappa\right)-6\right)}{\phi}\right)^{1/\kappa},$$

and for each $1 \leq i \leq k$, let U_i and ρ_i be the set U and the parameter ρ respectively at the end of the i-th iteration. Then, for each $1 \leq i \leq k$,

$$U_i = \{u_j : 1 \leq j \leq i\},$$

$$\rho_i = 1 + \left(\frac{\sigma \left(16\zeta \left(\kappa - 1\right) + 8\zeta \left(\kappa\right) - 6\right)}{\phi \left(1 - \left(\left\| a_i \right\| / R\right)^{\kappa}\right)} \right)^{1/\kappa}.$$

It's easy to verify that for each $1 \leq i \leq k$,

$$\rho_i = 1 + \frac{\rho_0 - 1}{\left(1 - \left(\left\| a_i \right\| / R\right)^{\kappa}\right)^{1/\kappa}}$$

and $\rho_0 \leq \rho_1 \leq \rho_2 \leq \cdots \leq \rho_k$.

The theorem below asserts the correctness of our algorithm.

Theorem 1. *The output set I is independent.*

Proof. It's sufficient to show that for any $1 \leq i \leq k$, $RI\left(U \setminus \{u_i\}; a_i\right) \leq 1$. Fix an integer i between 1 and k. Since a_i is not discarded, $RI\left(U_{i-1}; a_i\right) \leq 1 - \phi$. Since none of $a_i, a_{i+1}, \cdots, a_k$ is discarded, for each $i \leq j < j' \leq k$, we have $\left\| u_j u_{j'} \right\| \geq \rho_j \left\| a_j \right\| \geq \rho_i \left\| a_i \right\|$. So, the mutual distances of the nodes $U \setminus U_{i-1}$ are at least $\rho_i \left\| a_i \right\|$. By Lemma 1, $RI\left(U \setminus U_i; a_i\right) \leq \phi$. Therefore,

$$RI\left(U \setminus \{u_i\}; a_i\right) = RI\left(U_{i-1}; a_i\right) + RI\left(U \setminus U_i; a_i\right) \leq 1.$$

So, the theorem holds.

Let

$$\mu_1 = \left\lfloor \frac{2\pi}{\sqrt{3}} \left(\frac{\rho_0}{\sigma^{1/\kappa} - 1} \right)^2 + \pi \frac{\rho_0}{\sigma^{1/\kappa} - 1} \right\rfloor + 1,$$
$$\mu_2 = 5 \left\lceil \left((1 - \phi)^{-\frac{1}{\kappa}} + 2\sigma^{-\frac{1}{\kappa}} \right)^{\kappa} \right\rceil.$$

The theorem below presents an approximation bound of our algorithm.

Theorem 2. *I is an $(1 + \mu_1 + \mu_2)$-approximation.*

We give a brief overview on the proof. For each $1 \leq j \leq k$, S'_j and S''_j be the sets S' and S'' respectively discarded from S in the j-th iteration. Let $A' = S'_1 \cup S'_2 \cup \cdots \cup S'_k$ and $A'' = S''_1 \cup S''_2 \cup \cdots \cup S''_k$. Then, I, S' and S'' form a partition of A. Consider an optimal solution OPT. We will prove in Lemma 3 that for each $1 \leq i \leq k$, $|OPT \cap S'_i| \leq \mu_1$, and in Lemma 4 that $|OPT \cap A''| \leq \mu_2 k$. Then,

$$|OPT \cap A'| = \sum_{j=1}^{k} |OPT \cap S'_i| \leq \mu_1 k.$$

So,

$$|OPT| = |OPT \cap I| + |OPT \cap A'| + |OPT \cap A''| \leq \left(1 + \mu_1 + \mu_2\right) k,$$

and Theorem 2 follows.

