Coverage by Randomly Deployed Wireless Sensor Networks

Peng-Jun Wan, *Member, IEEE,* and Chih-Wei Yi, *Member, IEEE*

***Abstract*—One of the main applications of wireless sensor networks is to provide proper coverage of their deployment regions. A wireless sensor network k-covers its deployment region if every point in its deployment region is within the coverage ranges of at least k sensors. In this paper, we assume that the sensors are deployed as either a Poisson point process or a uniform point process in a square or disk region, and study how the probability of the k-coverage changes with the sensing radius or the number of sensors. Our results take the complicated boundary effect into account, rather than avoiding it by assuming the toroidal metric as done in the literature.**

***Index Terms*—Asymptotics, connectivity, k-coverage, node density, sensing radius, wireless ad hoc sensor networks.**

I. Introduction

One of the main applications of wireless sensor networks is to provide proper coverage of their deployment regions. Typically, the sensing range of a sensor is a (closed or open) circular disk centered at the sensor, whose radius is termed as the *sensing radius* of the sensor. For any positive integer k, a point is said to be k-covered by a sensor network if it falls in the sensing ranges of at least k sensors, and a region is said to be k-covered if each point in this region is k-covered. In this paper, we study how the probability of a deployment region being k-covered by randomly deployed sensors changes with the sensing radius or the number of sensors. A precise description of the problems is given below.

Let $X_1, X_2, \ldots$ be independent and uniformly distributed random points on a bounded region A in the plane. Given a positive integer n, the point process $\{X_1, X_2, \ldots, X_n\}$ is referred to as the *uniform n-point process* on A, and is denoted by $\mathcal{X}_n(A)$. Given a positive number λ, let $Po(\lambda)$ be a Poisson random variable with parameter λ, independent of $\{X_1, X_2, \ldots\}$. Then the point process $\{X_1, X_2, \ldots, X_{Po(\lambda)}\}$ is referred to as the Poisson point process with mean λ on A, and is denoted by $\mathcal{P}_\lambda(A)$. Let k be a fixed nonnegative integer, and Ω be the unit-area square or disk centered at the origin o. For any real number t, use $t\Omega$ to denote the set $\{tx : x \in \Omega\}$, i.e., the square or disk of area t^2 centered at the origin. Let $C_{n,r}$ (respectively, $C'_{n,r}$) denote the event that Ω is $(k+1)$-covered by the (open or closed) disks of radius r centered at the points in $\mathcal{P}_n(\Omega)$ (respectively, $\mathcal{X}_n(\Omega)$). Let $K_{s,n}$ (respectively, $K'_{s,n}$) denote the event that $\sqrt{s}\Omega$ is $(k+1)$-covered by the unit-area (closed or open) disks centered at the points in $\mathcal{P}_n(\sqrt{s}\Omega)$ (respectively, $\mathcal{X}_n(\sqrt{s}\Omega)$). Then, we would like to study the asymptotics of $\Pr[C_{n,r}]$ and $\Pr[C'_{n,r}]$ as n approaches infinity, and the asymptotics of $\Pr[K_{s,n}]$ and $\Pr[K'_{s,n}]$ as s approaches infinity.

To simplify the presentation of our results, we introduce some notation. Let η denote the peripheral of Ω, which is equal to 4 (respectively, $2\sqrt{\pi}$) if Ω is a square (respectively, disk). For any $\xi \in \mathbb{R}$, let

$$\alpha(\xi) = \begin{cases} \frac{\left(\frac{\sqrt{\pi}\eta}{2} + e^{-\frac{\xi}{2}}\right)^2}{16\left(2\sqrt{\pi}\eta + e^{-\frac{\xi}{2}}\right)} e^{-\frac{\xi}{2}}, & \text{if } k = 0 \\ \frac{\sqrt{\pi}\eta}{2^{k+6}(k+2)!} e^{-\frac{\xi}{2}}, & \text{if } k \geq 1 \end{cases}$$

and

$$\beta(\xi) = \begin{cases} 4e^{-\xi} + 2\left(\sqrt{\pi} + \frac{1}{\sqrt{\pi}}\right)\eta e^{-\frac{\xi}{2}}, & \text{if } k = 0 \\ \frac{\sqrt{\pi} + \frac{1}{\sqrt{\pi}}}{2^{k-1}k!}\eta e^{-\frac{\xi}{2}}, & \text{if } k \geq 1. \end{cases}$$

The mains results of this paper are summarized in the following two theorems.

