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Greedy approximations for minimum submodular cover with submodular cost

Peng-Jun Wan · Ding-Zhu Du · Panos Pardalos · Weili Wu

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Abstract It is well-known that a greedy approximation with an integer-valued polymatroid potential function f is $H(\gamma)$-approximation of the minimum submodular cover problem with linear cost where γ is the maximum value of f over all singletons and $H(\gamma)$ is the γ-th harmonic number. In this paper, we establish similar results for the minimum submodular cover problem with a submodular cost (possibly nonlinear) and/or fractional submodular potential function f.

Keywords Greedy approximations · Minimum submodular cover

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P.-J. Wan
Department of Computer Science, Illinois Institute of Technology, Chicago, IL 60616, USA
e-mail: wan@cs.iit.edu

D.-Z. Du · W. Wu
Department of Computer Science, University of Texas at Dallas, Richardson, TX 75083, USA

D.-Z. Du
e-mail: dzdu@utdallas.edu

P. Pardalos (✉)
Department of Industrial Engineering and System Science, University of Florida, Gainsville, FL 32611, USA
e-mail: pardalos@ufl.edu

1 Introduction

Consider a ground set E and a real function f defined on 2^E. f is *increasing* if for $X \subset Y$, $f(X) \leq f(Y)$. f is *submodular* if for any two subsets X and Y of E,

$$f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y).$$

The *marginal value* of $Y \subseteq E$ with respect to $X \subseteq E$ is defined by

$$\Delta_Y f(X) = f(X \cup Y) - f(X).$$

Similarly, the *marginal value* of an element $e \in E$ with respect to $X \subseteq E$ is defined by

$$\Delta_e f(X) = f(X \cup \{e\}) - f(X).$$

Both monotonicity and submodularity of a function f can be characterized in terms of the marginal values (see, e.g., [1, 4–6]). f is increasing if and only if $\Delta_e f(X) \geq 0$ for any $X \subseteq E$ and $e \in E \setminus X$. f is submodular if and only if for any $X \subseteq E$ and different $a, b \in E \setminus X$.

$$\Delta_a f(X) \geq \Delta_a f(X \cup \{b\}).$$

In addition, the following are equivalent:

- f is increasing and submodular.
- For any $X, Y \subseteq E$,

$$f(Y) - f(X) \leq \sum_{y \in Y \setminus X} \Delta_y f(X).$$

- For any $X \subseteq E$ and $a, b \in E \setminus X$,

$$\Delta_a f(X) \geq \Delta_a f(X \cup \{b\}).$$

A submodular and increasing function f is called a *polymatroid function* if $f(\emptyset) = 0$. Suppose that f is a polymatroid functions on 2^E. Then, a set $X \subseteq E$ is said to be a submodular cover of (E, f) if $f(X) = f(E)$. Suppose that both f and c are polymatroid functions on 2^E. The minimization problem

$$\min\{c(X) : f(X) = f(E), X \subseteq E\}$$

is known as a *Minimum Submodular Cover with Submodular Cost* (MSC/SC). A greedy approximation for it is described in Table 1. We remark that $|E|$ may be not polynomial. In this case, we assume that there is polynomial-time oracle to compute an $x \in E$ with maximum $\Delta_x f(X)/c(x)$ for any $X \subset E$ with polynomial $|X|$. When c is linear and f is integer-valued, it is well-known that the algorithm *GSC* produces an $H(\gamma)$-approximation solution, where

$$\gamma = \max_{x \in E} f(e)$$

Table 1 Greedy algorithm for Minimum Submodular Cover

Greedy Algorithm GSC
$X \leftarrow \emptyset$;
While $\exists e \in E$ such that $\Delta_e f(X) > 0$ do
select $x \in E$ with maximum $\Delta_x f(X)/c(x)$;
$X \leftarrow X \cup \{x\}$;
Output X.

and

$$H(k) = 1 + \frac{1}{2} + \cdots + \frac{1}{k}$$

is the k-th Harmonic number [5]. In this paper, we establish similar results for the minimum submodular cover problem with a submodular cost (possibly nonlinear) and/or fractional submodular potential function f. Define the curvature of the submodular cost c to be

$$\rho = \min_{S\text{:min-cost cover}} \frac{\sum_{e \in S} c(e)}{c(S)}.$$

Note that if c is linear (i.e., modular), then $\rho = 1$. This paper contains the following three contributions:

1. Analysis of the greedy algorithm for integral submodular cover with submodular cost. The charging argument is new and considerably simpler than all the known proofs for the linear-cost variant in the literature.
2. Analysis of the greedy algorithm for fractional submodular cover with submodular cost.
3. Application of the first result to obtaining a tighter approximation bound for a power assignment problem.

