J Comb Optim (2008) 16: 99–106
DOI 10.1007/s10878-007-9124-y

On minimum *m*-connected *k*-dominating set problem in unit disc graphs

Weiping Shang · Frances Yao · Pengjun Wan · Xiaodong Hu

Published online: 5 December 2007

Abstract Minimum m-connected k-dominating set problem is as follows: Given a graph $G=(V,E)$ and two natural numbers m and k, find a subset $S \subseteq V$ of minimal size such that every vertex in $V \setminus S$ is adjacent to at least k vertices in S and the induced graph of S is m-connected. In this paper we study this problem with unit disc graphs and small m, which is motivated by the design of fault-tolerant virtual backbone for wireless sensor networks. We propose two approximation algorithms with constant performance ratios for $m \leq 2$. We also discuss how to design approximation algorithms for the problem with arbitrarily large m.

Keywords k-dominating set · m-connectivity · Unit disc graph · Approximation algorithm · Wireless sensor networks

1 Introduction

A Wireless Sensor Network (WSN) consists of wireless nodes (transceivers) without any underlying physical infrastructure. In order to enable data transmission in such networks, all the wireless nodes need to frequently flooding control messages

This work was supported in part by the Research Grants Council of Hong Kong under Grant No. CityU 1165/04E, the National Natural Science Foundation of China under Grant No. 70221001, 10531070 and 10771209.

W. Shang · X. Hu (✉)
Institute of Applied Mathematics, Chinese Academy of Sciences, Beijing, China
e-mail: xdhu@amss.ac.cn

W. Shang · F. Yao
Department of Computer Science, City University of Hong Kong, Hong Kong, Hong Kong

P. Wan
Department of Computer Science, Illinois Institute of Technology, Chicago, USA

thus causing a lot of redundancy, contentions and collisions. To support various network functions such as multi-hop communication and area monitoring, some wireless nodes are selected to form a *virtual backbone*, which could be considered and used as the routing infrastructure of WSNs. In many existing schemes (e.g., Alzoubi et al. 2002) virtual backbone nodes form a Connected Dominating Set (CDS) of the WSN. With virtual backbones, routing messages are only exchanged between the backbone nodes, instead of being broadcasted to all the nodes. Prior work (e.g., Sinha et al. 2001) has demonstrated that virtual backbones could dramatically reduce routing overhead.

In WSNs, a node may fail due to accidental damage or energy depletion and a wireless link may fade away during node movement. Thus it is desirable to have several sensors monitor the same target, and let each sensor report via different routes to avoid losing an important event. Hence, how to construct a fault tolerant virtual backbone that continues to function when some nodes break down is an important research problem (Bredin et al. 2005; Koskinen et al. 2005; Kuhn et al. 2006).

As usual, we assume in this paper that all nodes have the same transmission range (scaled to 1). Under such an assumption, a WSN can be modeled as a Unit Disc Graph (UDG) that consists of all nodes in the WSN and there exists an edge between two nodes if the distance between them is at most 1. Fault tolerant virtual backbone problem can be formalized as a combinatorial optimization problem: Given a UDG $G = (V, E)$ and two nonnegative integers m and k, find a subset of nodes $S \subseteq V$ of minimum size that satisfies the following two conditions:

(i) each node u in $V \setminus S$ is *dominated* by at least k nodes in S;
(ii) S is *m-connected*, that is, there are at least m disjoint paths between each pair of nodes in S, i.e., G is still connected by removing any $m - 1$ nodes.

Every node in S is called a *backbone node* and every set S satisfying (i–ii) is called *m-connected k-dominating set*, which is simply denoted by (m, k)-CDS, and the problem is called *minimum m-connected k-dominating set* problem (in UDGs).

In this paper, we will first study the minimum m-connected k-dominating set problem for $m = 1, 2$, which is important for fault tolerant virtual backbone problem in WSNs. Clearly, when $m = 1$ and $k = 1$ the problem is reduced to minimum connected dominating set problem, which is a well-known NP-hard problem (Garey and Johnson 1979). We will propose two centralized approximation algorithms for computing m-connected k-dominating sets for $m = 1, 2$. We will also discuss how to design approximation algorithms for the problem with arbitrarily large $3 \le m \le k$.

The remainder of this paper is organized as follows: We first give some definitions in Sect. 2 and then present some related works in Sect. 3. In Sect. 4 we present our algorithms with theoretical analysis on guaranteed performances. In Sect. 5 we conclude the paper with remarks on future work.

