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Near-Optimal Conflict-Free Channel Set Assignments for an Optical Cluster-Based Hypercube Network

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Abstract. Recently a class of scalable multi-star optical networks is proposed. In this network class nodes are grouped into clusters. Each cluster employs a separate pair of broadcast and select couplers. The clusters are interconnected via fiber links according to a regular topology. This approach efficiently combines time and/or wavelength division with direct space division. One important issue for this network class is the conflict-free channel set assignment that maximizes spatial re-use of channels. This paper studies the conflict-free channel set assignment for the hypercube cluster interconnection topology. The approach in this paper is novel and is conjectured to be optimal.

Keywords:

1. Introduction

Emerging high bandwidth applications, such as voice/video services, distributed databases, and network super-computing, are driving the use of single-mode optical fibers as the communication media for the future (Burr, 1986). Optical passive stars (Dragone, 1988) provide a simple medium to connect nodes in a local or metropolitan area network. The single-star optical networks with time and/or wavelength division multiplexing have been extensively studied in the past (Chen et al., 1990; Hluchyj and Karol, 1991; Wan, —). However, the scalability of the single-star configuration is constrained by the number of wavelengths that can be coupled and separated while maintaining acceptable crosstalk and power budget levels. Recently a multi-star configuration which efficiently combines space with time and/or wavelength division was proposed in (Aly and Dowd, 1994) to overcome this constraint. In this network class nodes are grouped into clusters with time and/or wavelength multiplexing. Clusters are interconnected via fiber links to form a *cluster interconnection network* (CIN). If the cluster size is more than one, self cluster links are provided to enable connectivity among nodes in the same cluster. Wavelength spatial re-use is exploited in the channel set assignment to clusters. This network class has several advantages including low link density, nice scalability and desirable reconfigurability (Aly, 1993).

One important issue in the design of this network class is conflict-free channel set assignment to the transmitters. To reduce the number of required channels, an objective for

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conflict-free channel set assignment is to find the minimal number of disjoint channel sets required by the conflict-free communication. This optimal conflict-free channel set assignment problem has been studied for various CIN topologies in (Aly 1993). In this paper, we use the hypercube as the CIN topology. Two cases are studied. One is without self-cluster links and the other is with self-cluster links. For either case, a simple near-optimal conflict-free channel set assignment scheme is given. The number of channel sets used in either of the two scheme is in general very close to the lower-bound for the minimal number of required channel sets. For some particular values of the number of clusters in the network, the two scheme are optimal. We conjecture that the two assignment scheme are optimal.

The rest of this paper proceeds as follows. Section 2 describes the network configuration. Section 3 presents a graph-theoretic formulation of the conflict-free channel set assignment problems into two vertex coloring problems. Section 4 gives a simple and efficient coloring scheme for each vertex coloring problem. Finally a conclusive discussion is presented in Section 5.

2. Network configuration

In this section we introduce the network configuration of the cluster-based multi-star optical network. The network contains $M = m_0 m_1$ nodes, which are grouped into m_1 clusters. A node represents the lowest abstraction level and may consist of a single processor, multiple time-multiplexed processors, an interface to a space switch, or a broadband network interface unit. Each node possesses a single fixed-wavelength transmitter (light source) and a receiver that is capable to simultaneously monitor a subset of separable channels. A channel here can be a reserved time slot, a dedicated wavelength, or a reserved time slot over a given wavelength. The receiver can be realized using a multichannel acoustooptic tunable filter or a detector array with a passive (grating based) wavelength demultiplexer (Jump, 1992). Each cluster possesses its own broadcast and select domains realized by an output and an input star couplers, respectively. The *cluster interconnection network* (CIN) refers to the fiber connection pattern from output to input couplers. When $m_0 > 1$, each cluster is provided with a self link to enable connectivity among nodes in the same cluster. Figure 1 illustrates a cluster-based 3-cube network.

