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# The Steiner ratio for the dual normed plane

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**Abstract**

A minimum Steiner tree for a given set $X$ of points is a network interconnecting the points of $X$ having minimal possible total length. The Steiner ratio for a metric space is the largest lower bound for the ratio of lengths between a minimum Steiner tree and a minimum spanning tree on the same set of points in the metric space. Du et al. (1993) conjectured that the Steiner ratio on a normed plane is equal to the Steiner ratio on its dual plane. In this paper we show that this conjecture is true for $|X| \leqslant 5$.

## 1. Introduction

Given a compact, convex, centrally symmetric domain $D$ in the Euclidean plane $E^2$, one can define a norm $\|\cdot\|_D : E^2 \to R$ by setting $\|x\|_D = \lambda$ where $x = \lambda u$ and $u \in \partial D$, the boundary of $D$. We can then define a metric $d_D$ on $E^2$ by taking

$$d_D(x, y) = \|x - y\|_D.$$

Thus $\partial D = \{x \mid \|x\|_D = 1\}$. The resulting metric space $M = M(D) = \left(E^2, d_D\right)$ is often called a *Minkowski* or *normed* plane with unit disk $D$. We will usually suppress the explicit dependence of various quantities on $D$. For a finite subset $X \subset E^2$, a minimum spanning tree $S = S(X)$ consists of a collection of segments $AB$ with $A, B \in X$, which spans all the points of $X$, and such that the sum of all the lengths $\|AB\|_D$ is a minimum. We denote this minimum by $L_M(X)$. Further, we define

$$L_S(X) = \inf_{Y \supseteq X} L_M(X)$$

where $Y$ ranges over all finite subsets of $E^2$ containing $X$. It is not hard to show that there always exists $X' \supseteq X$ with $|X'| \leqslant 2|X| - 2$ having $L_S(X) = L_M(X')$. The minimum

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spanning tree $S(Y)$ will be called a *minimum Steiner tree* $T(X)$ for $X$. The points of $Y \backslash X$ are usually called *Steiner* points of $T(X)$; the points of $X$ are known as regular points of $T(X)$. A minimum Steiner tree is *full* if every regular point is a leaf (i.e., has degree one). A tree is called a *full Steiner tree* if, by varying its edge lengths, it can occur as a full minimum Steiner tree for the resulting set of endpoints.

Minimum Steiner trees have been the subject of extensive investigations during the past 25 years or so (see [4,10,12,17]). Most of this research has dealt with the Euclidean metric, with much of the remaining work concerned with the $L_1$ metric, or more generally, the usual $L_p$ metric or norm (see [6,3]). Cockayne [4] initiated a study of minimum Steiner trees in a generic metric space and Melzak [17] investigated this problem on a normed plane. Chakerian and Ghandehari [2] found interesting properties of minimum Steiner trees for three points in a normed space. Du et al. [6] established some fundamental properties of minimum Steiner trees on a normed plane. It has been shown, for example, that the determination of $L_S(X)$ in general is an NP-complete problem, both for the Euclidean as well as the $L_1$ case (cf. [10,11]).

The *Steiner ratio* $\rho(D)$ for $M(D)$ is defined by

$$\rho(D) := \inf_X \frac{L_S(X)}{L_M(X)}$$

Thus, $\rho(D)$ is a measure of how much the total length of a minimum spanning tree can be decreased by allowing additional (Steiner) points. More recently, Du et al. [6] conjectured that the Steiner ratio on a normed plane is equal to the Steiner ratio on its dual normed plane, i.e., $\rho(D) = \rho(D^*)$ where $D^* = \{x \mid x^T y \leqslant 1, \text{for all } y \in D\}$ is the polar dual of $D$.

For all $n \in N$, define

$$\rho_n(D) := \inf_{|X| \leqslant n} \frac{L_S(X)}{L_M(X)}.$$

Then $\rho(D) := \inf_{n \in N} \rho_n(D)$. In this paper, we show that for $n \leqslant 5$, $\rho_n(D) = \rho_n(D^*)$. Thus, this partially verifies the conjecture of Du et al. [6].

For prior results on minimum Steiner trees in normed planes, the reader can consult [1,2,8,18,20]. This note is organized in the following way. In Section 2, fundamental properties of minimum Steiner trees and dualities are presented. In Section 3, some new concepts about *-dual are introduced. In Section 4, the main result is proved.