2 Preliminaries

Let G be a graph with vertex-set $V(G)$ and edge-set $E(G)$. For any vertex $v \in V$, the neighborhood of v is defined by $N(v) \equiv \{u \in V(G) : uv \in E(G)\}$ and the closed neighborhood of v is defined by $N[v] \equiv \{u \in V(G) : uv \in E(G)\} \cup \{v\}$, i.e., $N[v] = N(v) \cup \{v\}$. The minimum degree of vertices in $V(G)$ is denoted by $\delta(G)$.

A subset $U \subseteq V$ is called an *independent set* of G if all vertices in U are pairwise non-adjacent, and it is further called a *Maximal Independent Set* (MIS) if each vertex $V \setminus U$ is adjacent to at least one vertex in U.

A *dominating set* of a graph $G = (V, E)$ is a subset $S \subseteq V$ such that each vertex in $V \setminus S$ is adjacent to at least one vertex in S. A dominating set is called a *Connected Dominating Set* (CDS) if it also induces a connected subgraph. A *k-dominating set* $S \subseteq V$ of G is a set of vertices such that each vertex $u \in V$ is either in S or has at least k neighbors in S.

A *cut-vertex* of a connected graph G is a vertex v such that the graph $G \setminus \{v\}$ is disconnected. A *block* is a maximal connected subgraph having no cut-vertex (so a graph is a block if and only if it is either 2-connected or equal to K_1 or K_2). The block-cut-vertex graph of G is a graph H where $V(H)$ consists of all cut-vertices of G and all blocks of G, with a cut-vertex v adjacent to a block G_0 if v is a vertex of G_0. The block-cut-vertex graph is always a forest. A *2-connected graph* is a graph without cut-vertices. Clearly, a block with more than three vertices is a 2-connected component. A *leaf block* of a connected graph G is a block with only one cut-vertex.

3 Related work

Lots of efforts have been made to design approximation algorithms for the minimum connected dominating set problem in UDGs. Wan et al. (2004) proposed a two-phase distributed algorithm for the problem that has an approximation performance ratio of 8. The algorithm first constructs a spanning tree, and then at the first phase, each vertex in a tree is examined to find an MIS and all the vertices in the MIS are colored black. At the second phase, more vertices are added (color blue) to connect those black vertices. Recently, Li et al. (2005) proposed another two-phase distributed algorithm with a better approximation ratio of $(4.8 + \ln 5)$. At the first phase, an MIS is computed as in (Wan et al. 2004). At the second phase, based on the property that any vertex in a UDG is adjacent to at most 5 independent vertices, a Steiner tree algorithm is used to connect vertices in the MIS.

Dai and Wu (2006) addressed the problem of constructing k-connected k-dominating virtual backbones. They proposed three localized algorithms, two of them, k-gossip algorithm and color-based (k, k)-CDS algorithm, are random ones, while the other, k-coverage condition algorithm, is a deterministic one. In k-gossip algorithm, each vertex decides its own backbone status with a probability based on the network size, deploying area size, transmission range, and k. In color-based (k, k)-CDS algorithm, each vertex randomly selects one of the k colors such that the network is divided into k-disjoint subsets based on vertex colors. For each subset of vertices, a CDS is constructed and (k, k)-CDS is the union of k CDS's. k-coverage condition algorithm only works in very dense networks and no upper bound on the size of returned backbone is obtained.

More recently, Wang et al. (2007) proposed a 64-approximation algorithm for the minimum $(2, 1)$-CDS problem. The basic idea of this centralized algorithm is as follows: (1) Construct a small-sized CDS C as a starting point of the backbone; (2) iteratively augment the backbone by adding new vertices to connect a leaf block in the backbone to other block (or blocks); (3) the augmentation process stops when all backbone nodes are in the same block, i.e., the backbone nodes are 2-connected. The augmentation process stops in at most $|C|-1$ steps and each step at most 8 nodes are added.

Most recently, in work (Shang et al. 2007) we studied minimum m-connected k-tuple dominating set problem. Give a graph $G=(V,E)$, a subset $S\subseteq V$ is called a m-connected k-tuple dominating set if it satisfies condition (ii) and every vertex in V is dominated by at least k vertices in S. Clearly, m-connected k-tuple domination is stronger than m-connected k-domination in the sense that every m-connected k-tuple dominating set is a m-connected k-dominating set. We proposed two algorithms for the cases of $m=1$ and $m=2$ with performance ratios less than $(6+\ln\frac{5}{2}(k-1)+\frac{5}{k})$ and $(6+\ln\frac{5}{2}(k-1)+\frac{25}{k})$, respectively. In this paper we will show that the m-connected k-domination version admits approximation algorithms with smaller performance ratios.