Lemma 3. *For each $1 \leq i \leq k$, $|OPT \cap S'_i| \leq \mu_1$.*

Proof. We first show that for any pair of links (u, v) and (u', v') in $OPT \cap S'_i$,

$$\frac{\|u'u\|}{\|a_i\|} \geq \frac{\sigma^{1/\kappa}}{\left(1 - (\|a_i\|/R)^\kappa\right)^{1/\kappa}} - 1.$$

Since

$$\sigma \left(\frac{\|uv\|}{\|u'v\|}\right)^\kappa \leq 1 - \left(\frac{\|uv\|}{R}\right)^\kappa,$$

we have

$$\frac{\|u'v\|}{\|uv\|} \geq \frac{\sigma^{1/\kappa}}{\left(1 - (\|uv\|/R)^\kappa\right)^{1/\kappa}} \geq \frac{\sigma^{1/\kappa}}{\left(1 - (\|a_i\|/R)^\kappa\right)^{1/\kappa}}.$$

Thus,

$$\frac{\|u'u\|}{\|a_i\|} \geq \frac{\|u'v\| - \|uv\|}{\|uv\|} = \frac{\|u'v\|}{\|uv\|} - 1 \geq \frac{\sigma^{1/\kappa}}{\left(1 - (\|a_i\|/R)^\kappa\right)^{1/\kappa}} - 1.$$

Now, we show that

$$\frac{\rho_i}{\frac{\sigma^{1/\kappa}}{(1-(\|a_i\|/R)^\kappa)^{1/\kappa}} - 1} < \frac{\rho}{\sigma^{1/\kappa} - 1}.$$

Since

$$\rho_i = 1 + \frac{\rho - 1}{\left(1 - (\|a_i\|/R)^\kappa\right)^{1/\kappa}},$$

we have

$$\frac{\rho_i}{\frac{\sigma^{1/\kappa}}{(1-(\|a_i\|/R)^\kappa)^{1/\kappa}} - 1} = \frac{1 + \frac{\rho-1}{(1-(\|a_i\|/R)^\kappa)^{1/\kappa}}}{\frac{\sigma^{1/\kappa}}{(1-(\|a_i\|/R)^\kappa)^{1/\kappa}} - 1} = \frac{\left(1 - (\|a_i\|/R)^\kappa\right)^{1/\kappa} + \rho - 1}{\sigma^{1/\kappa} - \left(1 - (\|a_i\|/R)^\kappa\right)^{1/\kappa}}$$
$$< \frac{1 + \rho - 1}{\sigma^{1/\kappa} - 1} = \frac{\rho}{\sigma^{1/\kappa} - 1}.$$

Finally, we show that $|OPT \cap S'_i| \leq \mu_1$. Note that the senders of all links in $OPT \cap S'_i$ lie in the disk of radius $\rho_i \|a_i\|$ centered at u_i and their mutual distances are at least

$$\left(\frac{\sigma^{1/\kappa}}{\left(1 - (\|a_i\|/R)^\kappa\right)^{1/\kappa}} - 1\right) \|a_i\|.$$

By Groemer's Inequality [5] on disk packing,

$$|OPT \cap S'_i| \leq \frac{2\pi}{\sqrt{3}} \left(\frac{\rho_i}{\frac{\sigma^{1/\kappa}}{(1-(\|a_i\|/R)^\kappa)^{1/\kappa}} - 1}\right)^2 + \pi \frac{\rho_i}{\frac{\sigma^{1/\kappa}}{(1-(\|a_i\|/R)^\kappa)^{1/\kappa}} - 1} + 1$$
$$< \frac{2\pi}{\sqrt{3}} \left(\frac{\rho}{\sigma^{1/\kappa} - 1}\right)^2 + \pi \frac{\rho}{\sigma^{1/\kappa} - 1} + 1.$$

Since $|OPT \cap S'_i|$ is an integer, we have

$$|OPT \cap S'_i| \leq \left\lfloor \frac{2\pi}{\sqrt{3}} \left(\frac{\rho}{\sigma^{1/\kappa} - 1} \right)^2 + \pi \frac{\rho}{\sigma^{1/\kappa} - 1} \right\rfloor + 1 = \mu_1.$$

So, the lemma holds.

Lemma 4. $|OPT \cap S''| \leq \mu_2 k$.