It's easy to see that the dimension of the output coupler is $m_0 : F$ and that of the input coupler is $F : m_0$, where

$$F = \text{ the degree of the CIN topology}$$

if $m_0 = 1$ and

$$F = 1 + \text{ the degree of the CIN topology}$$

if $m_0 > 1$.

The comunications between transmitters and receivers are as follows. Nodes in a cluster transmit over an ordered set of m_0 distinct channels through the output broadcast star coupler. At the input coupler side, several distinct channel sets are monitored depending on the CIN topology. Transmit channel sets are assigned to the output couplers such that no conflicts may happen at the input coupler. That is, the assignment is such that the channel sets which

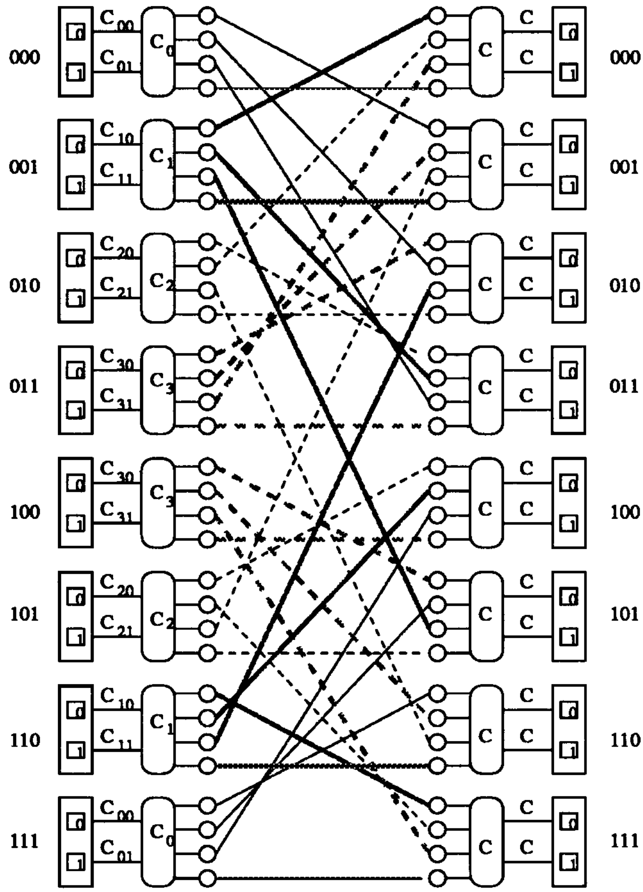

Figure 1. An 8 cluster 3-cube network with self links (2 nodes per cluster). Clusters 000 and 111 transmit over channel set $C_0 = \{C_{00}, C_{01}\}$, Clusters 001 and 110 transmit over channel set $C_1 = \{C_{10}, C_{11}\}$, Clusters 010 and 101 transmit over channel set $C_2 = \{C_{20}, C_{21}\}$, and Clusters 011 and 100 transmit over channel set $C_3 = \{C_{30}, C_{31}\}$. Input couplers receive conflict-free the set of all channels $C = C_0 \cup C_1 \cup C_2 \cup C_3$. Passive individual channel selection can be done at the cluster or node level.

can be listened to through any input coupler are disjoint to provide a collisionless environment. In figure 1, 4 channel sets are used by the conflict-free channel assignment for 3-cube.

The above cluster-based optical network efficiently combines space with time and/or wavelength division. It has several advantages including low link density, nice scalability and desirable reconfigurability. It reduces to an all-spaced network when $m_0 = 1$, and to a time and wavelength division multiplexed (TWDM) network when $m_1 = 1$.

3. Conflict-free channel set assignment: A vertex coloring formulation

Let C_{opt} be the minimal number of disjoint channel sets to satisfy the conflict-free communication condition for a given CIN topology. An objective is to find C_{opt} to reduce the number of required channels. Consequently, larger networks can be built with lower space complexity. Since an input cluster always listens to F output couplers, an immediate lower-bound for C_{opt} is given by the following lemma.

