Minimizing Drop Cost for SONET/WDM Networks with $\frac{1}{8}$ Wavelength Requirements

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SONET/WDM networks using wavelength add–drop multiplexing can be constructed using certain graph decompositions used to form a "grooming," consisting of unions of certain primitive rings. The existence of such decompositions when every pair of sites employs no more than $\frac{1}{8}$ of the wavelength capacity is determined, with few possible exceptions, when the ring size is a multiple of four. The techniques developed rely heavily on tools from combinatorial design theory. © 2001 John Wiley & Sons, Inc.

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1. THE COMBINATORIAL PROBLEM AND SONET/WDM NETWORKS

A *synchronous optical network* (*SONET*) *ring* on n sites is an optical interconnection device. The sites are arranged circularly. A *clockwise* or *right ring* connects the ith site to the $(i+1)$st, and a *counterclockwise* or *left ring* connects the ith site to the $(i-1)$st. This provides two *directions* in which traffic can be delivered between any two sites.

Each optical connection can carry multiple signals on different wavelengths. However, the number of wavelengths is limited, and the bandwidth on each wavelength is also limited. Typically, one goal is to minimize the number of wavelengths used. An equally important goal is to ensure that each wavelength has a sufficient bandwidth for the traffic it is to carry; we return to this later.

In this paper, we examine the situation in which every pair of sites needs a communication path, and each such path requires only a fixed fraction $1/g$ of the capacity of a wavelength. This problem has been studied primarily in the context of variable traffic requirements [4, 11, 10], but the case of fixed traffic requirements has served as an important special case [11, 12]. In this introduction, we establish that the wavelength assignment problem which arises can be reduced to a problem on partitioning complete graphs with loops into a small set of fixed subgraphs. Then, in the remainder of the paper, we develop techniques for solving the wavelength assignment when the SONET ring has a size which is a multiple of four.

We require a substantial set of mathematical preliminaries. Let n be a positive integer. Let $\mathbb{Z}_n$ denote the set $\{0, \ldots, n-1\}$; when the arithmetic is done on the elements of $\mathbb{Z}_n$, it is carried out modulo n. Let $\mathscr{P}_n$ be the set $\{(i, j) : i, j \in \mathbb{Z}_n,\ i \neq j\}$. Partition the sets of $\mathscr{P}_n$ into two sets, $\mathscr{L}$ and $\mathscr{R}$. Then, associate with each pair $P = (i, j) \in \mathscr{P}_n$ the set $S(P) = \{i, i+1, \ldots, j-1\}$ if $P \in \mathscr{R}$, or the set $S(P) = \{j+1, \ldots, i-1, i\}$ if $P \in \mathscr{L}$.

Partition $\mathscr{P}_n$ into s classes, $C_1, \ldots, C_s$. Compute the multiset union $M_i = \bigcup_{P \in \mathscr{R} \cap C_i} S(P)$ and the multiset union $N_i = \bigcup_{P \in \mathscr{L} \cap C_i} S(P)$. Let g be an integer. If, for every $1 \leq i \leq s$, the multisets M_i and N_i do not contain any symbol of $\mathbb{Z}_n$ more than g times, then $C_1, \ldots, C_s$ is an (s, g)-*assignment* for the partition $\mathscr{L}, \mathscr{R}$ of $\mathscr{P}_n$.

Since we are at liberty to choose the partition of $\mathscr{P}_n$ into $\mathscr{L}$ and $\mathscr{R}$, we define an (n, s, g)-*assignment* to be the partition together with the (s, g)-assignment for that partition. We shall be concerned with those (n, s, g)-assignments that minimize s for particular values of n and g. Among these assignments, we prefer certain ones realizing a minimality condition, described next.

Consider a particular (n, s, g)-assignment. Let $V_i = \{x, y : (x, y) \in C_i\}$. The *drop cost* of the assignment is defined to be $\sum_{i=1}^{s} |V_i|$. For specific choices of n and g, what is the smallest value of s that we can achieve? For (n, s, g)-assignments, what is the smallest drop cost

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that we can achieve? We address these two questions and describe an approach that uses techniques from combinatorial design theory and graph decompositions to obtain results on the existence of such assignments.

Let us first examine how the basic SONET situation is modeled in the combinatorial formulation. The sites of the SONET ring are the elements in $\mathbb{Z}_n$. Then, $\mathscr{P}_n$ simply indicates the pairs of sources and destinations for communication. The choice of a left or right ring on which traffic is to be routed is indicated by the partition into $\mathscr{L}$ and $\mathscr{R}$. Then, for $P \in \mathscr{P}_n$, the set $S(P)$ is precisely the originating site together with all the intermediate sites through which the traffic flows for the pair P. The number of wavelengths used is s, and the partition into classes $C_1, \ldots, C_s$ specifies the chosen wavelengths.

Suppose that the pairs $P_1, \ldots, P_t$ have all been assigned to the same direction and the same wavelength. Suppose further that some site $h \in \mathbb{Z}_n$ has the property that $h \in \bigcap_{i=1}^{t} S(P_i)$. Then, all traffic involving these source–destination pairs must be routed through site h. If the traffic requirement for these t pairs exceeds the bandwidth, then site h would be unable to handle the traffic. In the absence of specific information about the traffic requirements, we suppose that no pair has a traffic requirement exceeding $\frac{1}{g}$ times the bandwidth. Then, the condition on M_i and N_i ensures that a sufficient bandwidth is available on each wavelength in each direction.

In a communication, the source and destination sites typically convert between the electrical and optical domains, while intermediate sites are all-optical forwarding devices. To start or terminate a connection is more expensive. *Wavelength add–drop multiplex* (*WADM*) permits a wavelength to bypass a node without the costly termination when no traffic on the wavelength originates or terminates at the node; a *SONET add–drop multiplexer* (SONET ADM) accomplishes this task. Hence, the costs of a SONET ring configuration using WADM can often be lowered by reducing the number of different source and destination sites on each wavelength. The drop cost of the assignment defined earlier gives the number of SONET ADMs employed, and our interest is to minimize this number.

We have outlined an optical communications environment in which traffic from a particular source to a particular destination remains on a single wavelength. However, only for certain (s, g)-assignments can we partition each wavelength into g channels so that that each source–destination pair remains on a single channel. The implementation is substantially simplified when we can do so. If an (s, g)-assignment can be partitioned into channels in this way, then it arises from an $(sg, 1)$-assignment by forming s unions of g classes each. This special type of assignment is a *grooming*. We shall typically require that the assignments produced are, in fact, groomings.

2. PRIMITIVE RINGS

The case when $g = 1$ arises when each communication requires the entire bandwidth available on a wavelength. Two pairs P and P' can, nonetheless, share the bandwidth if they are on opposite (left and right) rings or if they are on the same ring and $S(P) \cap S(P') = \emptyset$. Our task is then to partition the set $\mathscr{P}_n$ into s wavelengths and two directions, so that within each we find each site at most once. It is then necessary that any such partition of the pairs $\mathscr{P}_n$ also partition the multiset $\mathscr{M} = \bigcup_{P \in \mathscr{P}_n} S(P)$ into s classes and two directions, each containing every symbol in $\mathbb{Z}_n$ at most once. Now, $S(P)$ depends upon the direction chosen for P, but let us suppose for the moment that $S(P)$ contains at most $n/2$ elements. This can be guaranteed *if* we choose the shorter direction around the ring. Then, $\mathscr{M}$, by an easy counting argument, has $(n^3 - n)/4$ elements when n is odd and $n^3/4$ when n is even. Since each wavelength accounts for at most $2n$ of the entries in $\mathscr{M}$, we require that $s \geq \lceil \frac{n^2-1}{8} \rceil$ wavelengths be available when n is odd and $s \geq \lceil \frac{n^2}{8} \rceil$ when n is even. See, for example, [2, 11].

