Asymptotic Distribution of The Number of Isolated Nodes in Wireless Ad Hoc Networks with Bernoulli Nodes

Chih-Wei Yi Peng-Jun Wan Xiang-Yang Li Ophir Frieder
Department of Computer Science, Illinois Institute of Technology, Chicago, IL 60616.
Emails: yichihw@iit.edu, wan@cs.iit.edu, xli@cs.iit.edu, ophir@cs.iit.edu

***Abstract*— Nodes in wireless ad hoc networks may become inactive or unavailable due to, for example, internal breakdown or being in the sleeping state. The inactive nodes cannot take part in routing/relaying and thus may affect the connectivity. A wireless ad hoc network containing inactive nodes is then said to be connected if each inactive node is adjacent to at least one active node and all active nodes form a connected network. This paper is the first installment of our probabilistic study of the connectivity of wireless ad hoc networks containing inactive nodes. We assume that the wireless ad hoc network consists of n nodes which are distributed independently and uniformly in a unit-area disk and are active (or available) independently with probability p for some constant $0 < p \leq 1$. We show that if all nodes have a maximum transmission radius $r_n = \sqrt{\frac{\ln n + c}{\pi p n}}$ for some constant c, then the total number of isolated nodes is asymptotically Poisson with mean e^{-c} and the total number of isolated active nodes is also asymptotically Poisson with mean pe^{-c}.**

I. Introduction

A wireless ad hoc network is a collection of radio devices (transceivers) located in a geographic region. Each node is equipped with an omni-directional antenna and has limited transmission power. A communication session is established either through a single-hop radio transmission if the communication parties are close enough, or through relaying by intermediate devices otherwise. Because of the no need for a fixed infrastructure, wireless ad hoc networks can be flexibly deployed at low cost for varying missions such as decision making in the battlefield, emergency disaster relief and environmental monitoring. In most applications, the ad hoc wireless devices are deployed in a large volume. The shear large number of devices deployed coupled with the potential harsh environment often hinders or completely eliminates the possibility of strategic device placement, and consequently, random deployment is often the only viable option. In some other applications, the ad hoc wireless devices may be continuously in motion or be dynamically switched to on or off. For all these applications, it is natural to represent the ad hoc devices by a finite random point process over the (finite) deployment region. Correspondingly, the wireless ad hoc network is represented by a random graph.

The classic random graph model due to Erdős and Rényi (1960) [4], in which each pair of vertices are joined by an edge *independently* and uniformly at some probability, is not suited to accurately represent networks of short-range radio nodes due to the presence of local correlation among radio links. This motivated Gilbert (1961) [5] to propose an alternative random graph model for radio networks. Gilbert's model assumes that all devices, represented by an *infinite* random point process over the entire plane, have the same maximum transmission radius r and two devices are joined by an edge if and only if their distance is at most r. For the modelling of wireless ad hoc networks which consist of finite radio nodes in a bounded geographic region, a bounded (or finite) variant of the standard Gilbert's model has been used by Gupta and Kumar (1998) [6] and others. In this variant, the random point precess representing the ad hoc devices is typically assumed to be a *uniform n-point process* $\mathcal{X}_n$ over a disk or a square of unit area by proper scaling, and the wireless ad hoc network, denoted by $\mathcal{G}(n,r)$, is exactly the r-disk graph over $\mathcal{X}_n$. To distinguish the random graph $\mathcal{G}(n,r)$ from the classic random graph due to Erdős and Rényi, it is referred to as a *random geometric graph*.

The connectivity of the random geometric graph $\mathcal{G}(n,r)$ has been studied by Dette and Henze (1989) [3] and Penrose (1997) [7]. For any constant c, Dette and Henze (1989) [3] showed that the graph $\mathcal{G}\left(n, \sqrt{\frac{\ln n + c}{\pi n}}\right)$ has no isolated nodes with probability $\exp(-e^{-c})$ asymptotically. Eight years later, Penrose (1997) [7] established that if a random geometric graph $\mathcal{G}(n,r)$ has no isolated nodes, then it is almost surely connected. These results are the exact analogue of the counterpart in classic random graphs. However, as pointed out by Bollobás (2001) [2], we should not be misled by the remembrance: the proof for the random geometric graph is much harder.

In this paper, we consider an extension to the random geometric graph $\mathcal{G}(n,r)$ by introducing an additional assumption that all nodes are active (or available) independently with probability p for some constant $0 < p \leq 1$. Such extension is motivated by the fault-tolerance of wireless ad hoc networks. In a practical wireless ad hoc network, a node may be inactive (or unavailable) due to either internal breakdown, or being in the sleeping state. In either case, the inactive nodes will not take part in routing/relaying and thus may affect the

0-7803-7700-1/03/$17.00 (C) 2003 IEEE

connectivity. It is natural to model the availability of the nodes by a Bernoulli model, and hence we call the nodes as Bernoulli nodes. A wireless ad hoc network of Bernoulli nodes is then said to be connected if each inactive node is adjacent to at least one active node and all active nodes form a connected network.