2 Integral submodular cover

In this section, we first show a general result on integral submodular cover, and then present a real-world problem as an example of submodular cover problem with submodular cost.

Theorem 2.1 *If f is integer-valued, then the greedy solution of GSC is a $\rho H(\gamma)$-approximation where $\gamma = \max_{e \in E} f(e)$.*

Proof Let $x_1, x_2, \ldots, x_k$ be the sequence of elements selected by the greedy algorithm, and S be a cover of minimum cost satisfying

$$\sum_{e \in S} c(e) = \rho \cdot c(S).$$

We prove

$$c(X) \le \rho H(\gamma) \cdot c(S)$$

by a charging argument. Set $X_0 = \emptyset$, and $X_i = \{x_j : 1 \le j \le i\}$ for each $1 \le i \le k$. Denote $\mu_0 = 0$ and $\mu_i = \frac{c(x_i)}{\Delta_{x_i} f(X_{i-1})}$ for each $1 \le i \le k$. The parameter μ_i is referred to as the average price per increment of coverage by x_i for each $1 \le i \le k$. We claim that

$$\mu_0 \le \mu_1 \le \mu_2 \le \cdots \le \mu_k.$$

Indeed, the first inequality is trivial. For any $1 \le i < k$,

$$\mu_i = \frac{c(x_i)}{\Delta_{x_i} f(X_{i-1})} \le \frac{c(x_{i+1})}{\Delta_{x_{i+1}} f(X_{i-1})} \le \frac{c(x_{i+1})}{\Delta_{x_{i+1}} f(X_i)} = \mu_{i+1},$$

where the first inequality follows from the greedy rule and the second inequality follows from the submodularity of f. Thus, our claim holds. Now for iteration i with $1 \le i \le k$, we charge each $e \in S$ with $\mu_i(\Delta_e f(X_{i-1}) - \Delta_e f(X_i))$. Then, the total charge on each $e \in S$ is

$$\sum_{i=1}^{k} \mu_i(\Delta_e f(X_{i-1}) - \Delta_e f(X_i)),$$

and the total charge on S is

$$\sum_{e \in S} \sum_{i=1}^{k} \mu_i(\Delta_e f(X_{i-1}) - \Delta_e f(X_i)).$$

We claim that

1. $\sum_{i=1}^{k} c(x_i)$ is no more than the total charge on S.
2. The total charge on $e \in S$ is at most $H(\gamma)c(e)$.

The first claim is true because

$$\begin{aligned}
&\sum_{i=1}^{k} c(x_i) \\
&\quad = \sum_{i=1}^{k} \mu_i \Delta_{x_i} f(X_{i-1}) \\
&\quad = \sum_{i=1}^{k} \mu_i (f(X_i) - f(X_{i-1})) \\
&\quad = \sum_{i=1}^{k} \mu_i ((f(S) - f(X_{i-1})) - (f(S) - f(X_i)))
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{k} \mu_i (f(S) - f(X_{i-1})) - \sum_{i=1}^{k} \mu_i (f(S) - f(X_i)) \\
&= \sum_{i=1}^{k} \mu_i (f(S) - f(X_{i-1})) - \sum_{i=1}^{k-1} \mu_i (f(S) - f(X_i)) \quad (\text{as } f(X_k) = f(S)) \\
&= \sum_{i=1}^{k} \mu_i (f(S) - f(X_{i-1})) - \sum_{i=2}^{k} \mu_{i-1} (f(S) - f(X_{i-1})) \\
&= \sum_{i=1}^{k} \mu_i (f(S) - f(X_{i-1})) - \sum_{i=1}^{k} \mu_{i-1} (f(S) - f(X_{i-1})) \quad (\text{as } \mu_0 = 0) \\
&= \sum_{i=1}^{k} (\mu_i - \mu_{i-1})(f(S) - f(X_{i-1})) \\
&\le \sum_{i=1}^{k} (\mu_i - \mu_{i-1}) \sum_{e \in S} \Delta_e f(X_{i-1}) \\
&= \sum_{e \in S} \sum_{i=1}^{k} (\mu_i - \mu_{i-1}) \Delta_e f(X_{i-1}) \\
&= \sum_{e \in S} \left(\sum_{i=1}^{k} \mu_i \Delta_e f(X_{i-1}) - \sum_{i=1}^{k} \mu_{i-1} \Delta_e f(X_{i-1}) \right) \\
&= \sum_{e \in S} \left(\sum_{i=1}^{k} \mu_i \Delta_e f(X_{i-1}) - \sum_{i=2}^{k} \mu_{i-1} \Delta_e f(X_{i-1}) \right) \quad (\text{as } \mu_0 = 0) \\
&= \sum_{e \in S} \left(\sum_{i=1}^{k} \mu_i \Delta_e f(X_{i-1}) - \sum_{i=1}^{k-1} \mu_i \Delta_e f(X_i) \right) \\
&= \sum_{e \in S} \left(\sum_{i=1}^{k} \mu_i \Delta_e f(X_{i-1}) - \sum_{i=1}^{k} \mu_i \Delta_e f(X_i) \right) \quad (\text{as } \Delta_e f(X_k) = 0) \\
&= \sum_{e \in S} \sum_{i=1}^{k} \mu_i (\Delta_e f(X_{i-1}) - \Delta_e f(X_i)).
\end{aligned}
$$