Wan [12] described a very useful set of "primitive rings" which provide the wavelength assignment. We review his method briefly: First, suppose that $n = 2m$ is even. Define the rings $Q^n_{ij} = \{(i, j), (j, i + m), (i + m, j + m), (j + m, i)\}$; when these rings are routed clockwise and $i < j < i + m$, the pairs of each ring can be placed on a single wavelength. Next, define the rings $R_i = \{(i, i + m), (i + m, i)\}$.

We place the pairs in the rings Q^n_{ij} in $\mathscr{R}$ when $0 \leq i < j < m$. Then, for $0 \leq i < \lceil n/4 \rceil$, we place the ring R_i in $\mathscr{R}$. All other pairs are placed in $\mathscr{L}$. Indeed, when Q^n_{ij} is placed in $\mathscr{R}$, we suppose that Q^n_{ji} is placed in $\mathscr{L}$. The rings Q^n_{ij} and Q^n_{ji} involve the same sites and can be placed on the same wavelengths in opposite directions.

It is easily verified that such a set of primitive rings minimizes the number of wavelengths (indeed, every wavelength is used at every single site). When $n = 2m + 1$ is odd, a simple variant can be used. Begin with the primitive rings described for the even order $n - 1$: We introduce a new symbol ∞ between $n - 1$ and 0. Rings Q^{n-1}_{ij} are placed as before. However, the rings R_i are removed, and each is replaced by two rings, $T^n_i = \{(\infty, i), (i, i + m), (i + m, \infty)\}$ in $\mathscr{R}$ and $T^n_{i+m} = \{(\infty, i + m), (i + m, i), (i, \infty)\}$ in $\mathscr{L}$.

Using this solution for $g = 1$, one way to produce an assignment with $g > 1$ is to combine up to g primitive rings, while minimizing the number of wavelengths and drop costs. A method for forming an assignment by taking unions of primitive rings is called a *grooming* of primitive rings. The most natural case to treat is when g is a power of two as a result of the bandwidth hierarchy available in SONET rings.

Wan [12] solved the cases when $g = 2$ and $g = 4$ and develops some general techniques. The solution when

$g = 2$ is straightforward using his model, while the solution for $g = 4$ is complicated by the necessity of constructing numerous small solutions for use in a recursive combinatorial technique. We review his method briefly: First, let $n = 2m + 1$ be odd. Form a graph LK_m, a complete graph with a loop on each vertex. Associate with edge $\{i, j\}$, $i < j$, the primitive ring Q_{ij}^{n-1} and with each loop $\{i\}$ the primitive ring T_i^n. Now choose a subgraph of LK_m having q edges and p vertices of nonzero degree. If we place the primitive rings corresponding to these edges in the clockwise direction on the same wavelength, the bandwidth suffices *exactly when* $g \geq q$. Indeed, this simply restates the requirement that every site be involved as a sender or intermediate site in at most g primitive rings on the same wavelength and same direction. What about the drop cost? Each edge $\{i, j\}$ with $i < j$ corresponds to a primitive ring involving four sites, while each loop corresponds to a primitive ring involving three. When a vertex appears in more than one edge, a savings results. When no loops are chosen, the *cost* for this wavelength (or subgraph) is easily calculated to be $2p$; when at least one loop is chosen, it is $2p + 1$.

When n is even, the situation is fundamentally the same, but some details differ. We examine this next. While for the case of odd n, the primitive rings fall naturally into pairs, the primitive rings R_i do not "naturally" pair when n is even. Indeed, if we select a graph in the decomposition whose number of nonloop edges plus *half* the number of loops does not exceed eight, then we can make a valid assignment to wavelengths as follows: On the clockwise ring, we place a primitive ring of size four, placing its reversal on the counterclockwise ring. However, for primitive rings of size two, we place half on the clockwise ring and half on the counterclockwise ring. Hence, in the even case, we treat loops differently than when the ring size is odd. In this case, the *cost* of a subgraph is always simply twice the number of vertices of nonzero degree in the subgraph.

Our task in both cases has been reduced to a *graph decomposition problem*. Partition the edges of LK_m into subgraphs, each containing at most g edges (counting loops as edges when the ring size is odd and as half-edges when the ring size is even) so that the number of subgraphs is minimized *and* so that the total cost of all chosen subgraphs is minimized. See [5, 9] for results on graph decompositions in general and for further references.

3. GROOMING WITH $g = 8$

We examine the problem of grooming when $g = 8$. If a subgraph on eight edges is chosen, we must minimize the number of vertices. Evidently, three or fewer vertices do not suffice to support eight edges. However, with four vertices, if we choose two, three, or four loops, we can place four, five, or six other edges to form a subgraph. When loops only contribute a half-count, including all four loops permits us to select all six nonloop edges. Hence, it appears that our "favorite" subgraphs are those with four vertices. However, each such subgraph uses loops and there is a small supply of loops. Hence, we also require some subgraphs on five vertices and eight edges. In the absence of loops, there are two such subgraphs, as shown in Figure 1. The names G_{20} and G_{21} follow the numbering in [3].

The existence problem for decompositions into either of these graphs is open, although substantial partial results are known [9]. Using techniques from combinatorial design theory, we are able to establish recursive constructions that enable us to employ such graph decompositions for small values of m to produce decompositions for larger values of m. Indeed, these constructions establish that if the graph decomposition problems can be solved in a small finite number of cases it can be then be solved for every m. This necessitates the generation of graph decompositions for "small" orders. To illustrate the type of problem encountered, we display in Table 1 a decomposition of K_{16} into isomorphic copies of G_{20} of Figure 1.

In general, a decomposition of K_m (or LK_m) contains $\frac{m^2}{16} + O(m)$ subgraphs, so finding and presenting decompositions can be lengthy even for relatively small values of m if no additional structure is assumed. Indeed, the decomposition of Table 1 was found only after an extensive backtrack search by computer. Fortunately, many decompositions do exhibit an additional structure. To illustrate this, we show a set of four graphs in Figure 2.

We interpret the vertex labels in each of these graphs as elements of $\mathbb{Z}_m$, where $m = 65$. Let us denote a copy of G_{20} as $[a, b, c, d|e]$, where all edges are present *except* for $\{c, e\}$ and $\{d, e\}$. For example, the first graph of Figure 2 is denoted by $[0, 1, 22, 29|18]$. To form the decomposition of K_m, which has $260 = \frac{1}{8}\binom{m}{2}$ graphs in it, we produce for each graph $[a, b, c, d|e]$ in Figure 2 the m graphs $[a + i, b + i, c + i, d + i|e + i]$ for $i \in \mathbb{Z}_m$. Vertex labels are reduced modulo m whenever needed. It is easy to verify that each edge appears in exactly one of the 260 graphs that result, as follows: To check that the pair $\{k, \ell\}$ is in exactly one of these graphs, let us

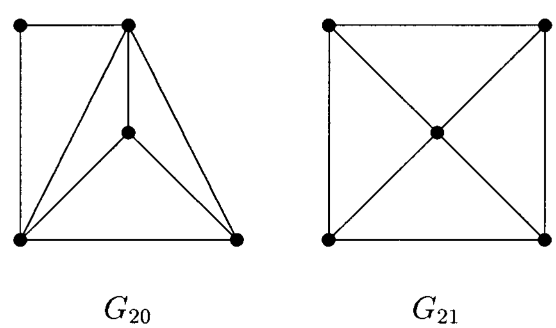

FIG. 1. The two graphs.

TABLE 1. Partition of K_{16} into G_{20}.