Our probabilistic study of the connectivity of the random geometric graph with Bernoulli nodes consists of two installments due to the lengthy analysis. The first intallment, which is the focus of this paper, addresses the distribution of the number of nodes without active neighbors. For convenience, a node is said to be *isolated from active nodes*, or simply *isolated*, it has no active neighbors. We shall prove that both the number of isolated nodes and the number of isolated active nodes have asymptotic Poisson distributions. The second installment, which will be reported in a separate paper, proves that if a random geometric graph with Bernoulli nodes has no isolated nodes, it is also connected almost surely.

In what follows, $\|x\|$ is the Euclidean norm of a point $x \in \mathbb{R}^2$, and $|A|$ is shorthand for 2-dimensional Lebesgue measure (or area) of a measurable set $A \subset \mathbb{R}^2$. All integrals considered will be Lebesgue integrals. The topological boundary of a set $A \subset \mathbb{R}^2$ is denoted by ∂A. The disk of radius r centered at x is denoted by $D(x,r)$. The special unit-area disk centered at the origin is denoted by Ω. For any set S and positive integer k, the k-fold Cartesian product of S is denoted by S^k. The symbols $O, o, \sim$ always refer to the limit $n \longrightarrow \infty$. To avoid trivialities, we tacitly assume n to be sufficiently large if necessary. For simplicity of notation, the dependence of sets and random variables on n will be frequently suppressed.

The remaining of this paper is organized as follows. In section II, we present several useful geometric results and integrals. In Section III, we derive both the distribution of the number of isolated nodes and the distribution of the number of isolated active nodes.

II. Geometry of Disks

The results in this section are purely geometric, with no probabilistic content. Let r be the transmission radius of the nodes. For any finite set of nodes $\{x_1, \cdots, x_k\}$ in Ω, we use $G_r(x_1, \cdots, x_k)$ to denote the graph over $\{x_1, \cdots, x_k\}$ in which there is an edge between two nodes if and only if their Euclidean distance is at most r. For any positive integers k and m with $1 \leq m \leq k$, let C_{km} denote the set of $(x_1, \cdots, x_k) \in \Omega^k$ satisfying that $G_{2r}(x_1, \cdots, x_k)$ has exactly m connected components.

We partition the unit-area disk Ω into three regions, $\Omega(0)$, $\Omega(1)$ and $\Omega(2)$ as shown in Fig. 1: $\Omega(0)$ is the disk of radius $1/\sqrt{\pi} - r$ centered at the origin; $\Omega(1)$ is the annulus of radii $1/\sqrt{\pi} - r$ and $\sqrt{1/\pi - r^2}$ centered at the origin; and $\Omega(2)$ is the annulus of radii $\sqrt{1/\pi - r^2}$ and $1/\sqrt{\pi}$ centered at the origin. Then,

$$|\Omega(0)| = \left(1 - \sqrt{\pi} r\right)^2, |\Omega(2)| = \pi r^2,$$
$$|\Omega(1)| = 2\pi r \left(1/\sqrt{\pi} - r\right).$$

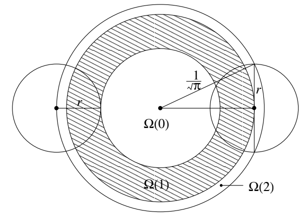

Fig. 1. The partition of the unit-area disk Ω.

For any set $S \subseteq \Omega$ and $r > 0$, the r-neighborhood of S is the set $\bigcup_{x \in S} D(x,r) \cap \Omega$. We use $\nu_r(S)$ to denote the area of the r-neighborhood of S, and sometimes by slightly abusing the notation, to denote the r-neighborhood of S itself. Obviously, for any $x \in \Omega$, $\nu_r(x) \geq \pi r^2/3$. If $x \in \Omega(0)$, $\nu_r(x) = \pi r^2$. If $x \in \Omega(1)$, we have the following tighter lower bound on $\nu_r(x)$.

Lemma 1: For any $x \in \Omega(1)$,

$$\nu_r(x) \geq \frac{\pi r^2}{2} + \left(\frac{1}{\sqrt{\pi}} - \|x\|\right) r.$$

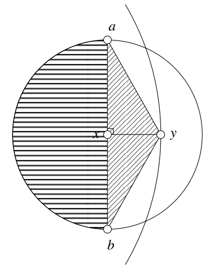

Fig. 2. The half-disk and the triangle.

Proof: Let y be the point in $\partial\Omega$ such that $\|y - x\| = \frac{1}{\sqrt{\pi}} - \|x\|$, and ab be the diameter of $D(x,r)$ perpendicular to xy (see Fig. 2). Then $\nu_r(x)$ contains a half-disk of $D(x,r)$ to the side of ab opposite to y, and the triangle aby. Since the area of the triangle aby is exactly $\left(\frac{1}{\sqrt{\pi}} - \|x\|\right) r$, the lemma follows. ■

The next lemma gives a lower bound on the area of the r-neighborhood of more than one nodes.