## 2. Preliminaries

Let $|\cdot|$ denote the Euclidean norm and $\|\cdot\|$ denote the norm determined by $D$, an arbitrary fixed compact, convex, centrally symmetric domain in $E^2$. The *dual norm* $\|\cdot\|^*$ is defined by

$$\|x\|^* = \max_y \frac{x^T y}{\|y\|}.$$

It is a well-known fact that the unit disk of the dual norm $\|\cdot\|^*$ is the *polar dual* of $D$, i.e., $D^* = \{x \mid x^T y \leqslant 1, \text{for all } y \in D\}$ and $D^{**} = D$.

The following lemma given in [9] states a relation of Steiner ratios on different normed planes.

**Lemma 2.1.** *Let $d(\partial D, \partial D')$ denote the maximum Euclidean distance between the two intersections of a ray from the origin with $\partial D$ and $\partial D'$. Then for any $\delta > 0$ and $n$, there exists $\varepsilon > 0$ such that $d(\partial D, \partial D') < \varepsilon$ implies $|\rho_n(D) - \rho_n(D')| < \delta$.*

From the above lemma, for every unit disk $D$ and $n$, we can choose a sequence of disks $D'$ with differentiable and strictly convex boundaries such that the $\rho_n(D')$ converge to $\rho_n(D)$ and the $\rho_n(D'^*)$ converge to $\rho_n(D^*)$. So we need only consider the unit disk $D$ with a differentiable and strictly convex boundary. For such a unit disk, we can introduce the concept of *dual point* as follows. First, for every point $x \in \partial D$, there exists a unique point $y$ such that $\|y\|^* = 1$ and $\overrightarrow{oy}$ is an outward normal vector to the unique line tangent to $\partial D$ at $x$. The point $y$ is called the *dual point* of $x$, and is denoted by $x^*$. In general, for any point $x$, its dual point is defined to be

$$x^* = \frac{1}{\|x\|}\left(\frac{x}{\|x\|}\right)^*.$$

We also say that the directions of $x$ and $x^*$ are *conjugate*. Clearly $(\lambda x)^* = (1/\lambda)x^*$.

The above facts and definitions can easily imply the following lemma about properties of dual points.

**Lemma 2.2.** *Suppose that $\partial D$ is differentiable and strictly convex. Then:*

(1) *for any point $x$, $x^T x^* = \|x\| \cdot \|x^*\|^* = 1$;*

(2) *for any point $x$ and $y$, $x^T y \leqslant \|x\| \cdot \|y\|^*$ with equality holding iff $y = \lambda x^*$ for some positive scalar $\lambda$;*

If for any two points $x$ and $y$, $\|x - y\| = \lambda \|x^* - y^*\|^*$ for some constant $\lambda$, then it follows immediately that the Steiner ratio for the normed plane is equal to the Steiner ratio for its dual normed plane. However, in general such a property does not hold for the duality. For a counterexample, we may consider the plane with $L_p$ norm. Its dual norm is the $L_q$ norm with $1/p + 1/q = 1$, and the dual point of $(x_1^{1/p}, x_2^{1/p})$ with $|x_1| + |x_2| = 1$ is $(x_1^{1/q}, x_2^{1/q})$. Consider three points $x = (1,0)$, $y = (0,1)$ and $z = ((1/2)^{1/p}, (1/2)^{1/p})$. Clearly $x^* = x$, $y^* = y$ and $z^* = ((1/2)^{1/q}, (1/2)^{1/q})$. If the expected property held, we would have

$$\frac{\|x - y\|_p}{\|x^* - y^*\|_q} = \frac{\|x - z\|_p}{\|x^* - z^*\|_q}.$$

However this would imply

$$\left[(2^{1/p} - 1)^p + 1\right]^{1/p} = \left[(2^{1/q} - 1)^q + 1\right]^{1/q},$$

which is impossible for $1 < p < 2$.

In fact, to the best of our knowledge, we do not know any bijection between a normed plane and its dual normed plane such that distance is preserved proportionally. So in order to prove our result, we will need to use some special properties of the minimum Steiner trees on the normed plane.

The minimum Steiner tree for three points has the following property (which can be found in [2,6]).

**Lemma 2.3.** *Suppose that $\partial D$ is differentiable and strictly convex and $A \in \partial D$. Then there exist unique points $B$ and $C$ on $\partial D$ such that $\{OA, OB, OC\}$ forms a minimum Steiner tree for the set $\{A, B, C\}$. Furthermore, the following are equivalent:*

*(1) $\{OA, OB, OC\}$ forms a minimum Steiner tree for the set $\{A, B, C\}$*

*(2) The triangle induced by the three tangent lines to $D$ at $A, B$ and $C$ has the property that the sum of the distances to the three sides of the triangle is the same for all points inside the triangle*

*(3) Let $A^*, B^*$ and $C^*$ be the dual points of $A, B$ and $C$, respectively, and let $A^{*\prime}, B^{*\prime}$ and $C^{*\prime}$ be the reflections of $A^*, B^*$ and $C^*$, respectively. Then the hexagon $A^*A^{*\prime}B^*B^{*\prime}C^*C^{*\prime}$ is partitioned into six congruent equilateral triangles of side length 1 (w.r.t the dual norm) by joining each of its six vertices to $O$.*

The directions of the set $\{\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}\}$ in the above lemma will be called a *consistent triple of directions.*

The following lemma states an important property of full minimum Steiner trees (which can be found in [6]).