Theorem 1: Let

$$r_n = \sqrt{\frac{\ln n + (2k+1)\ln\ln n + \xi_n}{\pi n}}.$$

If $\lim_{n\to\infty} \xi_n = \xi$ for some $\xi \in \mathbb{R}$, then

$$1 - \beta(\xi) \leq \lim_{n\to\infty} \Pr[C_{n,r_n}] \leq \frac{1}{1 + \alpha(\xi)} \tag{1}$$

and

$$1 - \beta(\xi) \leq \lim_{n\to\infty} \Pr\left[C'_{n,r_n}\right] \leq \frac{1}{1 + \alpha(\xi)}. \tag{2}$$

If $\lim_{n\to\infty} \xi_n = \infty$, then

$$\lim_{n\to\infty} \Pr[C_{n,r_n}] = \lim_{n\to\infty} \Pr\left[C'_{n,r_n}\right] = 1. \tag{3}$$

If $\lim_{n\to\infty} \xi_n = -\infty$, then

$$\lim_{n\to\infty} \Pr[C_{n,r_n}] = \lim_{n\to\infty} \Pr\left[C'_{n,r_n}\right] = 0. \tag{4}$$

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Theorem 2: Let

$$\mu(s) = \ln s + 2(k+1)\ln\ln s + \xi(s)$$

If $\lim_{s\to\infty}\xi(s) = \xi$ for some $\xi \in \mathbb{R}$, then

$$1-\beta(\xi) \le \lim_{s\to\infty}\Pr\left[K_{s,\mu(s)s}\right] \le \frac{1}{1+\alpha(\xi)}$$

and

$$1-\beta(\xi) \le \lim_{s\to\infty}\Pr\left[K'_{s,\mu(s)s}\right] \le \frac{1}{1+\alpha(\xi)}.$$

If $\lim_{s\to\infty}\xi(s) = \infty$, then

$$\lim_{s\to\infty}\Pr\left[K_{s,\mu(s)s}\right] = \lim_{s\to\infty}\Pr\left[K'_{s,\mu(s)s}\right] = 1.$$

If $\lim_{s\to\infty}\xi(s) = -\infty$, then

$$\lim_{s\to\infty}\Pr\left[K_{s,\mu(s)s}\right] = \lim_{s\to\infty}\Pr\left[K'_{s,\mu(s)s}\right] = 0.$$

We remark that the probabilistic studies of k-coverage by a random point process have been conducted for $k=1$ in [1] and arbitrary integer-valued constant k in [6] but with certain limitations. Both studies assume Poisson point processes on a square and use the *toroidal metric*, rather than the Euclidean metric which is more relevant to the applications. This renders their results hardly applicable to wireless sensor networks. Indeed, the smallest sensing radius or sensor density to ensure the k-coverage under the toroidal metric almost surely fails to guarantee the k-coverage under the Euclidean metric. The assumption of the toroidal metric technically eliminates the boundary effect under the Euclidean metric. As will be demonstrated later in this paper, the boundary effect is the major technical challenge which requires much delicate and involved analysis. Another related work is [2] in which a random sensor network is generated by a Poisson point process followed by a Bernoulli style active/inactive process. Here we note that a Poisson point process followed by a Bernoulli process is still a Poisson point process. Without explicit proof, authors of [2] claimed the boundary effect was handled, but their results are not consistent with ours.