Lemma 2: Assume that

$$r \leq \frac{1/\sqrt{\pi}}{12/\pi + \pi/12} \approx 0.245/\sqrt{\pi}.$$

Let $x_1, \cdots, x_k$ be a sequence of $k \geq 2$ nodes in Ω such that x_1 has the largest norm, and $\|x_i - x_j\| \leq 2r$ if and only if $|i - j| \leq 1$. Then

$$\nu_r(x_1, \cdots, x_k) \geq \nu_r(x_1) + \frac{\pi}{12} r \sum_{i=1}^{k-1} \|x_{i+1} - x_i\|.$$

Proof: We prove the lemma by induction on k. We begin with $k = 2$. Let $t = \|x_2 - x_1\|$ and $f(t) = |D(x_2,r) \setminus D(x_1,r)|$. We first show that $f(t) \geq (\pi/2) rt$. Let $y_1 y_2$ be the common chord of $\partial D(x_1,r)$ and $\partial D(x_2,r)$, and let $z_1 z_2$ be another chord of $\partial D(x_2,r)$ that is parallel to $y_1 y_2$ and has the same length as $y_1 y_2$ (see Fig. 3(a)). Then

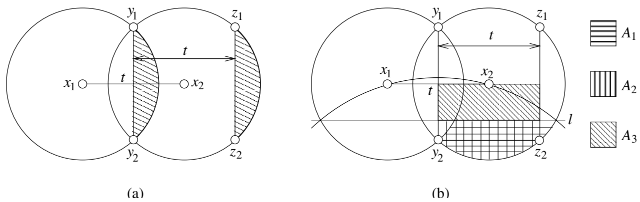

Fig. 3. The area of two intersecting disks.

$f(t)$ is also equal to the area of the portion of $D(x_2, r)$ between the two chords y_1y_2 and z_1z_2. Thus, $f'(t) = \|y_1y_2\|$, which is decreasing over $[0, 2r]$. Therefore, $f(t)$ is concave over $[0, 2r]$. Since $f(0) = 0$ and $f(2r) = \pi r^2$, we have $f(t) \geq (\pi/2) rt$.

Now we are ready to prove the lemma for $k = 2$. If $x_1 \in \Omega(0)$, then $\nu_r(x_1, x_2) - \nu_r(x_1)$ is exactly $f(t)$, and thus the lemma follows immediately from $f(t) \geq (\pi/2) rt$. So we assume that $x_1 \notin \Omega(0)$. Note that for the same distance t, $\nu_r(x_1, x_2) - \nu_r(x_1)$ achieves its minimum when both x_1 and x_2 are in $\partial\Omega$. It is sufficient to prove the lemma for $x_1, x_2 \in \partial\Omega$. Let y_1y_2 and z_1z_2 be the two chords of $\partial D(x_2, r)$ as above with $y_2 \in \Omega$, and ℓ be the line through the two intersection points between $\partial\Omega$ and $\partial(D(x_1, r) \cup D(x_2, r))$ (see Fig. 3(b)). We use A_1 to denote the portion of $D(x_2, r) \setminus D(x_1, r)$ which lie in the same side of ℓ as y_2; use A_2 to denote the portion of $D(x_2, r)$ which is surrounded by y_1y_2, z_1z_2, ℓ and the short arc between y_2 and z_2; and use A_3 to denote the rectangle surrounded by y_1y_2, z_1z_2, ℓ and the line through x_1 and x_2. Then

$$\nu_r(x_1, x_2) - \nu_r(x_1) \geq |A_1| = |A_2| = f(t)/2 - |A_3|.$$

Note that one side of A_3 is exactly t and the other side is at most

$$\frac{1}{\sqrt{\pi}} - \sqrt{\frac{1}{\pi} - (2r)^2} = \frac{4r^2}{\frac{1}{\sqrt{\pi}} + \sqrt{\frac{1}{\pi} - (2r)^2}}.$$

Thus,

$$\nu_r(x_1, x_2) - \nu_r(x_1) \geq \left(\frac{\pi}{4} - \frac{4r}{\frac{1}{\sqrt{\pi}} + \sqrt{\frac{1}{\pi} - (2r)^2}} \right) rt.$$

It is straightforward to verify that if

$$r \leq \frac{1/\sqrt{\pi}}{12/\pi + \pi/12} \approx 0.245/\sqrt{\pi},$$

then

$$\frac{\pi}{4} - \frac{4r}{\frac{1}{\sqrt{\pi}} + \sqrt{\frac{1}{\pi} - (2r)^2}} \geq \frac{\pi}{12},$$

and thereby the lemma follows.