{0,4}	{0,5}	{0,6}	{0,7}	{4,5}	{4,6}	{4,7}	{5,6}
{0,8}	{0,9}	{0,10}	{0,11}	{8,9}	{8,10}	{8,11}	{9,10}
{0,12}	{0,13}	{0,14}	{0,15}	{12,13}	{12,14}	{12,15}	{13,14}
{1,2}	{1,0}	{1,3}	{1,4}	{2,0}	{2,3}	{2,4}	{0,3}
{1,5}	{1,7}	{1,8}	{1,9}	{5,7}	{5,8}	{5,9}	{7,8}
{1,6}	{1,10}	{1,12}	{1,13}	{6,10}	{6,12}	{6,13}	{10,12}
{14,15}	{14,1}	{14,11}	{14,4}	{15,1}	{15,11}	{15,4}	{1,11}
{2,5}	{2,11}	{2,12}	{2,13}	{5,11}	{5,12}	{5,13}	{11,12}
{2,14}	{2,7}	{2,10}	{2,9}	{14,7}	{14,10}	{14,9}	{7,10}
{6,8}	{6,2}	{6,15}	{6,14}	{8,2}	{8,15}	{8,14}	{2,15}
{3,5}	{3,10}	{3,15}	{3,14}	{5,10}	{5,15}	{5,14}	{10,15}
{3,6}	{3,7}	{3,11}	{3,9}	{6,7}	{6,11}	{6,9}	{7,11}
{3,8}	{3,4}	{3,12}	{3,13}	{8,4}	{8,12}	{8,13}	{4,12}
{4,11}	{4,10}	{4,13}	{4,9}	{11,10}	{11,13}	{11,9}	{10,13}
{7,9}	{7,13}	{7,15}	{7,12}	{9,13}	{9,15}	{9,12}	{13,15}

observe that $\{k, \ell\}$ and $\{k+i, \ell+i\}$ are in the *same number* of graphs; in other words, every two edges whose endpoints have the same *difference* modulo m are in the same number of graphs. Thus, it suffices to check that, among the $\frac{m-1}{2}$ edges of the four graphs of Figure 2, every difference modulo m is represented once (an edge $\{k, \ell\}$ accounts for two differences, namely, $k-\ell \bmod m$ and $\ell - k \bmod m$). Since each of the graphs shown is used to generate further graphs (in this case, m graphs for each), we call each of the graphs shown a *base graph* (under the action of $\mathbb{Z}_m$, i.e., addition modulo m). This simplifies our task dramatically, since we can find and present just the base graphs. In fact, the decomposition of Figure 2 was easily found by a hand computation. We shall use similar techniques for numerous small cases.

These "difference" techniques have been used extensively, relating to the existence of cyclic designs and to various graph labelings; see [6].

4. $\mathcal{G}$-GROUP DIVISIBLE DESIGNS

In this section, we treat the cases when the ring size is a multiple of eight. To do this, we need not only consider decompositions of complete graphs, but also complete graphs with specified "missing" complete subgraphs. To make this precise, for a given number m of vertices, when $m = \sum_{i=1}^{s} g_i u_i$, we write the partition of a set of size m into u_i classes of size g_i for $1 \le i \le s$ by the exponential notation $g_1^{u_1} \cdots g_s^{u_s}$. We call the partition sizes the *group type*. Now, let $T = g_1^{u_1} \cdots g_s^{u_s}$ be a group type for order m. Then, we denote by $G(T)$ the graph on m vertices obtained by first identifying a partition of type T of the vertices, calling the equivalence classes of the partition the *groups*. Then, $G(T)$ contains precisely those edges whose endpoints are in different groups. Using graph theoretic nomenclature, $G(T)$ is a complete multipartite graph, with T representing the sizes of the classes in the partition.

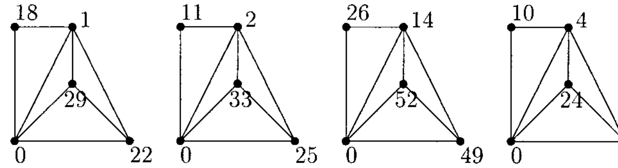

FIG. 2. Partition of K_{65} into G_{20}.

The basic idea of using decompositions of complete multipartite graphs is to partition "almost all" edges of LK_n, leaving behind only those on the groups. In this way, given partitions of the edges within each group, we can partition the larger graph. When groups are small, we need only develop partitions then for small complete graphs. This technique of "filling in holes" is a standard one in combinatorial design theory.

Let $\mathcal{G}$ be a class of graphs. A *$\mathcal{G}$-group divisible design* (*GDD* for short) of *type T* is a partition of all edges of $G(T)$ into graphs, so that each graph of the partition is isomorphic to a graph in the class $\mathcal{G}$. Our interest is in $\{G_{20}, G_{21}\}$-GDDs, so we assume that $\mathcal{G} = \{G_{20}, G_{21}\}$ unless explicitly stated otherwise. We permit group sizes to equal 0 and also the number u_i of groups of size g_i to be 0. We also permit that $g_i = g_j$ for $i \neq j$, so that, for example, 4^5 is the same as $4^4 4^1$. Let us consider an example: Let $m = 20$, $s = 1$, $g_1 = 4$, and $u_1 = 5$, so that the group type T is 4^5. On the vertex set $\mathbb{Z}_{20}$, define a set of five groups of size four as $\{i, i+5, i+10, i+15\}$ for $0 \le i \le 4$. Now use the graph of Figure 3 as the base graph under addition modulo 20 to form a set of 20 graphs. These 20 graphs partition $G(T)$, using the groups defined. (This is also easily checked. Edges whose endpoints have a difference which is a multiple of 5 are inside the groups; all other differences modulo 20 appear exactly once among the edges of the graph in Figure 3.)

A solution of a similar type is given for type 4^9 via the base graphs of Figure 4. Here, the groups are formed by edges having differences which are the multiples of 9.

If one attempts to employ this method for general $\mathcal{G}$-GDDs of type 4^n, it does not work. To see this, calculate the number of base graphs needed. Since $G(4^n)$ contains $\binom{n}{2} \cdot 4^2$ edges, the number of graphs required is $n(n-1)$. If addition modulo $4n$ is to generate these graphs from base graphs, then $\frac{n-1}{4}$ base graphs are needed so that this variety of decomposition can exist only when $n \equiv 1 \pmod 4$.

We therefore develop of variant of the technique to treat further small cases. Instead of choosing the vertex

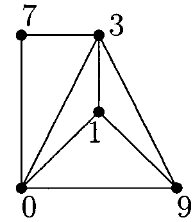

FIG. 3. Type 4^5 on $\mathbb{Z}_{20}$.

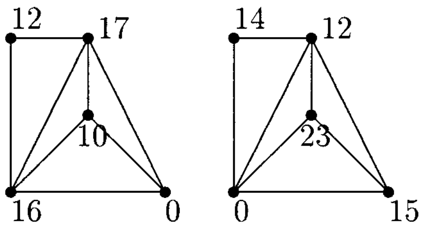

FIG. 4. Type 4^9 on $\mathbb{Z}_{36}$.

set to be $\mathbb{Z}_{4n}$, we employ the vertex set $\mathbb{Z}_{2n} \times \{0,1\}$, consisting of ordered pairs whose first component is from $\mathbb{Z}_{2n}$ and whose second component is from $\{0,1\}$. We write x_i for the element $(x,i) \in \mathbb{Z}_{2n} \times \{0,1\}$. We can again use base graphs, but in this case, each base graph generates $2n$ graphs, obtained by repeatedly adding 1 modulo $2n$ to the first component. Groups can be chosen as $\{x_i, (x+n)_i : i \in \{0,1\}\}$ for $0 \leq x < n$. Base graphs for a decomposition of type 4^7 of this variety are given in Figure 5; we leave the verification to the reader.