4 Approximation algorithms

We first prove the following lemma, which will be used in our performance analysis of proposed algorithms.

Lemma 1 *Let $G=(V,E)$ be a UDG and a natural number k such that $\delta(G)\geq k-1$. Let D_k^* be a minimum k-dominating set of G and S an MIS of G. Then $|S|\leq \max\{\frac{5}{k},1\}|D_k^*|$.*

Proof Let $S_0=S\cap D_k^*$, $X=S\setminus S_0$ and $Y=D_k^*\setminus S_0$. It is clear that X and Y are two disjoint subsets. For all $u\in X$, let $c_u=|N(u)\cap Y|$. As D_k^* is a k-dominating set of G, $c_u\geq k$ for each $u\in X$, and then we have $\sum_{u\in X}c_u\geq k|X|$. For each $v\in Y$, let $d_v=|N(v)\cap X|$. As G is a UDG, for each $v\in Y$, there are at most 5 independent vertices in its neighborhood and $d_v\leq 5$. Hence we have $5|Y|\geq\sum_{v\in Y}d_v$. Moreover, $\sum_{u\in X}c_u=|\{uv\in E: u\in X, v\in Y\}|=\sum_{v\in Y}d_v$, we have $|X|\leq\frac{5}{k}|Y|$. Thus $|S|=|X|+|S_0|\leq\frac{5}{k}|D_k^*\setminus S_0|+|S_0|\leq\max\{\frac{5}{k},1\}|D_k^*|$, which proves the lemma. □

Note that in the above lemma, if S is an independent set of G satisfying that $S\cap D_k^*=\emptyset$, then we have $|S|\leq\frac{5}{k}|D_k^*|$.

4.1 Algorithm for Computing $(1,k)$-CDS

To design an approximation algorithm for minimum m-connected k-dominating set problem for $m\geq 1$ and $k\geq 2$, in this subsection we will start with the simplest case of $m=1$. The basic idea of our algorithm for this case is as follows: First produce a CDS using method proposed in (Wan et al. 2004), and then sequentially produce an MIS $(k-1)$ times such that all vertices in $V\setminus D_A$ are k-dominated by vertices in set D_A. The algorithm is more formally presented as follows.

Algorithm A for computing $(1, k)$-CDS

1. Choose an MIS I_1 of G and a set C such that $I_1 \cup C$ is a CDS.
2. **for** $i := 2$ **to** k
3. Construct an MIS I_i in $G \setminus I_1 \cup \cdots \cup I_{i-1}$
4. **end-for**
5. $D_A := I_1 \cup \cdots \cup I_k \cup C$
6. **return** D_A

Theorem 1 *Algorithm A is an approximation algorithm for the minimum connected k-dominating set problem with performance ratios $(5 + \frac{5}{k})$ for $k \le 5$ and 7 for $k > 5$.*

Proof Suppose that Algorithm A, given graph $G = (V, E)$ and a natural number $k \ge 1$, returns $D_A = I_1 \cup \cdots \cup I_k \cup C$. Let D_k^* be a minimum k-dominating set of G. We will show that D_A is a connected k-dominating set of G. For all $u \in G \setminus D_A$, at the i-th iteration, u is not in I_i and thus it is dominated by one vertex of I_i. At the end, u is dominated by at least k different vertices of $I_1 \cup \cdots \cup I_k$. By the first step of Algorithm A, $C \cup I_1$ is a CDS and thus $I_1 \cup \cdots \cup I_k \cup C$ is connected. Therefore, D_A is a connected k-dominating set of G.

Let $S_i = I_i \cap D_k^*$ for $i = 1, 2, \ldots, k$. By the rule of Algorithm A, we have each $I_i \setminus S_i$ is an independent set and $(I_i \setminus S_i) \cap D_k^* = \emptyset$. Thus it follows from the remark given after Lemma 1 that $|I_i \setminus S_i| \le \frac{5}{k}|D_k^* \setminus S_i|$. To estimate the approximation ratio, we have

$$|I_1 \cup \cdots \cup I_k| = \sum_{i=1}^{k} |S_i| + \sum_{i=1}^{k} |I_i \setminus S_i| \le \sum_{i=1}^{k} |S_i| + \sum_{i=1}^{k} \frac{5}{k}|D_k^* \setminus S_i|$$
$$= \left(1 - \frac{5}{k}\right) \sum_{i=1}^{k} |S_i| + 5|D_k^*|.$$

Moreover, $\sum_{i=1}^{k} |S_i| \le |D_k^*|$. Hence we have $|I_1 \cup \cdots \cup I_k| \le 5|D_k^*|$ for $k \le 5$ and $|I_1 \cup \cdots \cup I_k| \le 6|D_k^*|$ for $k > 5$.