Proof. We prove the lemma by contradiction. Assume to the contrary that $|OPT \cap S''| > \mu_2 k$. Clearly, all links in $OPT \cap S''$ are disjoint. In addition, all links in $(OPT \cap S'') \cup S$ are also disjoint. Let W denote the set of senders of the links in $|OPT \cap S''|$, and let $l = \left\lceil \left((1-\phi)^{-\frac{1}{\kappa}} + 2\sigma^{-\frac{1}{\kappa}} \right)^{\kappa} \right\rceil$. We iteratively construct kl disjoint subsets W_{ij} for $1 \leq i \leq k$ and $1 \leq j \leq l$ as follows. Initialize W' to W and each W_{ij} to be the empty set. Repeat the following iterations for each $i = 1$ to k and for each $j = 1$ to q. Let w be a node in W' which is closest to u_i. Draw six (closed) 60^o-sectors originating at u_i such that one of six boundary rays goes through w. Add w to W_{ij}. For each of these four sectors not containing w, if it contains at least one node in W', we choose a node among these nodes in W' and in this sector which is nearest to u_i and add it to W_{ij}. After that we remove W_{ij} from W' and repeat the iterations.

By the construction, each W_{ij} contains at most 5 nodes. Hence,

$$\sum_{i=1}^{k} \sum_{j=1}^{l} |W_{ij}| \leq 5kl = \mu_2 k.$$

Thus, the set $W \setminus \bigcup_{i=1}^{k} \bigcup_{j=1}^{q} W_{ij}$ is non-empty. Pick an arbitrary node u in this set and suppose that (u, v) is the link in $OPT \cap S''$. For each $1 \leq i \leq k$ and $1 \leq j \leq l$, there is a node $w_{ij} \in W_{ij}$ such that $\|u_i w_{ij}\| \leq \|u_i u\|$ and the angle $\angle w u_i u \leq 60^o$ by the construction of W_{ij}. Let L_i denote the lune of u_i and u, which is the intersection of the two disks of radius $\|u_i u\|$ centered at u_i and u respectively. Then, each w_{ij} lies in L_i. Thus each L_i contains at least l nodes in W. Let $\rho' = 1 + \left(\frac{\sigma}{1-\phi} \right)^{\frac{1}{\kappa}}$. By Lemma 2,

$$\frac{(\|u_i u\| / \|uv\| + 1)^{\kappa}}{\sigma} > l = \left\lceil \left((1-\phi)^{-\frac{1}{\kappa}} + 2\sigma^{-\frac{1}{\kappa}} \right)^{\kappa} \right\rceil = \left\lceil \frac{(\rho' + 1)^{\kappa}}{\sigma} \right\rceil \geq \frac{(\rho' + 1)^{\kappa}}{\sigma},$$

and hence $\|u_i u\| / \|uv\| > \rho'$.

We claim that

$$\left(\frac{\|uv\|}{\|u_i v\|} \right)^{\kappa} \leq (1-\phi) \sum_{j=1}^{l} \left(\frac{\|uv\|}{\|w_{ij} v\|} \right)^{\kappa}.$$

Indeed, for each $1 \leq j \leq l$,

$$\frac{\|u_i v\|}{\|w_{ij} v\|} \geq \frac{\|u_i u\| - \|uv\|}{\|w_{ij} u\| + \|uv\|} \geq \frac{\|u_i u\| - \|uv\|}{\|u_i u\| + \|uv\|} = \frac{\|u_i u\| / \|uv\| - 1}{\|u_i u\| / \|uv\| + 1}$$
$$= 1 - \frac{2}{\|u_i u\| / \|uv\| + 1} \geq 1 - \frac{2}{\rho' + 1} = \frac{\rho' - 1}{\rho' + 1}.$$

Hence,

$$\frac{\sum_{j=1}^{l} \left(\frac{\|uv\|}{\|w_{ij} v\|} \right)^{\kappa}}{\left(\frac{\|uv\|}{\|u_i v\|} \right)^{\kappa}} = \sum_{j=1}^{l} \left(\frac{\|u_i v\|}{\|w_{ij} v\|} \right)^{\kappa} \geq l \left(\frac{\rho' - 1}{\rho' + 1} \right)^{\kappa}$$
$$\geq \frac{(\rho' + 1)^{\kappa}}{\sigma} \left(\frac{\rho' - 1}{\rho' + 1} \right)^{\kappa} = \frac{(\rho' - 1)^{\kappa}}{\sigma} = \frac{1}{1 - \phi}.$$

So, the claim holds.