**Lemma 2.4.** *Suppose that $\partial D$ is differentiable and strictly convex. Then every full Steiner minimum tree consists of three sets of parallel segments.*

A tree is called a *3-regular* tree if every vertex which is not a leaf has degree three. A consequence of Lemma 2.4 is that for strictly convex and differentiable norms, every minimum Steiner tree is a 3-regular tree.

In [7], Du and Hwang prove the following.

**Lemma 2.5.** *Suppose that $\partial D$ is differentiable and strictly convex. Then $\rho_n(D)$ is achieved by the vertex set of a polygon which can be triangulated into at most $n - 2$ interior-disjoint isomorphic equilateral triangles without adding new vertices. Furthermore, the vertex set has a full minimum Steiner tree.*

The polygon satisfying the conditions in the above lemma is said to be a *critical structure* and its vertex set is called as a *critical set*. Recall that a *maximal outerplanar graph* is a planar graph whose vertices all lie on the exterior ('outer') face and whose interior faces all have three vertices. Thus, a critical structure is a maximal outerplanar graph. Since every full minimum Steiner tree is 3-regular tree, it will be useful to establish a duality between maximal outerplanar graphs and 3-regular trees.

## 3. *-Dual

Every maximal outerplanar graph $G$ can be directed so that the boundary of each interior face is either clockwise or counterclockwise. Similarly, every 3-regular tree $T$ can be directed so that at each internal vertex, the three edges either all point to it or all leave it. In the following, we only consider directed maximal outerplanar graphs and 3-regular trees with such directions.

For every directed maximal outerplanar graph $G$, we define its **-dual* graph to be a directed 3-regular tree obtained in the following way:

1. First, for every finite region of $G$ we put a vertex in it. Connect the two vertices if the two regions in which the two vertices lie share a common portion of their boundaries.

2. Then, for every part of the boundary $l$ of the infinite region, we put a vertex in the infinite region so that the edge between it and any other vertex in a finite region, whose boundary also contains $l$, crosses $l$. Connect every pair of such two vertices. The resulting graph is a 3-regular tree.

3. Finally, direct the 3-regular tree so that at every internal vertex $v$, the directions of the three edges all leave (point to) it if the boundary of the interior region of $G$ in which $v$ lies is counterclockwise (clockwise).

The *-dual of $G$ is denoted by $G^*$. Fig. 1 shows examples of this duality.

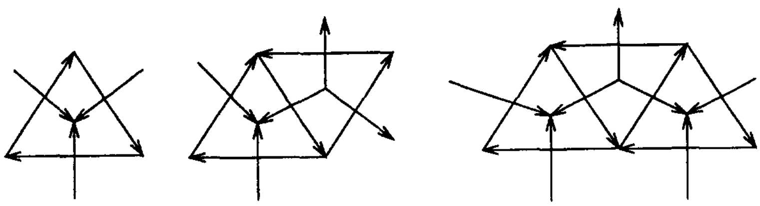
For every directed 3-regular tree $T$, we define its **-dual* graph to be a directed maximal outer planar graph in the following way:

1. First we add a star to the tree such that every leaf of the tree is identified with a leaf of the star and vice versa. The resulting graph is a planar graph.

2. Then, form the dual graph of the resulting planar graph. The dual graph is a maximal outerplanar graph and each internal vertex of $T$ lies in exactly one of its interior regions.

3. Finally, direct the maximal outer planar graph so that for each internal vertex $v$, if the directions of the three edges all leave (point to) $v$, the boundary of the interior region in which $v$ lies is counterclockwise (clockwise).

The *-dual of $T$ is denoted by $T^*$. Fig. 2 illustrates this duality.



n = 3 n = 4 n = 5

Fig. 1. *-Dual of maximal outer planar graphs.

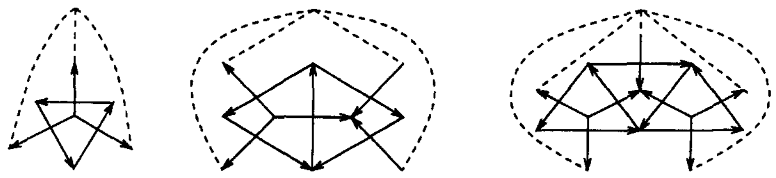



Fig. 2. *-Dual of 3-regular trees.