Next, we prove the second claim. Consider an arbitrary element $e \in S$. Let l be the first i such that $\Delta_e f(X_i) = 0$. For each $1 \le i \le l$, by the greedy rule,

$$
\mu_i = \frac{c(x_i)}{\Delta_{x_i} f(X_{i-1})} \le \frac{c(e)}{\Delta_e f(X_{i-1})}.
$$

Hence,

$$\begin{aligned}
&\sum_{i=1}^{k}\mu_i(\Delta_e f(X_{i-1})-\Delta_e f(X_i))\\
&\quad=\sum_{i=1}^{l-1}\mu_i(\Delta_e f(X_{i-1})-\Delta_e f(X_i))+\mu_l\Delta_e f(X_{l-1})\\
&\qquad(\text{as } \Delta_e f(X_i)=0 \text{ with } i\geq l)\\
&\quad\leq\sum_{i=1}^{l-1}\frac{c(e)(\Delta_e f(X_{i-1})-\Delta_e f(X_i))}{\Delta_e f(X_{i-1})}+\frac{c(e)\Delta_e f(X_{l-1})}{\Delta_e f(X_{l-1})}\\
&\quad=c(e)\left(1+\sum_{i=1}^{l-1}\frac{\Delta_e f(X_{i-1})-\Delta_e f(X_i)}{\Delta_e f(X_{i-1})}\right)\\
&\quad\leq c(e)\left(1+\sum_{i=1}^{l-1}\sum_{j=0}^{\Delta_e f(X_{i-1})-\Delta_e f(X_i)-1}\frac{1}{\Delta_e f(X_{i-1})-j}\right)\\
&\quad=c(e)\left(1+\sum_{i=1}^{l-1}\sum_{j=\Delta_e f(X_i)+1}^{\Delta_e f(X_{i-1})}\frac{1}{j}\right)\\
&\quad=c(e)\left(1+\sum_{j=\Delta_e f(X_{l-1})+1}^{\Delta_e f(X_0)}\frac{1}{j}\right)\\
&\quad=c(e)\left(1+\sum_{j=\Delta_e f(X_{l-1})+1}^{\Delta_e f(\emptyset)}\frac{1}{j}\right)\\
&\quad=c(e)\left(1+\sum_{j=1}^{\Delta_e f(\emptyset)}\frac{1}{j}-\sum_{j=1}^{\Delta_e f(X_{l-1})}\frac{1}{j}\right)\\
&\quad=c(e)(1+H(\Delta_e f(\emptyset))-H(\Delta_e f(X_{l-1})))\\
&\quad\leq c(e)(1+H(\gamma)-H(1))\\
&\quad=c(e)H(\gamma).
\end{aligned}$$

So, the second claim also holds. The two claims imply that

$$\sum_{i=1}^{k}c(x_i)\leq H(\gamma)\sum_{e\in S}c(e)=\rho H(\gamma)\cdot c(S).$$