In the next, we assume the lemma is true for at most $k-1$ nodes and we shall show that the lemma is true for k nodes. If $k = 3$, then

$$\begin{aligned} \nu_r(x_1, x_2, x_3) &\geq \nu_r(x_1) + \nu_r(x_3) \geq \nu_r(x_1) + \frac{\pi r^2}{3} \\ &= \nu_r(x_1) + \frac{\pi}{12} r \cdot 4r \geq \nu_r(x_1) + \frac{\pi}{12} r \sum_{i=1}^{2} \|x_{i+1} - x_i\|. \end{aligned}$$

If $k > 3$, then by the induction hypothesis

$$\begin{aligned} \nu_r(x_1, \cdots, x_k) &\geq \nu_r(x_1, \cdots, x_{k-2}) + \nu_r(x_k) \\ &\geq \nu_r(x_1) + \frac{\pi}{12} r \sum_{i=1}^{k-3} \|x_{i+1} - x_i\| + \frac{\pi r^2}{3} \\ &\geq \nu_r(x_1) + \frac{\pi}{12} r \sum_{i=1}^{k-1} \|x_{i+1} - x_i\|. \end{aligned}$$

Therefore, the lemma is true by induction. ■

Corollary 3: Assume that

$$r \leq \frac{1/\sqrt{\pi}}{12/\pi + \pi/12} \approx 0.245/\sqrt{\pi}.$$

Then for any $(x_1, \cdots, x_k) \in C_{k1}$ with x_1 being the one of the largest norm among $x_1, \cdots, x_k$,

$$\nu_r(x_1, \cdots, x_k) \geq \nu_r(x_1) + \frac{\pi}{12} r \max_{2 \leq i \leq k} \|x_i - x_1\|.$$

Proof: Without loss of generality, we assume that $\|x_k - x_1\|$ achieves $\max_{2 \leq i \leq k} \|x_i - x_1\|$. Let P be a min-hop path between x_1 and x_k in $G_{2r}(x_1, x_2, \cdots, x_k)$ and t be the total length of P. Then every pair of nodes in P that are not adjacent nodes in P are separated by a distance of more than $2r$. Thus by applying Lemma 2 to the nodes in P, we obtain

$$\nu_r(\{x_i \mid x_i \in P\}) \geq \nu_r(x_1) + \frac{\pi}{12} rt.$$

Since $\nu_r(x_1, \cdots, x_k) \geq \nu_r(\{x_i \mid x_i \in P\})$, and $t \geq \|x_k - x_1\|$, the corollary follows. ■

In the remaining of this section, we give the limits of several integrals.

Lemma 4: For any $z \in \left[0, \frac{1}{2}\right]$, $e^{-z-z^2} \leq 1 - z \leq e^{-z}$.

Proof: For any $z \geq 0$, $1 - z \leq e^{-z} \leq 1 - z + \frac{z^2}{2}$. If $z \in \left[0, \frac{1}{2}\right]$, then

$$\begin{aligned} e^{-z-z^2} &\leq \left(1 - z + \frac{z^2}{2}\right)\left(1 - z^2 + \frac{z^4}{2}\right) \\ &= 1 - z - z^2\left(\frac{1}{2} - z\right) - \frac{1}{4} z^5 (2 - z) \leq 1 - z. \end{aligned}$$

■

Lemma 5: Let $r = \sqrt{\frac{\ln n + \xi}{\pi p n}}$ for some constant ξ. Then

$$n \int_\Omega e^{-np\nu_r(x)} dx \sim e^{-\xi},$$

$$n \int_\Omega (1 - p\nu_r(x))^{n-1} dx \sim e^{-\xi}.$$

Proof: We only give the proof of the first asymptotic equality. The second one can be proved in the similar manner together with the inequalities in Lemma 4. First, we calculate the integration over $\Omega(0)$.

$$n\int_{\Omega(0)} e^{-np\nu_r(x)}dx = ne^{-np\pi r^2}\left|\Omega(0)\right| \sim ne^{-np\pi r^2} = e^{-\xi}.$$

Now, we calculate the integration over $\Omega(2)$.

$$n\int_{\Omega(2)} e^{-np\nu_r(x)}dx \le ne^{-\frac{1}{3}np\pi r^2}\left|\Omega(2)\right| = n\pi r^2 e^{-\frac{1}{3}np\pi r^2} = o(1).$$

Next, we calculate the integration over $\Omega(1)$. By Lemma 1,

$$\begin{aligned}
& n\int_{\Omega(1)} e^{-np\nu_r(x)}dx \\
&\le ne^{-\frac{np\pi r^2}{2}}\int_{\Omega(1)} e^{-npr\left(\frac{1}{\sqrt{\pi}}-\|x\|\right)}dx \\
&= 2\pi ne^{-\frac{np\pi r^2}{2}}\int_{\frac{1}{\sqrt{\pi}}-r}^{\sqrt{\frac{1}{\pi}-r^2}} \rho e^{-npr\left(\frac{1}{\sqrt{\pi}}-\rho\right)}d\rho \\
&\le 2\pi ne^{-\frac{np\pi r^2}{2}}\int_{\frac{1}{\sqrt{\pi}}-r}^{\frac{1}{\sqrt{\pi}}} \rho e^{-npr\left(\frac{1}{\sqrt{\pi}}-\rho\right)}d\rho \\
&\le 2\sqrt{\pi} ne^{-\frac{np\pi r^2}{2}}\int_{\frac{1}{\sqrt{\pi}}-r}^{\frac{1}{\sqrt{\pi}}} e^{-npr\left(\frac{1}{\sqrt{\pi}}-\rho\right)}d\rho \\
&= 2\sqrt{\pi} ne^{-\frac{np\pi r^2}{2}}\int_0^r e^{-nprt}dt \\
&\le \frac{2\sqrt{\pi}}{p}\frac{1}{r}e^{-\frac{np\pi r^2}{2}} = O(1)(\log n)^{-1/2} = o(1).
\end{aligned}$$