A decomposition obtained in a similar manner for type 4^{11} has the base graphs shown in Figure 6. Similar solutions for types 4^{15} and 4^{19} are given in Table 2.

Employing $\mathbb{Z}_{2n} \times \{0,1\}$ treats only those cases when n is odd. We therefore develop a similar strategy to treat small cases when n is even. To illustrate, let the point set be $\mathbb{Z}_{4(n-1)}$ together with four points $\{\beta_0, \beta_1, \beta_2, \beta_3\}$. We use the notation $(a,b,c,d)_e$ to denote the graph isomorphic to G_{21} missing the edges $\{\{a,c\},\{b,d\}\}$. When $a,b,c,d \in \mathbb{Z}_{4(n-1)}$, we denote by $(a,b,c,d)_\beta$ the base graph generating the set of graphs $(a+i, b+i, c+i, d+i)_{\beta_i}$, where the entries are reduced modulo $4(n-1)$ and the subscript on β is reduced modulo 4. Then, if a,b,c,d are all distinct modulo 4, this family of graphs contains every edge of the form $\{x, \beta_i\}$ for $x \in \mathbb{Z}_{4(n-1)}$ and $i \in \{0,1,2,3\}$ exactly once. The remaining base graphs only involve elements of $\mathbb{Z}_{4(n-1)}$ and are developed under addition modulo $4(n-1)$ as before. A decomposition of this type of type 4^8 is shown in Figure 7. It is essential that the graph involving β have all four remaining vertices distinct modulo 4, and we can check that $\{0,5,7,22\} \bmod 4 = \{0,1,3,2\}$. To be precise, the groups here are $\{i, i+(n-1), i+2(n-1), i+3(n-1)\}$ for $0 \leq i < n-1$, *along with* $\{\beta_0, \beta_1, \beta_2, \beta_3\}$.

A decomposition of type 4^{12} obtained in the same manner is shown in Figure 8.

We employ a variant to treat small cases when $n \equiv 2 \pmod 4$. We essentially combine the idea of employing a modulus for addition which is half the size with the use of certain "extra" points. Take as vertex set $\mathbb{Z}_{2(n-1)} \times \{0,1\}$ together with four "extra" vertices. When a base graph does not involve an extra point, it is treated as before. We use two conventions for specifying the extra points. Suppose that $a,b,c,d \in \mathbb{Z}_{2(n-1)} \times \{0,1\}$ so that $\{a,b,c,d\}$ contains one even element and one odd element from each of $\mathbb{Z}_{2(n-1)} \times \{0\}$ and $\mathbb{Z}_{2(n-1)} \times \{1\}$. Then, the notation $(a,b,c,d)_\alpha$ generates a base graph under addition modulo $2(n-1)$ in which the α denotes *two* extra elements, say α_0 and α_1. When addition is performed, the subscript of α is computed modulo 2. It is again easily checked that this results in all pairs involving α_0 and α_1 being employed exactly once. The second convention is that the symbol ∞ denotes an element which is fixed under the addition. When the symbol ∞ or α is used in different base graphs of the same solution, the multiple instances represent *distinct* symbols to be added; in this sense, ∞ and α are used as notational conveniences to indicate the way in which new symbols are added, rather than representing the actual symbols to be added. In each case, however, the extra elements are specified; they form one of the groups. A solution of this variety of type 4^{10} appears in Figure 9 and similar decompositions for types 4^{14}, 4^{26}, and 4^{30} appear in Table 3.

We now describe a different type of construction for $\mathscr{G}$-GDDs. Let G be a graph which is *regular* of degree d, that is, so that every vertex is in exactly d edges of G. A *1-factor* of G is a spanning subgraph of G which is regular of degree 1. A *1-factorization* of G is a partition $F_1, \ldots, F_d$ of the edges of G into 1-factors. Whenever gu is even, $G(g^u)$ has a 1-factorization [7]. We use this to establish the following:

Lemma 1. *Let $g \geq 1$ and $u \geq 2$ be integers. If gu is even, then there is a $\{G_{21}\}$-GDD of type $(2g)^u(g(u-1))^1$.*

Proof. Let $F_1, \ldots, F_{g(u-1)}$ be a 1-factorization of $G(g^u)$ on vertex set X with groups $G_1, \ldots, G_u$. We form the GDD on vertex set $(X \times \{0,1\}) \cup \{\gamma_1, \ldots, \gamma_{g(u-1)}\}$ as

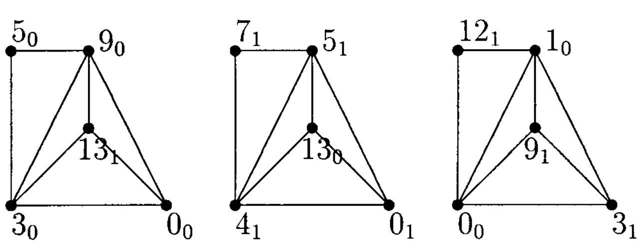

FIG. 5. Type 4^7 on $\mathbb{Z}_{14} \times \{0,1\}$.

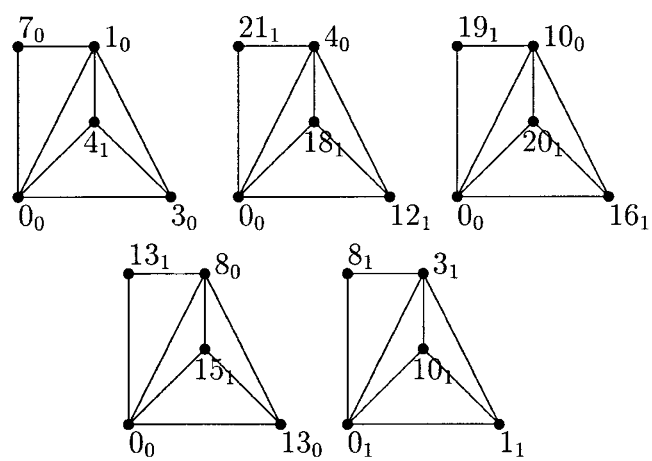

FIG. 6. Type 4^{11} on $\mathbb{Z}_{22} \times \{0,1\}$.

TABLE 2. Solutions of type 4^n, n odd.

$4^{15}, \mathbb{Z}_{30}$	$[0_0, 5_0, 14_0, 19_1 \mid 13_0]$	$[0_0, 7_0, 1_0, 3_0 \mid 11_1]$
	$[8_0, 18_0, 0_1, 11_1 \mid 4_1]$	$[13_0, 24_0, 0_1, 12_1 \mid 7_1]$
	$[5_0, 23_0, 0_1, 14_1 \mid 13_1]$	$[0_1, 3_1, 7_1, 13_1 \mid 5_1]$
	$[0_1, 1_1, 9_1, 29_0 \mid 3_0]$	
$4^{19}, \mathbb{Z}_{38}$	$[0_1, 1_1, 4_1, 11_1 \mid 14_1]$	$[32_1, 37_1, 0_0, 27_0 \mid 25_0]$
	$[4_1, 26_1, 0_0, 12_0 \mid 20_0]$	$[0_0, 18_0, 9_1, 35_1 \mid 11_1]$
	$[0_0, 30_0, 13_1, 33_1 \mid 8_1]$	$[0_0, 1_0, 5_0, 25_1 \mid 16_0]$
	$[0_0, 7_0, 36_0, 34_1 \mid 13_0]$	$[0_1, 17_1, 2_1, 8_1 \mid 15_0]$
	$[0_0, 18_1, 3_0, 17_0 \mid 28_0]$	

follows: Whenever $\{a, b\}$ is an edge of the 1-factor F_c, form the graph $(a_0, b_0, a_1, b_1)_{\gamma_c}$ in the GDD. Edges of the form $\{\gamma_c, \gamma_d\}$ do not appear in any such graph, so use such elements to form the group of size $g(u-1)$. Similarly, if $x, y \in G_z$, we find no edges of the form $\{x_i, y_j\}$ in the graphs, so let $G_i \times \{0, 1\}$ be groups for $1 \le i \le u$. It is now easily verified that edges inside groups appear in none of the chosen graphs, but that every other edge appears in exactly one. ∎