By the submodularity of c, we have

$$c(X) \leq \sum_{i=1}^{k} c(x_i) \leq \rho H(\gamma) \cdot c(S).$$

Thus, the theorem follows. □

Next, we give an application of above theorem. Let $D = (V, A; w)$ be any arc-weighted digraph with $w(e) > 0$ for any $e \in A$. Any subgraph H induces a power assignment p_H to V defined as follows: For each node $u \in V$ which is the tail node of at least one arc in H, $p_H(u) = \max_{uv \in H} w(uv)$; otherwise, $p_H(u) = 0$. The power cost of H is defined by $p(H) = \sum_{v \in V} p_H(v)$. We will treat each subgraph H of D as a subset of arcs in A. For any $B \subseteq A$, the power cost of B, denoted by $p(B)$, is the power cost of the subgraph of D induced by B. It's easy to verify that p is an increasing and submodular function on 2^A and $p(\emptyset) = 0$. The undirected version of a digraph D, denoted by $\overline{D}$, is the undirected graph obtained from D by ignoring the orientations of the arcs in D and then removing multiple edges between any pair of nodes. D is said to be weakly-connected if $\overline{D}$ is connected. The bidirected version of an undirected graph G, denoted by $\overrightarrow{G}$, is the digraph obtained from G by replacing every edge uv of G with two oppositely oriented arcs uv and vu. A digraph $D = (V, A)$ is said to be bidirected if $uv \in A$ implies $vu \in A$. Now, we introduce the problem *Min-Power Spanning Tree in Digraphs*. An instance of this problem is an arc-weighted, connected and bidirected graph $D = (V, A; w)$ with $w(e) > 0$ for any $e \in A$. The objective is to find a spanning tree T of $\overline{D} = (V, E)$ with minimum $p(\overrightarrow{T})$. This problem arises from the algorithmic study of maximum-life power scheduling for connectivity in wireless ad hoc networks [2]. It is at least as hard as SET COVER [3] and a $2(1 + \ln(n - 1))$-approximation for this problem was reported in [2]. In this section, we will apply Theorem 2.1 to obtain a greedy $2H(\Delta)$-approximation, where Δ is the maximum degree of $\overline{D}$ (or equivalently, the maximum in-degree or out-degree of D). The problem *Min-Power Spanning Tree in Digraphs* can be cast as a problem of MSC. Indeed, let r be the graphic matroid rank of $\overline{D}$, which is defined as follows. For any $F \subseteq E$, denote by $\kappa(V, F)$ the number of connected components of the graph (V, F), then

$$r(F) = |V| - \kappa(V, F).$$

For any $F \subseteq E$, define $c(F)$ to be the power cost of the bidirected version of the graph (V, F). Then, both r and c are increasing and submodular functions on 2^E with $r(\emptyset) = c(\emptyset) = 0$, and (V, F) is a connected spanning graph of $\overline{D}$ if and only if $r(F) = |V| - 1 = r(E)$. Thus, the problem is exactly

$$\min\{c(F) : r(F) = r(E), F \subseteq E\}.$$

However, if we apply the greedy algorithm naively with E as the ground set, Theorem 2.1 can only imply an upper bound Δ on the approximation ratio. Indeed, when E is the ground set, $\gamma = 1$ as r is a matroid rank, but the curvature of c can be as large as Δ. For example, we consider an instance of D in which $V = \{v_0, v_1, \ldots, v_n\}$,

$A = \{v_0 v_i : 1 \le i \le n\} \cup \{v_i v_0 : 1 \le i \le n\}$, $w(v_0 v_i) = 1$ and $w(v_i v_0) = \varepsilon$ for $1 \le i \le n$. Then, $\overline{D}$ is a star, which is also the (unique) optimal solution, and $\Delta = n$. Since $c(\overline{D}) = p(D) = 1 + \Delta\varepsilon$ and $c(e) = 1 + \varepsilon$ for every edge e of $\overline{D}$, the curvature of c is $\frac{\Delta(1+\varepsilon)}{1+\Delta\varepsilon}$ which tends to Δ as ε tends to 0. Thus, the approximation ratio is bounded by Δ. In the following, we describe how to apply the same Theorem 2.1 to obtain a greedy logarithmic approximation for the above problem. Instead of choosing E as the ground set, we choose the set $\mathcal{S}$ of all stars in $\overline{D}$ as the ground set. The ground set $\mathcal{S}$ may have exponential cardinality, but the greedy algorithm only uses it implicitly. We can extend both r and c to $2^{\mathcal{S}}$ in the straightforward manner. Specifically, for any subset of stars $\mathcal{F} \subseteq \mathcal{S}$, $r(\mathcal{F})$ is defined to be the graphic matroid rank of the union of the stars in $\mathcal{F}$, and $c(\mathcal{F})$ is defined to be the power cost of the bidirected version of the union of the stars in $\mathcal{F}$. Then, both r and c are also increasing and submodular functions on $2^{\mathcal{S}}$ with $r(\emptyset) = c(\emptyset) = 0$, and the union of the stars in a subset $\mathcal{F} \subseteq \mathcal{S}$ is a connected spanning graph of $\overline{D}$ if and only if $r(\mathcal{F}) = |V| - 1 = r(\mathcal{S})$. Thus, the problem can be formulated as

$$\min\{c(\mathcal{F}) : r(\mathcal{F}) = r(\mathcal{S}), \mathcal{F} \subseteq \mathcal{S}\}.$$