Therefore,

$$n\int_{\Omega} e^{-np\nu_r(x)}dx \sim e^{-\xi}.$$

■

Lemma 6: Let $r = \sqrt{\frac{\ln n+\xi}{\pi pn}}$ for some constant ξ. Then for any fixed integer $k \ge 2$,

$$n^k\int_{C_{k1}} e^{-np\nu_r(x_1,x_2,\cdots,x_k)}\prod_{i=1}^k dx_i = o(1),$$

$$n^k\int_{C_{k1}} \left(1-p\nu_r(x_1,x_2,\cdots,x_k)\right)^{n-k}\prod_{i=1}^k dx_i = o(1).$$

Proof: Since

$$\left(1-p\nu_r(x_1,x_2,\cdots,x_k)\right)^{n-k} \le \frac{e^{-np\nu_r(x_1,x_2,\cdots,x_k)}}{\left(1-pk\pi r^2\right)^k},$$

the second equality would follow from the first one. Hence, we only have to prove the first one. Let S denote the set of $(x_1,x_2,\cdots,x_k)\in C_{k1}$ satisfying that x_1 is the one with largest norm among $x_1,\cdots,x_k$ and x_2 is the one with longest distance from x_1 among $x_2,\cdots,x_k$. Then

$$\begin{aligned}
& n^k\int_{C_{k1}} e^{-np\nu_r(x_1,x_2,\cdots,x_k)}\prod_{i=1}^k dx_i \\
&\le k(k-1)n^k\int_S e^{-np\nu_r(x_1,x_2,\cdots,x_k)}\prod_{i=1}^k dx_i.
\end{aligned}$$

So it suffices to prove

$$n^k\int_S e^{-np\nu_r(x_1,x_2,\cdots,x_k)}\prod_{i=1}^k dx_i = o(1).$$

Note that for any $(x_1,x_2,\cdots,x_k)\in S$,

$$\nu_r(x_1) + cr\|x_2-x_1\| \le \nu_r(x_1,x_2,\cdots,x_k) \le k\pi r^2$$

for some constant c by Corollary 3, and

$$\begin{aligned}
& x_i \in B(x_1,\|x_2-x_1\|), 3\le i\le k; \\
& x_2 \in B(x_1, 2(k-1)r).
\end{aligned}$$

Thus,

$$\begin{aligned}
& n^k\int_S e^{-np\nu_r(x_1,x_2,\cdots,x_k)}\prod_{i=1}^k dx_i \\
&\le n^k\int_S e^{-np(\nu_r(x_1)+cr\|x_2-x_1\|)}\prod_{i=1}^k dx_i \\
&\le n^k\int_\Omega e^{-np\nu_r(x_1)}dx_1\int_{B(x_1,2(k-1)r)} e^{-npcr\|x_2-x_1\|}dx_2 \\
&\quad \prod_{i=3}^k\int_{B(x_1,\|x_2-x_1\|)}dx_i \\
&= n^k\int_\Omega e^{-np\nu_r(x_1)}dx_1 \\
&\quad \int_{B(x_1,2(k-1)r)} e^{-npcr\|x_2-x_1\|}\left(\pi\|x_2-x_1\|^2\right)^{k-2}dx_2 \\
&= 2\pi^{k-1}\left(n\int_\Omega e^{-np\nu_r(x_1)}dx_1\right) \\
&\quad \left(n^{k-1}\int_0^{2(k-1)r} e^{-npcr\rho}\rho^{2k-3}d\rho\right) \\
&< 2\pi^{k-1}\left(n\int_\Omega e^{-np\nu_r(x_1)}dx_1\right) \\
&\quad \left(n^{k-1}\int_0^\infty e^{-npcr\rho}\rho^{2k-3}d\rho\right) \\
&= \frac{(2k-3)!2\pi^{k-1}n^{k-1}}{(npcr)^{2k-2}}\left(n\int_\Omega e^{-np\nu_r(x_1)}dx_1\right) \\
&= O(1)\frac{n\int_\Omega e^{-np\nu_r(x_1)}dx_1}{(\ln n)^{k-1}} = o(1),
\end{aligned}$$

where the last equality follows from Lemma 5. ■

Lemma 7: Let $r = \sqrt{\frac{\ln n+\xi}{\pi pn}}$ for some constant ξ. Then for any fixed integers $2 \le m < k$.

$$n^k\int_{C_{km}} e^{-np\nu_r(x_1,x_2,\cdots,x_k)}\prod_{i=1}^k dx_i = o(1),$$

$$n^k\int_{C_{km}} \left(1-p\nu_r(x_1,x_2,\cdots,x_k)\right)^{n-k}\prod_{i=1}^k dx_i = o(1).$$