Corollary 1. *There is a $\mathcal{G}$-GDD of type 4^4.*

Proof. Apply Lemma 1 with $g = 2$ and $u = 3$ to produce a $\mathcal{G}$-GDD of type $4^3 4^1$, that is, 4^4. ∎

It is not difficult to verify that no $\mathcal{G}$-GDD of type 4^2 or 4^3 exists, and we have now produced a number of other $\mathcal{G}$-GDDs of type 4^n. However, we need a new technique to handle the case of general values of n. To do this, we employ another class of combinatorial designs. A *pairwise balanced design* of *order* v and *blocksizes* K, denoted (v, K)-PBD, is a pair $(X, \mathcal{B})$. X is a set of v elements, and $\mathcal{B}$ is a set of subsets (*blocks*) of X for which $|B| \in K$ for each $B \in \mathcal{B}$. For every 2-subset of elements $\{x, y\} \subset X$, there is exactly one block containing x and y. See [1, 8] for more information on PBDs. We employ PBDs as follows:

Lemma 2. *If a (n, K)-PBD exists, and for each $k \in K$ there exists a $\mathcal{G}$-GDD of type 4^k, then there exists a $\mathcal{G}$-GDD of type 4^n.*

Proof. Let $(X, \mathcal{B})$ be the PBD. We construct the GDD on $X \times \{0, 1, 2, 3\}$ with groups $\{x\} \times \{0, 1, 2, 3\}$ for $x \in X$ as follows: For each block $B \in \mathcal{B}$, place on the vertices in $B \times \{0, 1, 2, 3\}$ a copy of the $\mathcal{G}$-GDD of type $4^{|B|}$, aligning its groups on $\{x\} \times \{0, 1, 2, 3\}$ for $x \in B$. The verification is routine. ∎

Lemma 2 is a simple version of a powerful technique known as *Wilson's Fundamental Construction* (see [8]). To use it, we need some PBDs. Let $K = \{4, 5, 7, 8, 9, 10, 11, 12\}$, noting that we have presented $\mathcal{G}$-GDDs of type 4^k for each $k \in K$ already. Then, an (n, K)-PBD exists for all positive integers n *except* when n is 2, 3, 6, 14, 15, 18, 19, 23, 26, 27, or 30; see [1]. We have seen examples for 14, 15, 19, 26, and 30; solutions for 18, 23, and 27 appear later in Corollary 3 and Corollary 4. Applying Lemma 2, we establish the needed result:

Theorem 1. *A $\mathcal{G}$-GDD of type 4^n exists when n is a positive integer except when n is 2 or 3 and, possibly, when $n = 6$.*

Let us at long last return to the SONET/WDM application and examine the consequences of Theorem 1. A $\mathcal{G}$-GDD of type 4^n contains $n(n-1)$ graphs on five vertices and eight edges each. These account in the SONET ring of size $8n$ for $n(n-1)$ wavelengths at a drop cost of 10 each. On the vertices of each group, we place an LK_4 to handle the remaining edges and loops. This yields n further graphs (wavelengths), each having drop cost 8. We conclude the following:

Theorem 2. *Let n be a positive integer, $n \notin \{2, 3, 6\}$. Then, there is an $(8n, n^2, 8)$-assignment obtained from grooming primitive rings, which has a total drop cost of $10n^2 - 2n$.*

Theorem 2 represents a savings, for ring sizes a multiple of eight, of $22n^2 - 14n$ over the worst possible grooming having a drop cost of $32n^2 - 16n$. This worst cost is obtained by choosing all graphs associated with wavelengths to have all edges disjoint. But does our approach represent the best savings? When groomings are restricted to the same primitive rings, the answer is affirmative. This can be verified by an exhaustive search to establish that on nine or fewer points, cyclically ordered, there is no way to place eight edge-disjoint primitive rings each of length four. In addition, there is no way to place eight primitive rings of any kind on fewer than eight points. It follows that the best savings is obtained by choosing n unions consisting of six 4-cycles and two 2-cycles and the remainder consisting of eight 4-cycles. This corresponds to the savings demonstrated here.

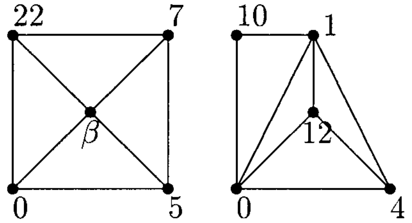

FIG. 7. Type 4^8 on $\mathbb{Z}_{28}$.

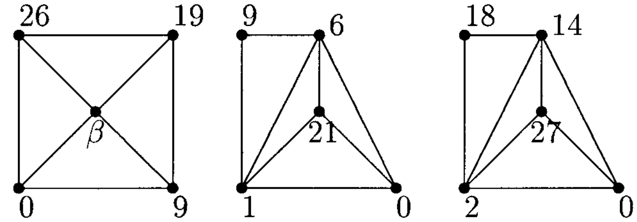

FIG. 8. Type 4^{12} on $\mathbb{Z}_{44}$.

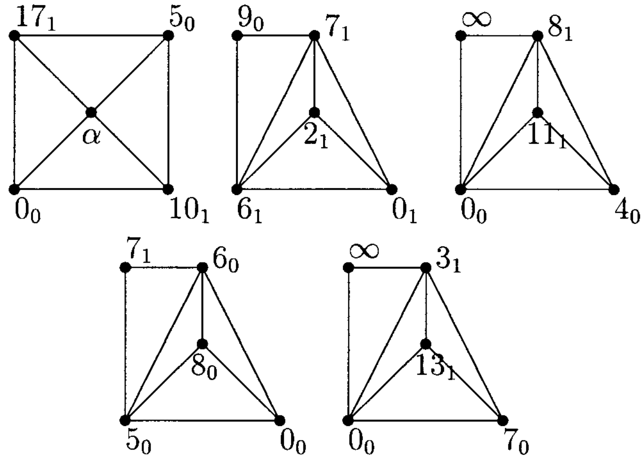

FIG. 9. Type 4^{10} on $\mathbb{Z}_{18} \times \{0, 1\}$.

It may happen that groomings of other primitive rings yield a better savings. However, in the unions of eight primitive rings to be formed, the average number of pairs to be directly connected is $32(n-1)/n$ and so approaches 32 as n increases. To see this, divide the $n(n-1)/2$ pairs to be connected in one direction among the $n^2/64$ wavelengths used; the number of wavelengths is determined by dividing the number of wavelengths used for the primitive rings, $n^2/8$, by the number of connections to the groomed onto a wavelength, $g = 8$. An exhaustive search establishes that if the union of eight primitive rings, each of arbitrary length at least three, employs nine or fewer points, then the number of pairs directly connected is at most 31. This serves as some evidence that the unions employed above, having 10 points and 32 pairs directly connected, yield good groomings.