There are three important properties of this *-duality.

1. For every directed maximal outer planar graph $G$, $G^{**} \simeq G$. For every directed 3-regular tree $T$, $T^{**} \simeq T$.

2. There is bijection between the edge sets of the graph and its *-dual such that two corresponding edges cross each other. Let $\pi$ denote this bijection.

3. The number of vertices of a directed maximal outer planar graph is equal to the number of leaves of its *-dual, a directed 3-regular tree, and vice versa.

The topologies of two *-dual graphs are also called *-dual to each other, and the *-dual of a topology $t$ is denoted by $t^*$. The above bijection $\pi$ induces a bijection between two *-dual topologies and we will also denote this induced bijection by $\pi$.

Let $G$ be a maximal outerplanar graph. Then the outside boundary of $G$, called the *circumference* of $G$, forms a cycle going through each vertex once. The cyclic order of the vertices on the circumference will be referred as the *circumferential order.* Let $T$ be a 3-regular tree. Inflate the edges of $T$ to have positive width. Then the outside boundary of the inflated $T$, also called the *circumference* of $T$, forms a cycle going through each leaf once and each edge twice. The cyclic order of the leaves on the circumference will also be referred as the *circumferential order.*

Now we define the **-product* of two graphs $G$ and $T$ having *-dual topologies as

$$G * T := \sum_{e \in E(G)} \|e\| \cdot \|\pi(e)\|^*.$$

**Lemma 3.1.** *Suppose that $G$ is a directed maximal outerplanar graph with topology $g$, and $x_1, x_2, \ldots, x_n$ are vertices of $G$ in the clockwise circumferential order. Let $T$ be a directed* 3*-regular tree with topology $g^*$ and $y_1, y_2, \ldots, y_n$ be leaves of $T$ in the clockwise circumferential order such that for each $1 \leqslant i \leqslant n$, the edge incident to $y_i$ is $\pi(x_i x_{i+1})$ (where $x_{n+1} = x_1$). Then*

$$G * T \geqslant y_1^T(x_1 - x_2) + y_2^T(x_2 - x_3) + \cdots + y_n^T(x_n - x_1).$$

*Moreover, equality holds iff for every $e \in E(G)$, $e$ and $\pi(e)$ are in conjugate directions.*

**Proof.** We first prove by induction on $n$ that

$$\sum_{e \in E(G)} e^T \pi(e) = y_1^T(x_1 - x_2) + y_2^T(x_2 - x_3) + \cdots + y_n^T(x_n - x_1). \tag{1}$$

Consider the case $n = 3$. In this case $G$ is a triangle. Let $z$ be the internal vertex of $T$. If the boundary of $G$ is clockwise (see Fig. 3), then

$$\begin{aligned}
&\sum_{e \in E(G)} e^T \pi(e) \\
&= (x_2 - x_1)^T(z - y_1) + (x_3 - x_2)^T(z - y_2) + (x_1 - x_3)^T(z - y_3) \\
&= [(x_2 - x_1) + (x_3 - x_2) + (x_1 - x_3)]^T z - (x_2 - x_1)^T y_1 \\
&\quad -(x_3 - x_2)^T y_2 - (x_1 - x_3)^T y_3 \\
&= -(x_2 - x_1)^T y_1 - (x_3 - x_2)^T y_2 - (x_1 - x_3)^T y_3 \\
&= y_1^T(x_1 - x_2) + y_2^T(x_2 - x_3) + y_3^T(x_3 - x_1).
\end{aligned}$$

If the boundary of $G$ is clockwise (see Fig. 4), then

$$\begin{aligned}
&\sum_{e \in E(G)} e^T \pi(e) \\
&= (x_1 - x_2)^T(y_1 - z) + (x_2 - x_3)^T(y_2 - z) + (x_3 - x_1)^T(y_3 - z) \\
&= [(x_1 - x_2) + (x_2 - x_3) + (x_3 - x_1)]^T z + (x_1 - x_2)^T y_1 \\
&\quad +(x_2 - x_3)^T y_2 + (x_3 - x_1)^T y_3 \\
&= (x_1 - x_2)^T y_1 + (x_2 - x_3)^T y_2 + (x_3 - x_1)^T y_3 \\
&= y_1^T(x_1 - x_2) + y_2^T(x_2 - x_3) + y_3^T(x_3 - x_1).
\end{aligned}$$

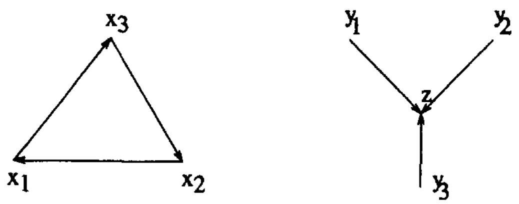



Fig. 3.