Lemma 1. (*Lower-bound*) $C_{\text{opt}} \geq F$.

To find the tight upper-bound for C_{opt}, we will formulate the conflict-free channel set assignment problems into vertex coloring problems. The formulation is valid for all CIN topologies which can be represented as undirected regular graphs. When an undirected regular graph, such as hypercube, is used as the CIN topology, we actually treat each edge as a bidirectional link. Before we present the vertex coloring formulation, we first give a characterization of the conflict-free communication condition.

Lemma 2. *Suppose that the CIN topology is an undirected regular graph.*
1. *If $m_0 = 1$, then a channel set assignment is conflict-free if and only if any two clusters with distance 2 in the CIN topology have disjoint channel sets.*
2. *If $m_0 > 1$, then a channel set assignment is conflict-free if and only if any two clusters with distance at most 2 in the CIN topology have disjoint channel sets.*

The proof of this lemma is straightforward and we omit the proof here. Now we consider the following two vertex coloring problems.

2-**VC.** Given a regular graph G, a vertex coloring scheme is called a 2-VC of G if no two vertices with distance of two have the same color. The minimal number of colors required by any 2-VC of G is denoted by $\chi_2(G)$.

$\bar{2}$-**VC.** Given a regular graph G, a vertex coloring scheme is called a $\bar{2}$-VC of G if no two vertices with distance of at most two have the same color. The minimal number of colors required by any $\bar{2}$-VC of G is denoted by $\chi_{\bar{2}}(G)$.

If we regard clusters as vertices and the channel sets as colors, then from Lemma 2, we can establish the equivalence between the conflict-free channel set assignment problems and the above vertex coloring problems as stated in the next lemma.

Lemma 3. *Suppose that the CIN topology G is an undirected graph.*
1. *If $m_0 = 1$, then $C_{\text{opt}} = \chi_2(G)$.*
2. *If $m_0 > 1$, then $C_{\text{opt}} = \chi_{\bar{2}}(G)$.*

So in order to find the optimal conflict-free channel set assignment for a given CIN topology G, we only need to find the values of $\chi_2(G)$ and $\chi_{\bar{2}}(G)$. In the next section, we will give tight upper-bounds for $\chi_2(C_n)$ and $\chi_{\bar{2}}(C_n)$ where C_n is the n-dimensional hypercube or n-cube in short.

4. Coloring schemes for *n*-cube

In a binary n-dimensional hypercube C_n, we index each vertex by a binary n-vector. For each $1 \leq i \leq n$, the ith entry of a binary n-vector v is denoted by v_i. So v can be represented as $(v_1, v_2, \ldots, v_n)$. The sum of two binary n-vectors is defined by the modulo two sum (or exclusive or). The *Hamming distance* between two binary n-vectors u and v, denoted by $d(u, v)$, is defined as the number of positions at which the entries of u and v are different. It's well known that the distance between two vertices u and v in an n-cube is equal to the Hamming distance between u and v.

Before we describe the coloring schemes, we introduce a special class of matrices called *binary representation matrices* (BRM). The binary representation matrix BRM_k of order k is of dimension $k \times \lceil \log_2 k \rceil$, in which row i is the binary $\lceil \log_2 k \rceil$-vector of the number i for $1 \leq i \leq k$. For example, the matrix

$$\mathrm{BRM}_5 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

is the binary representation matrix of order 5.

In the next, we will describe our coloring schemes. The colors are also indexed by binary vectors. For any binary n-vector v, we use $\hat{v}$ to denote the binary $(n-1)$-vector obtained from v by removing the last entry of v, i.e., $\hat{v} = (v_1, v_2, \ldots, v_{n-1})$.

2-Scheme. For any vertex v, coloring v with the color $\hat{v}\mathrm{BRM}_{n-1}$.
$\bar{2}$-Scheme. For any vertex v, coloring v with the color $v\mathrm{BRM}_n$.

We will show that 2-Scheme is a 2-VC of C_n, and $\bar{2}$-Scheme is a $\bar{2}$-VC of C_n.