In the end, let C be the set constructed from the first step of Algorithm A. By using the argument for the proof of Lemma 10 in (Wan et al. 2004), we can deduce $|C| \le |I_1|$. Hence it follows from Lemma 1 that $|C| \le \max\{\frac{5}{k}, 1\}|D_k^*|$, and the size of connected k-dominating set D_A is upper bounded by $(5 + \frac{5}{k})|D_k^*|$ for $k \le 5$ and $7|D_k^*|$ for $k > 5$. The size of the optimal solution of connected k-dominating set is at least $|D_k^*|$. The proof is then finished. □

4.2 Algorithm for computing $(2, k)$-CDS

In this subsection, we will study how to design an approximation algorithm for the minimum two-connected k-dominating set problem using Algorithm A for computing $(1, k)$-CDS. Similar to the method proposed in (Wang et al. 2007), our algorithm essentially consists of following four steps:

Step 1. Apply Algorithm A to construct a connected k-dominating set D_A.
Step 2. Compute all the blocks in D_A by computing the 2-connected components through the depth first search.
Step 3. Produce the shortest path in the original graph such that it can connect a leaf block in D_A with other part of D_A but does not contain any vertices in D_A except the two endpoints. Then add all intermediate vertices in this path to D_A.
Step 4. Repeat Step 2 and Step 3 until D_A is 2-connected.

In Step 2, we can apply the standard algorithm proposed in (Tarjan 1972) to compute all blocks in D_A, denote the number of blocks in D_A by ComputeBlock(D_A). The algorithm is more formally presented as follows:

Algorithm B for computing a 2-connected k-dominating set ($k \geq 2$)

```
Produce a connected k-dominating set D_A using Algorithm A
D_B := D_A and B := ComputeBlocks(D_B)
while B > 1 do
    Choose a leaf block L
    for vertex v ∈ L that is not a cut-vertex do
        for vertex u ∈ V \ L do
            Construct G' from G by deleting all vertices in D_B except u and v
            P_uv := shortestPath(G'; v, u) and P := P ∪ P_uv
        end-for
    end-for
    P_ij := the shortest path in P
    D_B := D_B ∪ {the intermediate vertices in P_ij}
    ComputeBlocks(D_B)
end-while
return D_B
```

Lemma 2 *For $k \geq 2$, at most two new vertices are added into D_B at each augmenting step.*

Proof Suppose that L is a leaf block of D_B and w is the cut-vertex. Consider two vertices u and v in D_B with $u \in L \setminus \{w\}$ and $v \in V \setminus L$, let P_{uv} be the shortest path that connects u and v. We claim that P_{uv} has at most two intermediate vertices. Suppose, by contradiction, that P_{uv} contains $u, u_1, u_2, \ldots, u_l, v$, where $l \geq 3$. Since each vertex u_i has at least 2 neighbors in D_B and $N(u_i) \cap D_B \subseteq L$ or $N(u_i) \cap D_B \subseteq (V \setminus L) \cup \{w\}$, $N(u_1) \cap D_B \subseteq L$. If $N(u_2) \cap D_B \subseteq L$, u_2 must have a neighbor s in $L \setminus \{w\}$, then the path between sv has a shorter distance than P_{uv}. Otherwise $N(u_2) \cap D_B \subseteq (V \setminus L) \cup \{w\}$, u_2 must have a neighbor s in $V \setminus L$, then the path between us has a shorter distance than P_{uv}, which contradicts that P_{uv} has the shortest distance. The proof is then finished. □

Lemma 3 *The number of cut-vertices in the connected k-dominating set D_A by Algorithm A is no bigger than the number of vertices in $I_1 \cup C$ generated at the first step of Algorithm A.*

Proof Let $S = I_1 \cup C$ be the connected domination set produced at the first step of Algorithm A. We will show that no vertex in $D_A \setminus S$ is a cut-vertex. For any two vertices $u, v \in S$, there is a path P_{uv} between them that contains only vertices in S. Since any vertex in $D_A \setminus S$ is dominated by at least one vertex in S, Hence, for any two vertices $u, v \in D_A$, there is a path P_{uv} between them that contains only vertices in $S \cup \{u, v\}$. Hence any vertex in $D_A \setminus S$ is not a cut-vertex. The proof is then finished. □

Theorem 2 *Algorithm B is an approximation algorithm for the minimum 2-connected k-dominating set problem with performance ratios* $(5 + \frac{25}{k})$ *for* $2 \le k \le 5$ *and* 11 *for* $k > 5$.