Fig. 4.

Thus the desired equality holds for $n = 3$. Now we assume that it holds for $n-1$ and consider the case $n$. Without loss generality, we may assume that $x_1$ is of degree 2 and the internal vertex of $T$ corresponding to the region $x_1x_2x_n$ is $z$. Consider $G' = \{x_2, x_3, \ldots, x_n\}$ and $T' = \{y_2, y_3, \ldots, y_{n-1}, z\}$. Then $G'$ and $T'$ are *-dual to each other and by the induction hypothesis,

$$\sum_{e \in E(G')} e^T \pi(e) = y_2^T(x_2 - x_3) + \cdots + y_{n-1}^T(x_{n-1} - x_n) + z^T(x_n - x_2).$$

If the boundary of $x_1x_2x_n$ is clockwise (see Fig. 5), then

$$\begin{aligned}
&\sum_{e \in E(G)} e^T \pi(e) \\
&\quad = (x_2 - x_1)^T(z - y_1) + (x_1 - x_n)^T(z - y_n) + \sum_{e \in E(G')} e^T \pi(e) \\
&\quad = y_1^T(x_1 - x_2) + y_n^T(x_n - x_1) + z^T(x_2 - x_n) \\
&\qquad + y_2^T(x_2 - x_3) + \cdots + y_{n-1}^T(x_{n-1} - x_n) + z^T(x_n - x_2) \\
&\quad = y_1^T(x_1 - x_2) + y_2^T(x_2 - x_3) + \cdots + y_n^T(x_n - x_1).
\end{aligned}$$

If the boundary of $x_1x_2x_n$ is counterclockwise (see Fig. 6), then

$$\begin{aligned}
\sum_{e \in E(G)} e^T \pi(e) &= (x_1 - x_2)^T(y_1 - z) + (x_n - x_1)^T(y_n - z) + \sum_{e \in E(G')} e^T \pi(e) \\
&= y_1^T(x_1 - x_2) + y_n^T(x_n - x_1) - z^T(x_n - x_2) \\
&\quad + y_2^T(x_2 - x_3) + \cdots + y_{n-1}^T(x_{n-1} - x_n) + z^T(x_n - x_2) \\
&= y_1^T(x_1 - x_2) + y_2^T(x_2 - x_3) + \cdots + y_n^T(x_n - x_1).
\end{aligned}$$

Thus the desired equality holds for $n$. Therefore, by induction (1) holds for every $n \geqslant 3$.

Now we prove Lemma 3.1. By Lemma 2.2 and (1), we have

$$\begin{aligned}
G * T &= \sum_{e \in E(G)} \|e\| \cdot \|\pi(e)\|^* \geqslant \sum_{e \in E(G)} e^T \pi(e) \\
&= y_1^T(x_1 - x_2) + y_2^T(x_2 - x_3) + \cdots + y_n^T(x_n - x_1).
\end{aligned}$$

and equality holds iff for every $e \in E(G)$, $e$ and $\pi(e)$ are in conjugate directions. □

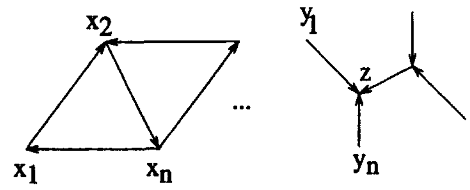



Fig. 5.

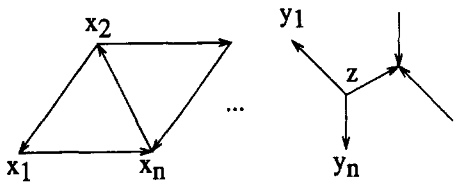



Fig. 6.

## 4. Main result

The main result of this paper is the following theorem.

**Theorem 4.1.** *For* $n = 3, 4,$ *and* 5

$$\rho_n(D) = \rho_n(D^*)$$

This result partially verifies the conjecture given by Du et al. [6] that the Steiner ratio for a normed plane is equal to the Steiner ratio for its dual normed plane. Before we give the proof of this theorem, let us first look at the following conjecture.