Proof: Since

$$\left(1-p\nu_r(x_1,x_2,\cdots,x_k)\right)^{n-k} \le \frac{e^{-np\nu_r(x_1,x_2,\cdots,x_k)}}{\left(1-pk\pi r^2\right)^k},$$

the second equality would follow from the first one, and thus we only have to prove the first one. For any m-partition $\Pi = \{K_1, K_2, \cdots, K_m\}$ of $\{1, 2, \cdots, k\}$, let $\Omega^k(\Pi)$ denote the set of $(x_1, x_2, \cdots, x_k) \in \Omega^k$ such that for any $1 \le j \le m$, the nodes $\{x_i : i \in K_j\}$ form a connected component of $G_{2r}(x_1, x_2, \cdots, x_k)$. Then C_{km} is the union of $\Omega^k(\Pi)$ over all m-partitions Π of $\{1, 2, \cdots, k\}$. So it is sufficient to show that for any m-partition Π of $\{1, 2, \cdots, k\}$,

$$n^k \int_{\Omega^k(\Pi)} e^{-np\nu_r(x_1,x_2,\cdots,x_k)} \prod_{i=1}^{k} dx_i = o(1)$$

Now fix a m-partition $\Pi = \{K_1, K_2, \cdots, K_m\}$ of $\{1, 2, \cdots, k\}$, and let $l_j = |K_j|$ for $1 \le j \le m$. Then,

$$\Omega^k(\Pi) \subseteq \prod_{j=1}^{m} C_{l_j 1},$$

and for any $(x_1, x_2, \cdots, x_k) \in \Omega^k(\Pi)$,

$$\nu_r(x_1, x_2, \cdots, x_k) = \sum_{j=1}^{m} \nu_r(\{x_i \mid i \in K_j\}).$$

Thus,

$$\begin{aligned}
& n^k \int_{\Omega^k(\Pi)} e^{-np\nu_r(x_1,x_2,\cdots,x_k)} \prod_{i=1}^{k} dx_i \\
&= n^k \int_{\Omega^k(\Pi)} e^{-np\sum_{j=1}^{m} \nu_r(\{x_i|i\in K_j\})} \prod_{i=1}^{k} dx_i \\
&= n^k \int_{\Omega^k(\Pi)} \prod_{j=1}^{m} e^{-np\nu_r(\{x_i|i\in K_j\})} \prod_{i=1}^{k} dx_i \\
&\le n^k \prod_{j=1}^{m} \int_{C_{l_j 1}} e^{-np\nu_r(\{x_i|i\in K_j\})} \prod_{i\in K_j} dx_i \\
&= \prod_{j=1}^{m} \left(n^{l_j} \int_{C_{l_j 1}} e^{-np\nu_r(\{x_i|i\in K_j\})} \prod_{i\in K_j} dx_i \right) = o(1),
\end{aligned}$$

where the last equality follows from Lemma 6 and the fact that at least one $l_j \ge 2$. ∎

Lemma 8: Let $r = \sqrt{\frac{\ln n + \xi}{\pi p n}}$ for some constant ξ. Then for any fixed integer $k \ge 2$,

$$n^k \int_{C_{kk}} e^{-np\nu_r(x_1,x_2,\cdots,x_k)} \prod_{i=1}^{k} dx_i \sim e^{-k\xi},$$

$$n^k \int_{C_{kk}} (1 - p\nu_r(x_1, x_2, \cdots, x_k))^{n-k} \prod_{i=1}^{k} dx_i \sim e^{-k\xi}.$$

Proof: We again only give the proof of the first asymptotic equality and remark that he second one can be proved in the similar manner together with the inequalities in Lemma 4. For any $(x_1, x_2, \cdots, x_k) \in C_{kk}$,

$$\nu_r(x_1, x_2, \cdots, x_k) = \sum_{i=1}^{k} \nu_r(x_i).$$

Thus,

$$\begin{aligned}
& n^k \int_{C_{kk}} e^{-np\nu_r(x_1,x_2,\cdots,x_k)} \prod_{i=1}^{k} dx_i \\
&= n^k \int_{C_{kk}} e^{-np\sum_{i=1}^{k} \nu_r(x_i)} \prod_{i=1}^{k} dx_i \\
&= n^k \int_{\Omega^k} e^{-np\sum_{i=1}^{k} \nu_r(x_i)} \prod_{i=1}^{k} dx_i \\
&\quad - n^k \int_{\Omega^k \setminus C_{kk}} e^{-np\sum_{i=1}^{k} \nu_r(x_i)} \prod_{i=1}^{k} dx_i.
\end{aligned}$$