5. RING SIZE A MULTIPLE OF FOUR

In this section, we address the remaining cases when the SONET ring has a size which is a multiple of four but not eight. Our task is to partition LK_m when $m \equiv 2 \pmod 4$. Now, calculating the number of edges plus half the number of loops in LK_m, we obtain $\frac{m^2}{2}$. If we write $m = 4t + 2$, this number is $(2t + 1)(4t + 2) = 8t^2 + 8t + 2$. We therefore need $t^2 + t + 1$ graphs in total. We can employ a partition of m into t parts of size 4 and one part of size 2 and place LK_4 on each of the first t parts and LK_2 on the last. This accounts for $t + 1$ graphs using $8t + 2$ from the edge total and contributing $10t + 4$ to the drop cost. More importantly, to complete the grooming, we now need only produce a $\mathcal{G}$-GDD of type $4^t 2^1$, aligning the groups on the parts chosen earlier. Hence, we concentrate next on producing $\mathcal{G}$-GDDs of the type $4^t 2^1$. We use the same strategy as before for a number of examples; see Figures 10–11 and Table 4.

TABLE 3. Solutions of type 4^n, n even.

4^{14}, $\mathbb{Z}_{26}$	$(0_0, 1_0, 25_1, 20_1)_\alpha$	$(0_0, 9_0, 5_1, 16_1)_\alpha$
	$[10_1, 17_1, 0_0, 8_0 \| 22_0]$	$[0_0, 3_0, 6_1, 8_1 \| 7_1]$
	$[0_1, 4_1, 1_1, 10_1 \| 12_1]$	$[23_1, 0_0, 5_0, 12_0 \| 11_0]$
	$[25_1, 0_0, 6_0, 10_0 \| 24_0]$	
4^{26}, $\mathbb{Z}_{50}$	$(0_0, 5_0, 3_1, 2_1)_\alpha$	$(0_0, 13_0, 9_1, 4_1)_\alpha$
	$[0_0, 18_0, 1_0, 7_0 \| 21_0]$	$[0_0, 14_0, 15_1, 35_1 \| 8_1]$
	$[0_0, 40_1, 9_0, 24_0 \| 22_0]$	$[12_1, 34_1, 0_0, 2_0 \| 5_0]$
	$[0_0, 47_1, 8_0, 20_0 \| 10_0]$	$[13_1, 45_1, 0_0, 4_0 \| 25_0]$
	$[0_1, 6_1, 2_1, 19_1 \| 16_1]$	$[19_1, 30_1, 0_0, 16_0 \| 8_0]$
	$[3_1, 26_1, 0_1, 12_1 \| 8_1]$	$[0_0, 19_0, 36_1, 43_1 \| 42_1]$
	$[0_0, 23_0, 28_1, 49_1 \| 6_1]$	
4^{30}, $\mathbb{Z}_{58}$	$(0_0, 25_0, 22_1, 35_1)_\alpha$	$(0_0, 23_0, 19_1, 50_1)_\alpha$
	$[0_0, 10_0, 28_0, 37_1 \| 43_1]$	$[0_1, 12_1, 26_1, 38_0 \| 1_0]$
	$[0_0, 26_0, 28_1, 56_1 \| 42_1]$	$[0_0, 22_0, 25_1, 45_1 \| 5_1]$
	$[14_1, 38_1, 0_0, 7_0 \| 25_0]$	$[0_0, 16_0, 22_1, 40_1 \| 26_1]$
	$[0_0, 2_0, 19_1, 36_1 \| 51_1]$	$[0_0, 14_0, 8_1, 15_1 \| 18_1]$
	$[39_1, 48_1, 0_0, 27_0 \| 53_0]$	$[6_0, 9_0, 0_0, 17_0 \| 21_0]$
	$[0_1, 1_1, 3_1, 11_1 \| 23_1]$	$[4_1, 25_1, 0_1, 19_1 \| 20_1]$

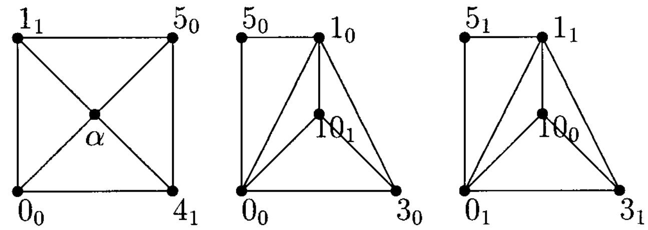

FIG. 10. Type $4^6 2^1$ on $\mathbb{Z}_{12} \times \{0, 1\}$.

In the decomposition of Figure 11, we employ a new notation, namely, $(a, b, c, d)_{\overline{\infty}}$. Whenever this notation is used, we require that (a, b, c, d) be of the form $(x_0, y_1, (x+t)_0, (y+t)_1)$, where addition is modulo $2t$. Then, adding t to each entry produces the *same* graph again, so this base graph only makes t different graphs, not $2t$. Hence, it does employ only one extra point, consistent with the choice of the symbol ∞.

A $\{G_{20}\}$-GDD of type $4^4 2^1$ can be produced as follows: Form the (16,{4})-PBD with blocks in Table 5. Now, form eight graphs by adding an element ∞ adjacent to the first two elements in each of the first eight quadruples given. Form eight further graphs by adding an element ∞' adjacent to the first two elements in each of the next eight quadruples given. The final four quadruples are deleted to form four groups of size four, and the group of size two is $\{\infty, \infty'\}$.

Taking $g = 2$ and $u = 2$ produces a $\mathcal{G}$-GDD of type $4^2 2^1$ using Lemma 1, leading to a useful corollary:

Corollary 2. *There is a $\mathcal{G}$-GDD of type $4^x(2x-2)^1$ whenever $x \geq 2$. If, in addition, x is even and there is a $\mathcal{G}$-GDD of type $4^{(x/2)-1}2^1$, then there is a $\mathcal{G}$-GDD of type $4^{(3x/2)-1}2^1$.*

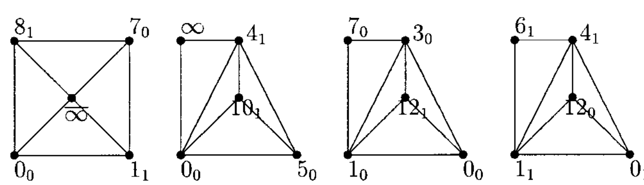

FIG. 11. Type $4^7 2^1$ on $\mathbb{Z}_{14} \times \{0, 1\}$.

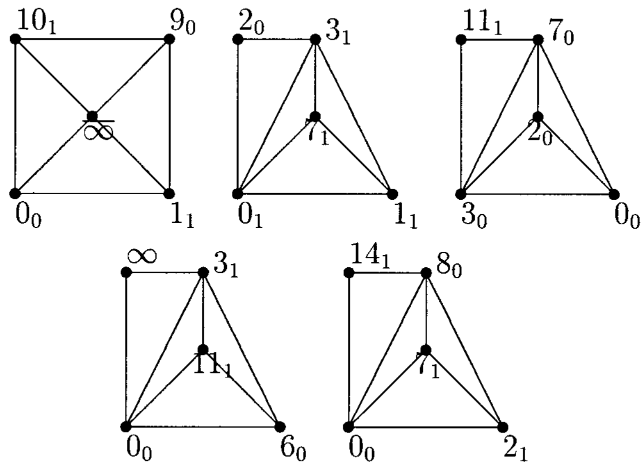

FIG. 12. Type $4^9 2^1$ on $\mathbb{Z}_{18} \times \{0, 1\}$.