**Conjecture 4.2.** Suppose that $\partial D$ is differentiable and strictly convex, and $G$ is a directed critical structure with topology $g$ having a full minimal Steiner tree $T$ with topology $t$. Then there exists a sequence of maximal outer planar graphs $\{H_\alpha\}$ satisfying:

(1) Each $H_\alpha$ has topology $t^*$ and has a directed full Steiner tree $S_\alpha$ with topology $g^*$ such that for each edge $e$ of $S_\alpha$, the corresponding edge $\pi(e)$ of $G$ is in the conjugate direction of $e$.

(2) As $\alpha \to \alpha_0$, $\{H_\alpha\}$ converges to a critical structure $H$ on the dual normed plane and $H$ has the same number of vertices as $G$.

(3) As $\alpha \to \alpha_0$, $\{S_\alpha\}$ converges to a tree $S$ spanning the vertices of $H$.

This conjecture plays an important role in our proof of Theorem 1. In fact, we have the following result.

**Lemma 4.3.** *If Conjecture* 4.2 *is true for all critical structures with at most n vertices, then*

$$\rho_n(D) = \rho_n(D^*).$$

**Proof.** First we suppose that $\partial D$ is differentiable and strictly convex. According to Lemma 2.5, $\rho_n(D)$ is achieved at the vertices of a directed critical structure $G$ with a full minimal Steiner tree $T$. Let $g$ be the topology of $G$ and $t$ be the topology of $T$. Let $H_\alpha$, $S_\alpha$, $H$ and $S$ be the graphs occurring in Conjecture 4.2. Suppose that $x_1, x_2, \ldots, x_m$ are vertices of $G$ for some $m \leqslant n$ and $y_{\alpha,1}, y_{\alpha,2}, \ldots, y_{\alpha,m}$ are leaves of $S$ with the order and relations as in Lemma 3.1. By Lemma 3.1,

$$\begin{aligned} G * S_\alpha &= y_{\alpha,1}^T(x_1 - x_2) + y_{\alpha,2}^T(x_2 - x_3) + \cdots + y_{\alpha,m}^T(x_m - x_1) \\ &= x_m^T(y_{\alpha,m} - y_{\alpha,m-1}) + x_{m-1}^T(y_{\alpha,m-1} - y_{\alpha,m-2}) + \cdots + x_1^T(y_{\alpha,1} - y_{\alpha,m}) \\ &\leqslant H_\alpha * T. \end{aligned}$$

Let $L(T)$ and $L^*(T)$ denote the lengths of a tree $T$ on a normed plane and its dual normed plane, respectively. Let $L_M(G)$ and $L_M^*(G)$ denote the lengths of a minimum spanning tree of the vertices of a critical structure $G$ on a normed plane and its dual normed plane, respectively. Note that in a critical structure, all edges are of equal length. Thus

$$\begin{aligned} G * S_\alpha &= \sum_{e \in E(G)} \|e\| \cdot \|\pi(e)\|^* \\ &= \frac{L_M(G)}{m-1} \cdot L^*(S_\alpha) \\ &\to \frac{L_M(G)}{m-1} \cdot L^*(S) \quad \text{as } \alpha \to \alpha_0 \end{aligned}$$

and

$$\begin{aligned} H_\alpha * T &= \sum_{e \in E(H_\alpha)} \|e\|^* \cdot \|\pi(e)\| \\ &\to \sum_{e \in E(H)} \|e\|^* \cdot \|\pi(e)\| \quad (\text{as } \alpha \to \alpha_0) \\ &= \frac{L_M^*(H)}{m-1} \cdot \sum_{e \in E(H)} \|\pi(e)\| \\ &\leqslant \frac{L_M^*(H)}{m-1} \cdot \sum_{e \in E(H_\alpha)} \|\pi(e)\| \\ &= \frac{L_M^*(H)}{m-1} \cdot L(T). \end{aligned}$$

The above inequality follows from the following fact:

$$\{\pi(e) \mid e \in E(H)\} \subseteq \{\pi(e) \mid e \in E(H_\alpha)\}.$$

Hence we have

$$L_M(G) \cdot L^*(S) \leqslant L_M^*(H) \cdot L(T)$$

and thus

$$\rho_n(D^*) \leqslant \frac{L^*(S)}{L_M^*(H)} \leqslant \frac{L(T)}{L_M(G)} = \rho_n(D).$$

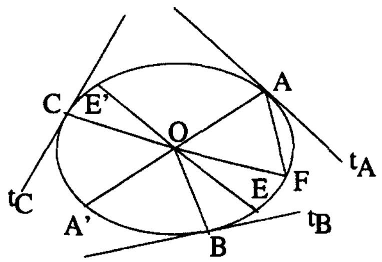



Fig. 7.

Since $D^{**} = D$, we also have

$$\rho_n(D) = \rho_n(D^{**}) \leqslant \rho_n(D^*).$$

Therefore, $\rho_n(D) = \rho_n(D^*)$.