We show the first term is asymptotically equal to $e^{-k\xi}$, and the second term is asymptotically negligible. Indeed,

$$\begin{aligned}
& n^k \int_{\Omega^k} e^{-np\sum_{i=1}^{k} \nu_r(x_i)} \prod_{i=1}^{k} dx_i \\
&= n^k \int_{\Omega^k} \prod_{i=1}^{k} e^{-np\nu_r(x_i)} \prod_{i=1}^{k} dx_i \\
&= \prod_{i=1}^{k} \int_{\Omega} e^{-np\nu_r(x_i)} dx_i \sim e^{-k\xi},
\end{aligned}$$

where the last equality follows from Lemma 5. Note that for any $(x_1, x_2, \cdots, x_k) \in \Omega^k \setminus C_{kk}$,

$$\nu_r(x_1, x_2, \cdots, x_k) \le \sum_{i=1}^{k} \nu_r(x_i).$$

Thus,

$$\begin{aligned}
& n^k \int_{\Omega^k \setminus C_{kk}} e^{-np\sum_{i=1}^{k} \nu_r(x_i)} \prod_{i=1}^{k} dx_i \\
&\le n^k \int_{\Omega^k \setminus C_{kk}} e^{-np\nu_r(x_1,x_2,\cdots,x_k)} \prod_{i=1}^{k} dx_i \\
&= \sum_{m=1}^{k-1} \int_{C_{km}} e^{-np\nu_r(x_1,x_2,\cdots,x_k)} \prod_{i=1}^{k} dx_i = o(1),
\end{aligned}$$

where the last equality follows from Lemma 6. ∎

III. Asymptotic Distribution of The Number of Isolated Nodes

The main result of this paper is the following theorem.

Theorem 9: Suppose that all nodes have a maximum transmission radius $r = \sqrt{\frac{\ln n + \xi}{\pi p n}}$ for some constant ξ. Then the total number of isolated nodes is asymptotically Poisson with mean $e^{-\xi}$, and the total number of isolated active nodes is also asymptotically Poisson with mean $pe^{-\xi}$.

The above theorem will be proved by using *Brun's sieve* in the form described, for example, in [1], Chapter 8, which is an implication of the Bonferroni inequalities.

Theorem 10: Let $B_1, \cdots, B_n$ be events and Y be the number of B_i that hold. Suppose that for any set $\{i_1, \cdots, i_k\} \subseteq \{1, \cdots, n\}$

$$\Pr\left(B_{i_1} \wedge \cdots \wedge B_{i_k}\right) = \Pr\left(B_1 \wedge \cdots \wedge B_k\right),$$

and there is a constant μ so that for any fixed k

$$n^k \Pr\left(B_1 \wedge \cdots \wedge B_k\right) \sim \mu^k.$$

Then Y is also asymptotically Poisson with mean μ.

For applying Theorem 10, let B_i be the event that X_i is isolated for $1 \leq i \leq n$ and Y be the number of B_i that hold. Then Y is exactly the number of isolated nodes. Similarly, let B'_i be the event that X_i is isolated and active for $1 \leq i \leq n$ and Y' be the number of B_i that hold. Then Y' is exactly the number of isolated active nodes. Obviously, for any set $\{i_1, \cdots, i_k\} \subseteq \{1, \cdots, n\}$,

$$\Pr\left(B_{i_1} \wedge \cdots \wedge B_{i_k}\right) = \Pr\left(B_1 \wedge \cdots \wedge B_k\right),$$
$$\Pr\left(B'_{i_1} \wedge \cdots \wedge B'_{i_k}\right) = \Pr\left(B'_1 \wedge \cdots \wedge B'_k\right).$$

In addition,

$$\Pr\left(B'_1 \wedge \cdots \wedge B'_k\right) = p^k \Pr\left(B_1 \wedge \cdots \wedge B_k\right).$$

Thus, in order to prove Theorem 9, it suffices to show that if $r = \sqrt{\frac{\ln n + \xi}{\pi p n}}$ for some constant ξ, then for any fixed k,

$$n^k \Pr\left(B_1 \wedge \cdots \wedge B_k\right) \sim e^{-k\xi}. \quad (1)$$

The proof of this asymptotic equality will use the following two lemmas.

Lemma 11: For any $x \in \Omega$,

$$\Pr\left(B_1 \mid X_1 = x\right) = \left(1 - p v_r(x)\right)^{n-1}.$$

Proof: For any $x \in \Omega$,

$$\begin{aligned}
&\Pr\left(B_1 \mid X_1 = x\right)\\
&= \Pr\left(\forall 2 \leq i \leq n, X_i \text{ is either outside } v_r(x) \text{ or inactive}\right)\\
&= \sum_{i=0}^{n-1} q^i \binom{n-1}{i} \left(1 - v_r(x)\right)^{n-1-i} v_r(x)^i \, dx\\
&= \left(1 - v_r(x) + q v_r(x)\right)^{n-1} = \left(1 - p v_r(x)\right)^{n-1}.
\end{aligned}$$

■

Lemma 12: For any $k \geq 2$ and $(x_1, \cdots, x_k) \in \Omega^k$,

$$\begin{aligned}
&\Pr\left(B_1 \wedge \cdots \wedge B_k \mid X_i = x_i, 1 \leq i \leq k\right)\\
&\leq \left(1 - p\nu_r(x_1, \cdots, x_k)\right)^{n-k}.
\end{aligned}$$

In addition, the equality is achieved for $(x_1, \cdots, x_k) \in C_{kk}$.