To conclude this section, we setup some notation which applies throughout the rest of the paper. $\|x\|$ is the Euclidean norm of a point $x \in \mathbb{R}^2$, and $|A|$ is shorthand for two-dimensional (2-D) Lebesgue measure (or area) of a measurable set $A \subset \mathbb{R}^2$. All integrals considered will be Lebesgue integrals. The topological boundary of a set $A \subset \mathbb{R}^2$ is denoted by ∂A. The (closed or open) disk of radius r centered at x is denoted by $D_r(x)$. For any $x, y \in \Omega$, let

$$v_r(x) = |D_r(x) \cap \Omega|$$
$$v_r(y\backslash x) = |(D_r(y) \setminus D_r(x)) \cap \Omega|$$
$$\phi_{n,r}(x) = \sum_{i=0}^{k}\frac{(nv_r(x))^i}{i!}e^{-nv_r(x)}$$
$$\phi_{n,r}(y\backslash x) = \sum_{i=0}^{k}\frac{(nv_r(y\backslash x))^i}{i!}e^{-nv_r(y\backslash x)}.$$

An event is said to be asymptotic almost sure (abbreviated by a.a.s.) if it occurs with a probability converges to one as $n \to \infty$. The symbols O, o, Θ,[1] $\sim$, $\lesssim$, $\gtrsim$[2] refer to either the limit $n \to \infty$ or the limit $s \to \infty$ depending on the context. To avoid trivialities, we tacitly assume n and s to be sufficiently large if necessary. For simplicity of notation, the subscripts will be frequently suppressed.

II. Geometric Ingredients

The results in this section are purely geometric, with no probabilistic content. In the following, we will give three lemmas that will be used in the next section. To increase readability, we leave their proofs as an appendix .

Lemma 3: Let S be a disk of radius s. For any $r > s$, let $\digamma$ be the set of points $(x,y) \in \mathbb{R}^2 \times \mathbb{R}^2$ satisfying that there exists $z \in S$ such that $\|x-z\| = \|y-z\| = r$ and $\overrightarrow{xy} \times \overrightarrow{xz} \ge 0$. (Here $\overrightarrow{xy} \times \overrightarrow{xz}$ is a cross product of 2D vectors and given by the determinant $\det\left(\overrightarrow{xy}, \overrightarrow{xz}\right)$.) Then the Lebesgue measure of $\digamma$ is $(4\pi r^2)|S|$.

If Ω is the disk region, then $\phi_{n,r}(z)$ has equal value for all $z \in \partial\Omega$. The next lemma provides a simplified asymptotic expression of this value.

Lemma 4: Assume Ω is the disk region and let r_n be such that $n\pi r_n^2 \to \infty$ and $n\pi r_n^3 \to 0$. Ω is the disk region. Then for any point $z \in \partial\Omega$

$$\phi_{n,r_n}(z) \sim \frac{\left(\frac{n\pi r_n^2}{2}\right)^k}{k!}e^{-\frac{n\pi r_n^2}{2}}.$$

The following lemma gives the asymptotic behavior of two integrals.

Lemma 5: Let

$$r_n = \sqrt{\frac{\ln n + (2k+1)\ln\ln n + \xi_n}{\pi n}}.$$

with $\lim \xi_n = \xi$ for some $\xi \in \mathbb{R}$. Then,

$$n\left(n\pi r_n^2\right)\int_\Omega \phi_{n,r_n}(x)\,dx \sim \begin{cases} e^{-\xi} + \frac{\sqrt{\pi}\eta}{2}e^{-\frac{\xi}{2}}, & \text{if } k=0 \\ \frac{\sqrt{\pi}\eta}{2^{k+1}k!}e^{-\frac{\xi}{2}}, & \text{if } k\ge 1 \end{cases}. \quad (5)$$

and

$$n^2\left(n\pi r_n^2\right)^2 \int_{\substack{x,y\in\Omega,\\ \|x-y\|\le 2r_n}} \phi_{n,r_n}(x)\,\phi_{n,r_n}(y\backslash x)\,dxdy \lesssim \begin{cases} 16e^{-\xi} + 32\sqrt{\pi}\eta e^{-\frac{\xi}{2}}, & \text{if } k=0 \\ \frac{(k+1)(k+2)\sqrt{\pi}\eta}{2^{k-4}k!}e^{-\frac{\xi}{2}}, & \text{if } k\ge 1 \end{cases}. \quad (6)$$