If $\partial D$ is not differentiable and strictly convex, then by Lemma 2.1 we can approximate it by a sequence of differentiable and strictly convex disks. Taking limits, the theorem follows. □

From Lemma 4.3, in order to prove Theorem 4.1, we only need to prove that Conjecture 4.2 is true for all critical structures with 3, 4, or 5 vertices. We first present the following result.

**Lemma 4.4.** *Suppose that $\partial D$ is differentiable and strictly convex, and $A, B$ and $C$ are three distinct points in $\partial D$ such that $\{OA, OB, OC\}$ forms a minimum Steiner tree for the set $\{A, B, C\}$. Let $A' \in \partial D$ be the reflection of $A$ w.r.t. $O$, and let $F$ be the unique point in $\partial D$ such that $F$ and $B$ lie on the same side of $AA'$, and $\triangle AOF$ is equilateral* (*see Fig.* 7). *Then $OF$ lies between $OA$ and $OB$.*

**Proof.** Let $t_A, t_B$ and $t_C$ be three tangent lines to $\partial D$ at $A, B$ and $C$, respectively. Let $EE'$ be a diameter of $D$ such that $EE'$ is parallel to $t_A$, and $E$ and $B$ lie on the same side of $AA'$. For any point $x$ and any line $\ell$, we denote by $\text{dist}(x, \ell)$ the distance from $x$ to $\ell$. Since

$$\|EA\| > \text{dist}(E, t_A) = \|OA\| = \|FA\|,$$

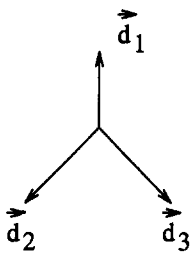
$OF$ must lie between $OA$ and $OE$. By Lemma 2.3,

$$\begin{aligned}&\text{dist}(B, t_A) + \text{dist}(B, t_B) + \text{dist}(B, t_C)\\ &\quad = \text{dist}(O, t_A) + \text{dist}(O, t_B) + \text{dist}(O, t_C) = 1 + 1 + 1 = 3.\end{aligned}$$

Since $\text{dist}(B, t_B) = 0$, $\text{dist}(B, t_A) + \text{dist}(B, t_C) = 2$. Noting that $\text{dist}(B, t_A) < 2$ and $\text{dist}(B, t_C) < 2$, we have $1 < \text{dist}(B, t_A) < 2$. This implies that $OF$ must lie between $OA'$ and $OE$. Hence $OF$ lies between $OA$ and $OB$. □

For a full Steiner tree $T$, let $P(T)$ denote its set of endpoints (i.e., vertices of degree 1).

**Lemma 4.5.** *Suppose that $\vec{d_1}, \vec{d_2}$ and $\vec{d_3}$ are a consistent triple of directions. Then there is full Steiner tree $T$ of the form*

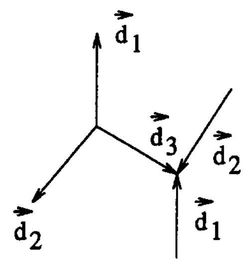



*such that $P(T)$ is a critical set.*

**Proof.** Let $l_1, l_2$ and $l_3$ be three rays at $O$ in the directions of $\vec{d_1}, \vec{d_2}$ and $\vec{d_3}$, respectively. We take a fixed point $A$ in $l_1$, and we let $B$ be an arbitrary point on $l_2$. Let $C$ be the unique point on the side of $l_3$ such that $\triangle ABC$ is an equilateral triangle (see Fig. 8).

By Lemma 4.4, as $B \to O$ along $l_2$, $C$ lies above $l_3$, and as $B \to \infty$ along $l_2$, $C$ lies below $l_3$. Thus by continuity, there exists a position of $B$ on $l_2$ such that $C$ lies on $l_3$, and thus the $\triangle ABC$ is a desired equilateral triangle. $\square$

**Lemma 4.6.** *Suppose that $\vec{d_1}, \vec{d_2}$ and $\vec{d_3}$ are a consistent triple of directions. Then there is full Steiner tree $T$ of the form*




*such that $P(T)$ is a critical set.*

**Proof.** Let $l_1, l_2$ and $l_3$ be three rays as in Lemma 4.5. We choose a fixed point $A$ on $l_1$, and we let $B$ be an arbitrary point on $l_2$. Let $\triangle ABC$ be the equilateral triangle along the side of $l_3$. Let $E$ be the intersection of $l_3$ and the line through $C$ which is parallel to $l_1$ (see Fig. 9).