Proof: For any $(x_1, \cdots, x_k) \in \Omega^k$,

$$\begin{aligned}
&\Pr\left(B_1 \wedge \cdots \wedge B_k \mid X_i = x_i, 1 \leq i \leq k\right)\\
&\leq \Pr\left(\begin{array}{l} \nu_r(x_1, \cdots, x_k) \text{ contains no active node} \\ \text{in } X_{k+1}, \cdots, X_n \end{array}\right)\\
&= \sum_{j=0}^{n-k} q^j \binom{n-k}{j} \left(1 - \nu_r(x_1, \cdots, x_k)\right)^{n-k-j}\\
&\qquad \nu_r(x_1, \cdots, x_k)^j\\
&= \left(1 - p\nu_r(x_1, \cdots, x_k)\right)^{n-k}.
\end{aligned}$$

For any $(x_1, \cdots, x_k) \in C_{kk}$,

$$\begin{aligned}
&\Pr\left(B_1 \wedge \cdots \wedge B_k \mid X_i = x_i, 1 \leq i \leq k\right)\\
&= \Pr\left(\begin{array}{l} \forall 1 \leq i \leq k,\ \nu_r(x_i) \text{ contains no active node} \\ \text{in } X_{k+1}, \cdots, X_n \end{array}\right)\\
&= \sum_{m_1 + \cdots + m_k = 0}^{n-k} \Pr\left(\begin{array}{l} \forall 1 \leq i \leq k,\ \nu_r(x_i) \text{ contains } m_i \\ \text{inactive nodes and no active nodes} \\ \text{in } X_{k+1}, \cdots, X_n \end{array}\right)\\
&= \sum_{m_1 + \cdots + m_k = 0}^{n-k} \binom{n-k}{m_1, \cdots, m_k} \left(\prod_{i=1}^{k} \left(q^{m_i} \nu_r(x_i)^{m_i}\right)\right)\\
&\qquad \left(1 - \nu_r(x_1, \cdots, x_k)\right)^{n-k-\sum_{i=1}^{k} m_i}\\
&= \left(1 - p\nu_r(x_1, \cdots, x_k)\right)^{n-k}.
\end{aligned}$$

■

Now we are ready to prove the asymptotic equality (1). From Lemma 11 and Lemma 5,

$$n \Pr\left(B_1\right) = n \int_{\Omega} \left(1 - p v_r(x)\right)^{n-1} dx \sim e^{-\xi}.$$

So the asymptotic equality (1) is true for $k = 1$. Now we fix $k \geq 2$. From Lemma 12, Lemma 6 and Lemma 7,

$$\begin{aligned}
&n^k \Pr\left(B_1 \wedge \cdots \wedge B_k, \text{and } (X_1, \cdots, X_k) \in \Omega^k \setminus C_{kk}\right)\\
&\leq n^k \int_{\Omega^k \setminus C_{kk}} \left(1 - p\nu_r(x_1, \cdots, x_k)\right)^{n-k} \prod_{i=1}^{k} dx_i = o(1).
\end{aligned}$$

From Lemma 12 and Lemma 8,

$$\begin{aligned}
&n^k \Pr\left(B_1 \wedge \cdots \wedge B_k, \text{and } (X_1, \cdots, X_k) \in C_{kk}\right)\\
&= n^k \int_{C_{kk}} \left(1 - p\nu_r(x_1, \cdots, x_k)\right)^{n-k} \prod_{i=1}^{k} dx_i \sim e^{-k\xi}.
\end{aligned}$$

Thus, the asymptotic equality (1) is also true for any fixed $k \geq 2$. This completes the proof of Theorem 9.

REFERENCES

[1] N. Alon, and J.H. Spencer, The Probabilistic Method, Second Edition, Wiley, New York, (2000).

[2] B. Bollobás, *Random Graphs*, Second Edition, Cambridge University Press, 2001.

[3] H. Dette, and N. Henze, The limit distribution of the largest nearest neighbor link in the unit *d*-cube, *J. Appl. Probab.*, vol. 26, pp. 67-80, 1989.

[4] P. Erdős, and A. Rényi, On the evolution of random graphs, *Publ. Math. Inst. Hung. Acad. Sci.* 5, 17-61, 1960.

[5] E.N. Gilbert, Random plane networks, *Journal of the Society for Industrial and Applied Mathematics* 9:4 (1961), 533-543.

[6] P. Gupta, and P.R. Kumar, Critical power for asymptotic connectivity in wireless networks, *Stochastic Analysis, Control, Optimization and Applications: A Volume in Honor of W.H. Fleming*, W. M. McEneaney, G. Yin, and Q. Zhang (Eds.) (1998).

[7] M.D. Penrose, The longest edge of the random minimal spanning tree, *Ann. Appl. Probab.*, vol. 7, pp. 340-361, 1997.