Proof. For the first statement, apply Lemma 1 with $u = x$ and $g = 2$. Place the GDD of type $4^{(x/2)-1}2^1$ on the elements of the group of size $2x - 2$. ∎

Taking $x \in \{6, 10, 16\}$ yields a $\mathcal{G}$-GDD of type $4^t 2^1$ for $t \in \{8, 14, 23\}$. Next, we employ a variant of Lemma 1:

Lemma 3. *Let t be an integer, and suppose that a $\mathcal{G}$-GDD of type $4^t y^1$ exists. Then, there exists a $\mathcal{G}$-GDD of type $4^{2t}(2t + y)^1$. If, in addition, $2t + y = 4s + z$ and a $\mathcal{G}$-GDD of type $4^s z^1$ exists, then a $\mathcal{G}$-GDD of type $4^{2t+s} z^1$ exists.*

Proof. Let $F_1, \ldots, F_{2t}$ be a 1-factorization of the complete bipartite graph $K_{2t,2t}$, with vertex classes A and B each of size $2t$. Let $C = \{\gamma_1, \ldots, \gamma_{2t}\}$ be disjoint from A and B. We form a set of graphs on $((A \cup B) \times \{0, 1\}) \cup C$ with groups $A \times \{0, 1\}$, $B \times \{0, 1\}$, and C, as follows: For each edge $\{a, b\}$ in the $K_{2t,2t}$, find the 1-factor F_c containing the edge. Then, add the graph $(a_0, b_0, a_1, b_1)_{\gamma_c}$. At this point, we have formed a $\mathcal{G}$-GDD of type $(4t)^2(2t)^1$.

Now, let D be a set of y further vertices. On $(A \times \{0, 1\}) \cup D$, and also on $(B \times \{0, 1\}) \cup D$, place the graphs of a $\mathcal{G}$-GDD of type $4^t y^1$ aligning the group of size y on D in each case. The result is a $\mathcal{G}$-GDD of type $4^{2t}(2t+y)^1$. For the final statement, place on $C \cup D$ the graphs of a $\mathcal{G}$-GDD of type $4^s z^1$. ∎

Corollary 3. *There exist $\mathcal{G}$-GDDs of type $4^{10}2^1$, $4^{12}2^1$, 4^{18}, 4^{23}, and $4^{33}2^1$.*

Proof. Apply Lemma 3 with (t, y, s, z) = (4,2,2,2), (5,0,2,2), (7,2,4,0), (8,12,7,0), and (13,4,7,2). We have seen all of the ingredients used except the $\mathcal{G}$-GDD of type $4^8 12^1$. To produce it, apply Lemma 3 with $t = 4$, $y = 4$, $s = 0$, and $z = 12$. ∎

We now introduce the most powerful construction that we need:

Lemma 4. *Let $m \geq 4$ be a positive integer, $m \neq 6, 10$. Let y be an even integer satisfying $0 \leq y \leq 8m - 2$. Then, there exists a $\mathcal{G}$-GDD of type $4^{4m} y^1$.*

Proof. First, we collect some needed ingredients: We write $y = z + \gamma$, where $\gamma \in \{0, 2m - 2\}$ and $0 \leq z \leq 6m$. Then, we write $z = z_1 + z_2 + \cdots + z_m$, where $z_i \in \{0, 2, 4, 6\}$ for $1 \leq i \leq m$. Then, we have seen $\mathcal{G}$-GDDs of type $4^4 z_i^1$. In addition, Theorem 1 gives a $\mathcal{G}$-GDD of type 4^m, and Lemma 3 gives one of type $4^m(2m - 2)^1$.

To perform the construction, we start with a $\{K_5\}$-GDD of type m^5 (this exists under the stated conditions on m; indeed, it is equivalent to "three mutually orthogonal latin squares of side m"; see [5]). Let A, B, C, D, E be the five groups of size m, and let $E = \{e_1, \ldots, e_m\}$. We begin the construction on vertex set $((A \cup B \cup C \cup D) \times \{0, 1, 2, 3\}) \cup \{\{e_i\} \times \{1, 2, \ldots, z_i\} : 1 \leq i \leq m\}$. When $z_i = 0$, the product $\{e_i\} \times \{1, 2, \ldots, z_i\}$ contains *no* vertices. For each graph K_5 on vertex set $\{a, b, c, d, e_i\}$ in the $\{K_5\}$-GDD, we place the graphs of a $\mathcal{G}$-GDD of

TABLE 4. Solutions of type $4^n 2^1$ over $\mathbb{Z}_{2n}$.

$4^{11}2^1$	$(0_0, 1_1, 11_0, 12_1)_{\overline{\infty}}$	$[0_0, 9_0, 6_1, 16_1 \mid 18_1]$
	$[0_0, 2_1, 10_0, 15_1 \mid \infty]$	$[0_0, 8_0, 1_0, 21_1 \mid 3_1]$
	$[0_0, 2_0, 6_0, 10_1 \mid 5_0]$	$[1_1, 3_1, 0_1, 8_1 \mid 7_1]$
$4^{13}2^1$	$(0_0, 1_1, 13_0, 14_1)_{\overline{\infty}}$	$[0_0, 17_1, 5_0, 20_1 \mid \infty]$
	$[0_0, 1_0, 12_0, 23_1 \mid 6_1]$	$[0_0, 10_0, 8_0, 3_1 \mid 9_1]$
	$[0_0, 3_0, 9_0, 7_1 \mid 7_0]$	$[0_1, 6_1, 8_1, 16_0 \mid 8_0]$
	$[0_1, 1_1, 5_1, 12_1 \mid 10_1]$	
$4^{15}2^1$	$(0_0, 1_1, 15_0, 16_1)_{\overline{\infty}}$	$[0_0, 22_1, 9_0, 18_1 \mid \infty]$
	$[3_0, 4_0, 0_0, 28_1 \mid 5_1]$	$[0_0, 2_0, 5_1, 19_1 \mid 6_1]$
	$[12_1, 24_1, 0_0, 4_0 \mid 1_0]$	$[2_1, 13_1, 0_1, 3_0 \mid 6_0]$
	$[1_0, 13_0, 0_0, 6_0 \mid 21_0]$	$[6_1, 9_1, 0_1, 1_1 \mid 16_1]$
$4^{17}2^1$	$(0_0, 1_1, 17_0, 18_1)_{\overline{\infty}}$	$[0_0, 6_1, 2_0, 15_0 \mid \infty]$
	$[2_1, 9_1, 0_1, 5_1 \mid 15_1]$	$[11_1, 19_1, 0_1, 1_1 \mid 8_0]$
	$[0_0, 5_0, 12_0, 27_1 \mid 6_0]$	$[0_0, 14_0, 10_0, 33_1 \mid 16_1]$
	$[0_1, 14_1, 4_0, 20_0 \mid 22_0]$	$[0_0, 3_0, 11_0, 24_1 \mid 8_1]$
	$[0_1, 12_1, 5_0, 14_0 \mid 3_0]$	
$4^{19}2^1$	$(0_0, 1_1, 19_0, 20_1)_{\overline{\infty}}$	$[0_0, 10_1, 6_0, 24_1 \mid \infty]$
	$[0_0, 18_0, 8_0, 35_1 \mid 16_1]$	$[7_0, 16_0, 0_0, 37_1 \mid 1_1]$
	$[0_0, 14_0, 1_0, 29_1 \mid 26_1]$	$[12_1, 18_1, 0_1, 25_0 \mid 4_0]$
	$[0_1, 2_1, 17_1, 33_0 \mid 29_0]$	$[0_1, 1_1, 11_1, 5_0 \mid 36_0]$
	$[2_0, 17_0, 0_0, 5_0 \mid 6_0]$	$[0_1, 3_1, 7_1, 16_1 \mid 8_1]$
$4^{21}2^1$	$(0_0, 1_1, 21_0, 22_1)_{\overline{\infty}}$	$[0_0, 3_1, 13_0, 40_1 \mid \infty]$
	$[3_0, 20_0, 0_0, 11_0 \mid 37_1]$	$[0_0, 4_0, 14_0, 19_0 \mid 16_0]$
	$[1_1, 20_1, 0_1, 13_1 \mid 28_1]$	$[2_1, 6_1, 0_1, 16_1 \mid 33_0]$
	$[0_0, 1_0, 6_1, 24_1 \mid 36_1]$	$[0_0, 2_0, 9_1, 20_1 \mid 16_1]$
	$[8_1, 25_1, 0_0, 6_0 \mid 38_0]$	$[0_0, 7_0, 4_1, 37_1 \mid 33_1]$
	$[28_1, 31_1, 0_0, 18_0 \mid 32_0]$	
$4^{25}2^1$	$(0_0, 1_1, 25_0, 26_1)_{\overline{\infty}}$	$[0_0, 31_1, 12_0, 34_1 \mid \infty]$
	$[0_0, 1_0, 9_1, 17_1 \mid 48_1]$	$[0_0, 4_0, 10_0, 23_0 \mid 18_0]$
	$[0_0, 3_0, 11_0, 20_0 \mid 24_0]$	$[0_1, 2_1, 7_1, 16_1 \mid 17_1]$
	$[7_0, 22_0, 0_0, 49_1 \mid 1_1]$	$[11_1, 24_1, 0_1, 22_0 \mid 37_0]$
	$[12_1, 22_1, 0_1, 29_0 \mid 9_0]$	$[0_1, 1_1, 19_1, 15_0 \mid 21_1]$
	$[0_0, 2_0, 14_1, 20_1 \mid 40_1]$	$[11_1, 15_1, 0_0, 5_0 \mid 20_0]$
	$[23_1, 46_1, 0_0, 16_0 \mid 41_0]$	
$4^{31}2^1$	$(0_0, 1_1, 31_0, 32_1)_{\overline{\infty}}$	$[0_0, 24_1, 11_0, 60_1 \mid \infty]$
	$[0_0, 3_0, 10_0, 29_0 \mid 20_0]$	$[0_0, 4_0, 9_0, 27_0 \mid 28_0]$
	$[0_1, 2_1, 7_1, 30_1 \mid 29_1]$	$[0_1, 1_1, 12_1, 25_1 \mid 20_1]$
	$[0_0, 1_0, 4_1, 10_1 \mid 56_1]$	$[7_1, 16_1, 0_0, 2_0 \mid 1_0]$
	$[29_1, 44_1, 0_0, 8_0 \mid 61_0]$	$[50_1, 53_1, 0_0, 12_0 \mid 28_0]$
	$[48_1, 52_1, 0_0, 13_0 \mid 6_0]$	$[0_0, 14_0, 33_1, 54_1 \mid 61_1]$
	$[20_1, 37_1, 0_0, 25_0 \mid 48_0]$	$[0_0, 30_0, 11_1, 27_1 \mid 58_1]$
	$[0_0, 6_0, 21_0, 23_1 \mid 22_0]$	$[10_1, 18_1, 0_1, 54_0 \mid 32_1]$