Theorem 1. *2-Scheme is a 2-VC of C_n.*

Proof: Let u and v be any two vertices with $d(u, v) = 2$. Suppose that $u_i \neq v_i$ and $u_j \neq v_j$ for some $1 \leq i < j \leq n$. We consider two cases.

Case 1. $j = n$. In this case,

$$\hat{u}\mathrm{BRM}_{n-1} - \hat{v}\mathrm{BRM}_{n-1} = (\hat{u} - \hat{v})\mathrm{BRM}_{n-1} = \mathrm{BRM}_{n-1_i}$$

which is the binary $\lceil \log_2 n \rceil$-vector of the number i. Therefore

$$\hat{u}\mathrm{BRM}_{n-1} \neq \hat{v}\mathrm{BRM}_{n-1}$$

which means u and v have different colors.

Case 2. $j < n$. In this case,

$$\hat{u}\mathrm{BRM}_{n-1} - \hat{v}\mathrm{BRM}_{n-1} = (\hat{u} - \hat{v})\mathrm{BRM}_{n-1} = \mathrm{BRM}_{n-1_i} + \mathrm{BRM}_{n-1_j} \neq 0$$

since $i \neq j$. This implies that u and v have different colors.

So in either case, u and v have different colors. Therefore, 2-Scheme is a 2-VC of C_n. □

A vertex coloring scheme is said to be *blanced* if each color is assigned to the same number of vertices. Balance is a desired property for vertex colorings. The next lemma shows that the 2-Scheme has this nice property.

Lemma 4. *The 2-Scheme uses $2^{\lceil \log_2 n \rceil}$ colors and it is a balanced vertex coloring scheme.*

Proof: It's easy to see that 2-Scheme uses $2^{\lceil \log_2 n \rceil}$ colors. So we only need to prove that each color is assigned to $2^{n-\lceil \log_2 n \rceil}$ vertices. Since the rank of the matrix BRM_{n-1} is equal to $\lceil \log_2 n \rceil$, the dimension of the null space of BRM_{n-1} is equal to $n-1-\lceil \log_2 n \rceil$. Notice that each binary $(n-1)$-vector $\hat{v}$ corresponds to two binary n-vectors. Therefore any color is assigned to $2 \cdot 2^{n-1-\lceil \log_2 n \rceil} = 2^{n-\lceil \log_2 n \rceil}$ vertices of C_n. □

When $n = 2^k$ for some k, the 2-Scheme uses n colors, which is the lower-bound for $\chi_2(C_n)$. Thus we can obtain the following corollary.

Corollary 1. *If $n = 2^k$ for some k, then 2-Scheme is an optimal 2-VC of C_n.*

In general, since

$$\chi_2(C_n) \le 2^{\lceil \log_2 n \rceil} \le 2n \le 2\chi_2(C_n),$$

2-Scheme is a near-optimal 2-VC for C_n.

Now we prove that $\bar{2}$-Scheme is a $\bar{2}$-VC for C_n.

Theorem 2. *$\bar{2}$-Scheme is a $\bar{2}$-VC for C_n.*

Proof: Let u and v be any two vertices with $d(u, v) \le 2$. We consider two cases.

Case 1. $d(u, v) = 1$. Suppose that $u_i \ne v_i$ for some $1 \le i \le n$. Then

$$u\mathrm{BRM}_n - v\mathrm{BRM}_n = (u - v)\mathrm{BRM}_n = \mathrm{BRM}_{ni}$$

which is the binary $\lceil \log_2(n+1) \rceil$-vector of the number i. Therefore

$$u\mathrm{BRM}_n \ne v\mathrm{BRM}_n$$

which means u and v have different colors.

Case 2. $d(u, v) = 2$. Suppose that $u_i \ne v_i$ and $u_j \ne v_j$ for some $1 \le i < j \le n$. Then

$$u\mathrm{BRM}_n - v\mathrm{BRM}_n = (u - v)\mathrm{BRM}_n = \mathrm{BRM}_{ni} + \mathrm{BRM}_{nj} \ne 0$$

since $i \ne j$. This implies that u and v have different colors.