For such formulation, we claim that (1) $\gamma = \max_{S \in \mathcal{S}} r(S) = \Delta$ and (2) the curvature ρ of c is at most 2. The first claim follows from the fact that for any star S with the degree of the center equal to d, $r(S) = d$. The second claim follows from a decomposition argument. Let T be an optimal minimum spanning tree. Root T at an arbitrary node, and let U be the set of internal nodes in T and the root of T. For each $u \in U$, let T_u be the star consisting of the edges between u and its children in T. Then, $\{T_u : u \in U\}$ is a partition of T into stars. It's easy to show that

$$\sum_{u \in U} c(T_u) \le 2c(T).$$

Thus, the second claim holds. By Theorem 2.1, the approximation ratio of the greedy algorithm is at most $2H(\Delta)$. In the remaining of this section, we describe a polynomial-time oracle which computes a star $S \subset \overline{D}$ with maximum $\Delta_S r(H)/c(S)$ for any disconnected spanning subgraph H of $\overline{D}$. Suppose that H is a disconnected spanning subgraph of $\overline{D}$. Let V' be the set of nodes v with at least one neighbor in a different connected component of H from v. For each $v \in V'$, let $\Gamma(v)$ be the set of neighbors of v in $\overline{D}$ not belonging to the connected component of H containing v, and let

$$Q(v) = \{w(vu) : u \in \Gamma(v)\}.$$

For each $q \in Q(v)$, let

$$\Gamma(v, q) = \{u \in \Gamma(v) : w(vu) \le q\}.$$

In each connected component of H which contains at least one node in $\Gamma(v, q)$, choose a node $u \in \Gamma(v, q)$ with minimum $w(uv)$. Let $u_1, u_2, \ldots, u_l$ be those chosen nodes with

$$w(u_1 v) \le w(u_2 v) \le \cdots \le w(u_l v).$$

Compute $1 \le j \le l$ maximizing $j/(q + \sum_{i=1}^{j} w(u_i v))$, and let $S(v, q)$ be the star connecting v to $u_1, u_2, \ldots, u_j$. Then, compute $q \in Q(v)$ maximizing $\Delta_{S(v,q)} r(H)/c(S(v, q))$, and let $S(v)$ be the star $S(v, q)$. Finally, compute $v \in V'$ maximizing $\Delta_{S(v)} r(H)/c(S(v))$, and let S be the star $S(v)$. We claim that S is a star in $\overline{D}$ with maximum $\Delta_S r(H)/c(S)$. Indeed, consider an optimal star S' with head v_0 and j leaves $v_1, v_2, \ldots, v_j$ satisfying

$$w(v_1 v_0) \le w(v_2 v_0) \le \cdots \le w(v_j v_0).$$

Clearly, all the nodes in S' must belong to distinct connected components of H, and hence all the leaves of S' must belong to $\Gamma(v_0)$. Let

$$q' = \max_{1 \le i \le j} w(v_0 v_i).$$

Then, $q' \in Q(v_0)$ and all the leaves of S' belong to $\Gamma(v_0, q')$. Since

$$\frac{\Delta_{S'} r(H)}{c(S')} = \frac{j}{q' + \sum_{i=1}^{j} w(v_i v_0)},$$

the sum $\sum_{i=1}^{j} w(v_i v_0)$ must achieve the minimum over all sets of j nodes in $\Gamma(v_0, q')$ belonging to distinct connected components of H. Let $u_1, u_2, \ldots, u_j$ be the first j nodes chosen from $\Gamma(v_0, q')$ as in the above oracle. By the standard swapping argument, we can show that $w(v_i v_0) = w(u_i v_0)$ for each $1 \le i \le j$. Hence,

$$\frac{\Delta_{S'} r(H)}{c(S')} \le \frac{\Delta_{S(v_0,q')} r(H)}{c(S(v_0, q'))} \le \frac{\Delta_S r(H)}{c(S)}.$$

So, our claim holds.

3 Fractional submodular cover

In this section, we present a general result on fractional submodular cover.