TABLE 5. (16,{4})-PBD.

0,4,8,12	1,5,9,13	2,6,10,14	3,7,11,15
0,5,10,15	1,4,11,14	2,7,8,13	3,6,9,12
0,6,11,13	1,7,10,12	2,4,9,15	3,5,8,14
0,7,9,14	1,6,8,15	2,5,11,12	3,4,10,13
0,1,2,3	4,5,6,7	8,9,10,11	12,13,14,15

type $4^4 z_i^1$ on groups $\{a\} \times \{0,1,2,3\}$, $\{b\} \times \{0,1,2,3\}$, $\{c\} \times \{0,1,2,3\}$, $\{d\} \times \{0,1,2,3\}$, and $\{e_i\} \times \{1,2,\ldots,z_i\}$.

Next, we add γ further vertices V. For each $X \in \{A,B,C,D\}$, we place on $(X \times \{0,1,2,3\}) \cup V$ the graphs of a $\mathcal{G}$-GDD of type $4^m \gamma^1$, aligning groups of size four on $\{x\} \times \{0,1,2,3\}$ for $x \in X$ and the group of size γ on V. The verification is routine. ∎

A simple application follows:

Corollary 4. *There is a $\mathcal{G}$-GDD of type* 4^{27}.

Proof. Apply Lemma 4 with $m = 5$ and $y = 28$ to produce a $\mathcal{G}$-GDD of type $4^{20}28^1$. Then, fill the group of size 28 with a $\mathcal{G}$-GDD of type 4^7. ∎

We use primarily Lemma 4 to prove the main result:

Theorem 3. *There exists a $\mathcal{G}$-GDD of type $4^n 2^1$ for all $n \geq 0$ except when $n = 1$, $n = 3$, and possibly $n = 5$.*

Proof. We have presented such $\mathcal{G}$-GDDs for all $n < 16$ with the exception of 1, 3, and 5 and also for n =17, 19, 21, 23, 25, 31, and 33. Simple counting establishes nonexistence when n is 1 or 3. Now, whenever $n \geq 16$ and n is not 17, 19, 21, 23, 25, 31, or 33, it is an easy matter to choose integers m and s so that $n = 4m + s$, $0 \leq s \leq 2m-1$, $m \geq 4$ and $m \notin \{6, 10\}$ and a $\mathcal{G}$-GDD of type $4^s 2^1$ exists. The latter condition holds inductively unless s is 1, 3, or 5. Then, form a $\mathcal{G}$-GDD of type $4^{4m}(4s+2)^1$ using Lemma 4 and fill the group of size $4s + 2$ with a $\mathcal{G}$-GDD of type $4^s 2^1$. ∎

Now, we return again to the SONET/WDM application. We can calculate the drop cost of the groomings corresponding to the graph decompositions of Theorem 3 to obtain the following:

Corollary 5. *When $n \notin \{1, 3, 5\}$, there exists an $(8n + 4, n^2 + n + 1, 8)$-grooming of primitive rings with total drop cost $10n^2 + 8n + 4$.*

This appears to be the best drop cost that can be obtained from groomings of these primitive rings, when the ring size is a multiple of four but not eight.

6. CONCLUSIONS

The techniques developed here using combinatorial design theory are somewhat involved, but provide a general method for prescribing a grooming which realizes low drop cost. The cases when the ring size is odd can be treated by similar methods, but the structures involved are somewhat different from the case treated here. In the remaining cases when the ring size is even but not a multiple of four, the graph decompositions that arise are the same as those treated here. However, the need to produce a large number of small "ingredients" for that situation make its presentation much lengthier. In fact, the need for numerous solutions for small cases limits the general application of the methods developed here for determining exact minimum drop costs. We do not expect that it is reasonable to expect to generate all of the small solutions needed for general g. Nevertheless, the techniques developed can be much more easily used to obtain solutions whose drop cost is in some sense near the minimum. To illustrate this, with the solutions given here, it is easy to produce solutions for the ring sizes that are not multiples of four, which are "near" optimal in terms of drop cost. Indeed, a simple strategy is to first produce a solution for the next larger multiple of four, adding at most three "virtual" sites on the SONET ring. From this solution, we preserve all wavelength assignments that involve none of the virtual sites as drops. The connections involving the virtual sites are then deleted. The increase in ring size results in an increase in the number of wavelengths used; hence, as a final step, those remaining connections that involved one of the virtual sites as a drop are reassigned to wavelengths using the fewest wavelengths possible, but with no concern about the drop cost on these wavelengths. Since the number of wavelengths involving a virtual site is linear in the ring size, while the number that do not involve a virtual site grows quadratically, the dominant (quadratic) term in the drop cost is the same in the approximate solution and the exact solution. Hence, as ring size increases, this approximate solution exhibits a drop cost whose ratio to the minimum drop cost approaches 1. Using virtual sites, we can obtain approximate solutions without having complete exact solutions for all orders.

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