Proof Let D_k^* and D_{opt} be the optimal k-dominating set and 2-connected k-dominating set, respectively. It is clearly that $|D_k^*| \le |D_{opt}|$. After S is constructed in the first step of Algorithm A, by Lemmas 2-3, the algorithm terminates in at most $|C| + |I_1|$ steps, and in each step at most two vertices are added. Since $|C| + |I_1| \le 2|I_1| \le 2\max\{\frac{5}{k}, 1\}|D_k^*|$, we have $|D_B| \le |D_A| + 4\max\{\frac{5}{k}, 1\}|D_k^*|$. It follows from Theorem 1 that $|D_A| \le (5 + \frac{5}{k})|D_k^*|$ for $k \le 5$ and $|D_A| \le 7|D_k^*|$ for $k > 5$. Hence we obtain $|D_B| \le (5 + \frac{25}{k})|D_{opt}|$ for $2 \le k \le 5$ and $|D_B| \le 11|D_{opt}|$ for $k > 5$, which complete the proof. □

4.3 Algorithm for computing (m, k)-CDS

It turns out very difficult to design an approximation algorithm for the minimum m-connected k-dominating set problem for general m and k. In this subsection we will show that how to design such an algorithm assuming that we have algorithm $A_{(m,m)}$ for the case of $m = k$. The basic idea of our algorithm for the case of $k > m$ is as follows: Obtain a (m, m)-CDS by using $A_{(m,m)}$. After that sequentially choose an MIS $(k - m)$ times. The algorithm is more formally presented as follows.

Algorithm C for computing (m, k)-CDS

1. Produce an (m, m)-CDS S of G using algorithm $A_{(m,m)}$
2. **for** $i := 1$ **to** $k - m$
3. Construct an MIS I_i in $G \setminus S \cup I_1 \cup \cdots \cup I_{i-1}$
4. **end-for**
5. $D_C := I_1 \cup \cdots \cup I_{k-m} \cup S$
6. **return** D_C

Theorem 3 *If there exists an* α*-approximation algorithm* $A_{(m,m)}$ *for the case of* $m = k$, *then there exists an* $(\alpha + 6)$*-approximation algorithm for the case of* $k > m$.

Proof We first show that D_C is a (m, k)-CDS of G. For all $u \in G \setminus D_C$, u is not in S and thus it is dominated by at least m vertices of S. And at the i-th iteration, u is not in I_i and thus it is dominated by one vertex of I_i for $i = 1, \ldots, k - m$. At the end, u is dominated by at least k different vertices of D_C. To show that D_C is

m-connected, suppose that there exists a set X of $(m-1)$ vertices in D_C such that the induced subgraph D is disconnected by removing the $(m-1)$ vertices. For S is a (m,m)-CDS, $S \setminus X$ is a connected dominating set. So, $D_C \setminus X$ is connected, a contradiction! Hence D_C is a (m,k)-CDS of G.

Let D_{opt} be an optimal solution to the minimum m-connected k-dominating set problem. It is clear that $|S| \le \alpha|D_{opt}|$, and $|I_1 \cup \cdots \cup I_{k-m}| \le 6|D_{opt}|$ by using similar argument of Theorem 1. This shows that Algorithm C has performance ratio no bigger than $(\alpha+6)$ for $k > m$. The proof is then finished. $\square$

5 Conclusion

In this paper we have studied the minimum m-connected k-dominating set problem in unit disc graphs. We have proposed two approximation algorithms with constant performance ratios for $m \le 2$. We are unable to design such an algorithm for $m \ge 3$, but our study shows that we just need to focus on the case of $m = k$ in further study.

As the proposed algorithms in this paper are centralized ones, it is interesting to know if they can be extended to the distributed and localized methods, which are more applicable for the data transmission in wireless sensor networks. In addition, the performance analysis of the proposed algorithms is based on some geometrical properties of unit disc graphs, so it is a great challenge to study this problem in general case.

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