So in either case, u and v have different colors. Therefore, $\bar{2}$-Scheme is a $\bar{2}$-VC for C_n. □

Similar to the 2-Scheme, the $\bar{2}$-Scheme is also a balanced vertex coloring scheme.

Lemma 5. *The $\bar{2}$-Scheme uses $2^{\lceil \log_2 (n+1) \rceil}$ colors and it is a balanced vertex coloring scheme.*

The proof is similar to the proof for the 2-Scheme and we omit it here.

When $n = 2^k - 1$ for some k, the $\bar{2}$-Scheme uses $n + 1$ colors, which is the lower-bound for $\chi_{\bar{2}}(C_n)$. Thus we can obtain the following corollary.

Corollary 2. *If $n = 2^k - 1$ for some k, then $\bar{2}$-Scheme is an optimal $\bar{2}$-VC for C_n.*

In general, since

$$\chi_{\bar{2}}(C_n) \leq 2^{\lceil \log_2 (n+1) \rceil} \leq 2(n+1) \leq 2\chi_{\bar{2}}(C_n),$$

$\bar{2}$-Scheme is at least a suboptimal $\bar{2}$-VC for C_n.

From Lemma 1 and the above two theorems, we have

$$n \leq \chi_2(C_n) \leq 2^{\lceil \log_2 n \rceil},$$
$$n + 1 \leq \chi_{\bar{2}}(C_n) \leq 2^{\lceil \log_2 (n+1) \rceil}.$$

The above two corollaries say that the lower-bound can be achived for some special sizes of the network. One question is whether the lower-bound given in Lemma 1 is always achieveable. The following two examples give a negative answer.

Example 1. $\chi_2(C_3) = 2^{\lceil \log_2 3 \rceil} = 4$. This can be proved in the following way. Without loss of generality, assume that vertex 000 has color 1, vertex 011 has color 2 and vertex 101 has color 3. We consider the color of vertex 110. Since the vertex 110 has distance of 2 from each of the three vertices 000, 011 and 101, it must have some color different from color 1, 2 and 3. Therefore, $\chi_2(C_3) \geq 4$. On the other hand, 4 is an upper bound for $\chi_2(C_3)$. So $\chi_2(C_3) = 4$.

Example 2. $\chi_{\bar{2}}(C_2) = 2^{\lceil \log_2 3 \rceil} = 4$. The proof is straightforward, since in H_2 the distance between any pair of vertices is at most two.

From the above two examples, we raise the the following conjecture. We conjecture that

Conjecture 1. *For any positive integer n,*

$$\chi_2(C_n) = 2^{\lceil \log_2 n \rceil},$$
$$\chi_{\bar{2}}(C_n) = 2^{\lceil \log_2 (n+1) \rceil}.$$

If this conjecture is true, then 2-Scheme is an optimal 2-VC for C_n and $\bar{2}$-Scheme is an optimal $\bar{2}$-VC for C_n.

5. Conclusion

This paper studied a passive optical realization of binary hypercube networks. The considered configuration is cluster-based and has the potential of combining both time and wavelength division multiplexing with a space-connected structure to achieve efficient scalability. The focus of this paper is on deriving optimal conflict-free channel sets assignments. We formulate the problem of optimal conflict-free channel sets assignments into vertex coloring problems. Two near-optimal coloring schemes are given and are conjectured to be optimal.

This research suggests an interesting question in combinatorics. Given a regular graph G, what is the minimum number $\chi_{\bar{k}}(G)$ of vertex colors such that every two vertices with distance at most k have different colors. We studied $\chi_{\bar{2}}(C_n)$ for n-cube C_n in this paper. What is $\chi_{\bar{k}}(C_n)$ for $k \geq 3$? It is still open. In fact, our approach may not be able to apply to the case $k \geq 3$.

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