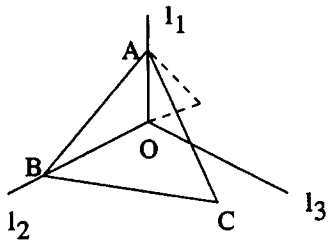



Fig. 8.

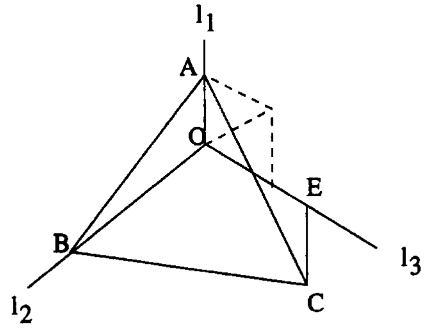



Fig. 9.

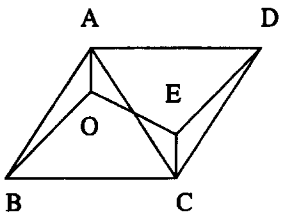
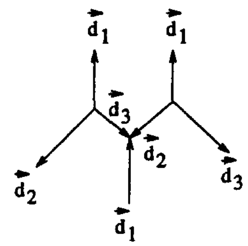



Fig. 10.

Then by Lemma 4.4, as $B \to O$ along $l_2$, $C$ lies above $l_3$ and as $B \to \infty$ along $l_2$, $C$ lies below $l_3$ and $\|CE\| \to \infty$. Thus by continuity, there exists a position of $B$ on $l_2$ such that $C$ lies below $l_3$ and $\|CE\| = \|OA\|$. Now let $D$ be the point such that $\overrightarrow{ED} = \overrightarrow{BO}$. Then by symmetry, the triangle $\triangle ACD$ is also an equilateral triangle (see Fig. 10).

Thus the quadrilateral $\{A, B, C, D\}$ is the desired critical set. $\square$

**Lemma 4.7.** *Suppose that $\vec{d_1}, \vec{d_2}$ and $\vec{d_3}$ are a consistent triple of directions. Then there exist a sequence of full Steiner trees $\{S_\alpha\}$ of the form*




*such that as $\alpha \to \alpha_0$, $\{S_\alpha\}$ converges to a tree $S$ and $\{P(S_\alpha)\}$ converges to a critical set.*

**Proof.** By Lemma 4.5, there is a critical set $H = \{A, B, C, D, E\}$ with a degenerate Steiner tree tree $S$ (see Fig. 11).

Let $\alpha$ be an arbitrary real positive number and $\alpha_0 = 0$. Let $C'$ be the point such that $\overrightarrow{C'C}$ is in the direction of $\vec{d_1}$ and $\|C'C\| = \alpha$. Let $H_\alpha$ and $S_\alpha$ be as shown in Fig. 12. Then as $\alpha \to \alpha_0$,$\{S_\alpha\}$ converges to the tree $S$ and $\{P(S_\alpha)\}$ converges to the critical set $H$. This proves the lemma. □

**Proof of Theorem 4.1.** W.l.o.g., suppose that $G$ has at least one interior region with counterclockwise boundary. Let $\vec{d_1}, \vec{d_2}$ and $\vec{d_3}$ be the vectors conjugate to the directions of edges of an interior region of $G$ with counterclockwise boundary. Then $\vec{d_1}, \vec{d_2}$ and

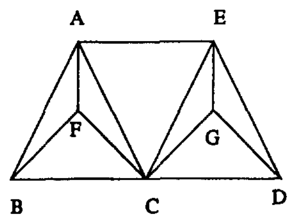



Fig. 11.

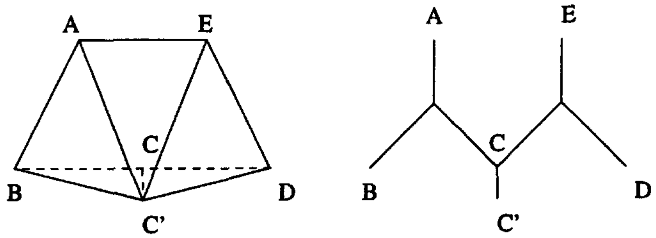



Fig. 12. $H_\alpha$ and $S_\alpha$.

$\vec{d_3}$ are a consistent triple of directions in the dual norm. Also it is obvious that when $n \leqslant 5$, $g$ and $t$ are unique under isomorphism and $g^* \simeq t, t^* \simeq g$. So by Lemmas 4.4–4.6, Conjecture 4.2 is true for all critical structures with at most 5 vertices. Thus by Lemma 4.3, for $n = 3, 4$, and 5,

$$\rho_n(D) = \rho_n(D^*). \qquad \square$$

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