Theorem 3.1 *Suppose that f is fractional and $f(E) \ge opt$ where opt is the cost of a minimum submodular cover. If in each iteration of the Greedy Algorithm GSC, the selected x always satisfies that $\Delta_x f(X)/c(x) \ge 1$, then the greedy solution is a $(1 + \rho \ln(f(E)/opt))$-approximation.*

Proof Let $x_1, x_2, \ldots, x_k$ be the sequence of elements selected by the greedy algorithm, and S be a cover of minimum cost satisfying

$$\sum_{e \in S} c(e) = \rho \cdot c(S) = \rho \cdot opt.$$

Set $X_0 = \emptyset$, and $X_i = \{x_j : 1 \le j \le i\}$ for each $1 \le i \le k$. For each $0 \le i \le k$, let $\ell_i = f(S) - f(X_i)$ be the "uncoverage" at the end of iteration i. We first claim that for each $1 \le i \le k$,

$$c(x_i) \le \min\left\{1, \frac{\rho \cdot opt}{\ell_{i-1}}\right\}(\ell_{i-1} - \ell_i).$$

Indeed,

$$\begin{aligned}
&\frac{\ell_{i-1} - \ell_i}{c(x_i)} \\
&\quad = \frac{\Delta_{x_i} f(X_{i-1})}{c(x_i)} \\
&\quad \ge \max_{e \in S} \frac{\Delta_e f(X_{i-1})}{c(e)} \\
&\quad \ge \frac{\sum_{e \in S} \Delta_e f(X_{i-1})}{\sum_{e \in S} c(e)} \\
&\quad \ge \frac{f(S) - f(X_{i-1})}{\sum_{e \in S} c(e)} \\
&\quad = \frac{f(S) - f(X_{i-1})}{\rho \cdot opt} \\
&\quad = \frac{\ell_{i-1}}{\rho \cdot opt}.
\end{aligned}$$

Thus,

$$c(x_i) \le \frac{\rho \cdot opt}{\ell_{i-1}}(\ell_{i-1} - \ell_i).$$

The other inequality $c(x_i) \le \ell_{i-1} - \ell_i$ follows from the assumption that $\Delta_{x_i} f(X_{i-1})/c(x_i) \ge 1$. Since

$$f(E) = \ell_0 > \ell_1 > \cdots > \ell_k = 0$$

and $f(E) \ge opt$ by assumption, there exists a unique index t satisfying $\ell_t \ge opt > \ell_{t+1}$. Using the inequalities

$$c(x_i) \le \frac{\rho \cdot opt}{\ell_{i-1}}(\ell_{i-1} - \ell_i)$$

for $1 \le i \le t+1$, we have

$$\begin{aligned}
&\sum_{i=1}^{t} c(x_i) + \frac{\ell_t - opt}{\ell_t - \ell_{t+1}} c(x_{t+1}) \\
&\quad \le \rho \cdot opt \left(\sum_{i=1}^{t} \frac{\ell_{i-1} - \ell_i}{\ell_{i-1}} + \frac{\ell_t - opt}{\ell_t} \right)
\end{aligned}$$

$$\le \rho \cdot opt \int_{opt}^{\ell_0} \frac{1}{y} dy$$
$$= \rho \cdot opt \ln \frac{\ell_0}{opt}$$
$$= \rho \cdot opt \ln \frac{f(E)}{opt}.$$

Using the inequalities $c(x_i) \le \ell_{i-1} - \ell_i$ for $t+1 \le i \le k$, we have

$$\frac{opt - \ell_{t+1}}{\ell_t - \ell_{t+1}} c(x_{t+1}) + \sum_{i=t+2}^{k} c(x_i)$$
$$\le opt - \ell_{t+1} + \sum_{i=t+2}^{k} (\ell_{i-1} - \ell_i)$$
$$= opt - \ell_k$$
$$= opt.$$

Hence,

$$\sum_{i=1}^{k} c(x_i)$$
$$= \sum_{i=1}^{t} c(x_i) + \frac{\ell_t - opt}{\ell_t - \ell_{t+1}} c(x_{t+1}) + \frac{opt - \ell_{t+1}}{\ell_t - \ell_{t+1}} c(x_{t+1}) + \sum_{i=t+2}^{k} c(x_i)$$
$$\le \rho \cdot opt \ln \frac{f(E)}{opt} + opt$$
$$= \left(1 + \rho \ln \frac{f(E)}{opt}\right) opt.$$

Thus, the theorem follows. □

In the study of Steiner trees, several greedy approximations have a fractional submodular potential function. In such a case, the above theorem may apply.

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