By the above claim, we have

$$\sigma \sum_{i=1}^{k} \left(\frac{\|uv\|}{\|u_i v\|} \right)^{\kappa} \leq (1 - \phi)\, \sigma \sum_{i=1}^{k} \sum_{j=1}^{l} \left(\frac{\|uv\|}{\|w_{ij} v\|} \right)^{\kappa} \leq (1 - \phi) \left(1 - \left(\frac{\|uv\|}{R} \right)^{\kappa} \right).$$

Therefore, the relative interference of $\{u_i : 1 \leq i \leq k\}$ to the link (u, v) is at most $1 - \phi$. This means (u, v) shouldn't have been removed, which is a contradiction. So, the lemma holds.

Finally, we choose the appropriate value of ϕ so that the overall approximation bound $\mu_1 + \mu_2 + 1$ is as small as possible. Clearly, μ_1 decreases with ϕ, and μ_2 increases with ϕ. So, it's not apparent where $\mu_1 + \mu_2 + 1$ can achieve its minimum. We will exploit the convexity to quickly find a suboptimal value of ϕ. Note that for $\kappa > 2$, both x^{κ} and $x^{-1/\kappa}$ are convex function of x on $(0, \infty)$. Using the fact the composite of two convex functions is also convex, we can assert the convexity of the next two functions in $(0, 1)$:

$$f_1(\phi) = \frac{2\pi}{\sqrt{3}} \left(\frac{\rho_0}{\sigma^{1/\kappa} - 1} \right)^2 + \pi \frac{\rho_0}{\sigma^{1/\kappa} - 1} + 1,$$
$$f_2(\phi) = 5 \left((1 - \phi)^{-\frac{1}{\kappa}} + 2\sigma^{-\frac{1}{\kappa}} \right)^{\kappa}.$$

By using standard binary search, we can obtain a short sub-interval of $(0, 1)$ containing the value of ϕ minimizing $f_1(\phi) + f_2(\phi)$. Then, we take a small number of samples of ϕ in this sub-interval. Among these samples, the one corresponding to the smallest value of $\mu_1 + \mu_2 + 1$ is then chosen as the value of ϕ.

For example, consider a scenario with $\kappa = 4$ and $\sigma = 16$. An appropriate choice of ϕ is 0.5. For such choice, $\rho = 6.145$, $\mu_1 = 156$, and $\mu_2 = 115$. So, the approximation bound of our algorithm is 272. In contrast, the approximation bounds obtained in [4] (in the absence of noise) and in [6] (restricted to short links) are

$$1+\left(2\left(2^5 3^2 \sigma \frac{\kappa-1}{\kappa-2}\right)^{1/\kappa}+1\right)^{\kappa}+5\cdot 3^{\kappa+1}$$

and

$$1+\left(2\left(2^5 3^2 \sigma \frac{\kappa-1}{\kappa-2}\right)^{1/\kappa}+1\right)^{\kappa}+5\cdot 2^{\kappa+1}\left(\left\lceil 3^{\kappa}\right\rceil/\sigma+1\right)$$

respectively. When $\kappa = 4$ and $\sigma = 16$, these two approximation bounds are at least 138135 and 137890 respectively, both of which are more than 500 times the approximation bound 272 of our algorithm.

4 Discussions

In this paper, we developed a constant-approximation algorithm for **MISL** under the assumption that the uniform transmission power by all nodes. Such assumption can be slightly relaxed to that all nodes transmit at different but fixed power levels and either the ratio of the maximum power level to the minimum power level or the number of power levels is bounded by a constant. It is also straightforward to extend our algorithm design and analysis to the case that the neworking nodes are located in a three-dimensional space.

There are two challenging variants of **MISL**. In the weighted variant of **MISL**, all input links are associated with some weights, and we would like to seek an independent set of input links whose total weight is maximized. In the power-adjustable variant, all nodes can freely adjust their transmission powers and the selection of the power levels is part of the problem. This variant is also NP-hard [1]. Neither of these two variants is known to have an logarithmic-approximation algorithm, let alone a constant-approximation algorithm. Any progress on these two variants towards constant-approximations would be significant.

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