Now, we introduce some notations. For any $r > 0$ and a unit-area square Ω, we partition Ω into three subregions $\Omega_r(0)$, $\Omega_r(1)$ and $\Omega_r(2)$ as illustrated in Fig. 1(a): $\Omega_r(0)$ consists of

[1]Without loss of generality, assume $f(n)$ and $g(n)$ are positive functions. If we write $f(n) = O(g(n))$, it means that there exist $c > 0$ and $n_0 > 0$ such that $f(n) \le cg(n)$ for any $n \ge n_0$. If we write $f(n) = \Theta(g(n))$, it means that there exist $c_1, c_2 > 0$ and $n_0 > 0$ such that $c_1 g(n) \le f(n) \le c_2 g(n)$ for any $n \ge n_0$. If we write $f(n) = o(g(n))$, it means that for any $\varepsilon > 0$, there exists $n_0 > 0$ such that $f(n) \le \varepsilon g(n)$ for any $n \ge n_0$.

[2]We use $f(n) \sim g(n)$ to denote $\lim_{n\to\infty} f(n) = \lim_{n\to\infty} g(n)$; $f(n) \lesssim g(n)$ to denote $\lim_{n\to\infty} f(n) \le \lim_{n\to\infty} g(n)$; and $f(n) \gtrsim g(n)$ to denote $\lim_{n\to\infty} f(n) \ge \lim_{n\to\infty} g(n)$.

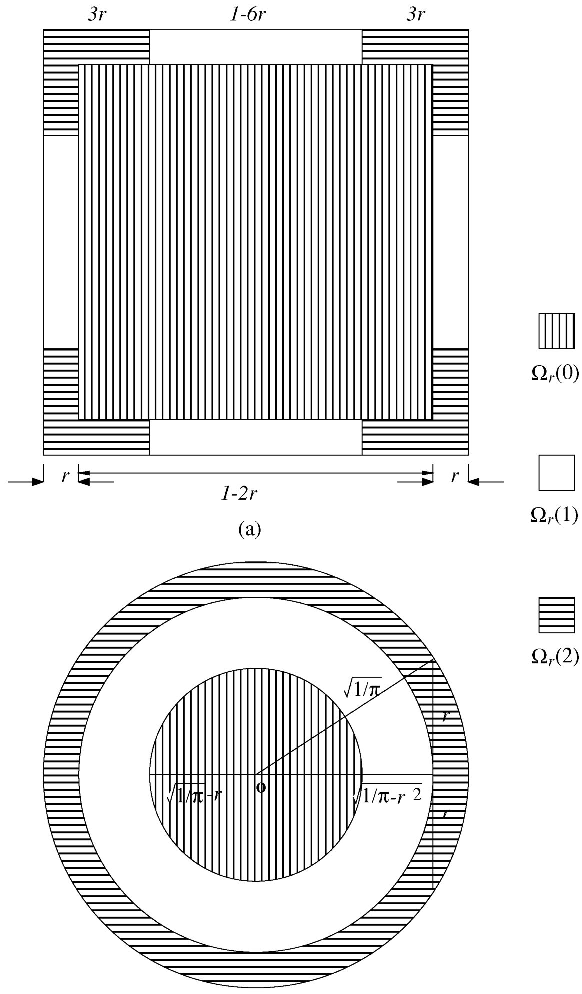


Fig. 1. Parition of Ω: (a) Ω is square and (b) Ω is disk.

all points in Ω apart from the sides of Ω by at least r, $\Omega_r(1)$ consists of all points in Ω apart from some side of Ω by less than r and from all other sides by at least $3r$, and $\Omega_r(2)$ consists of the rest points in Ω. The areas of these three regions are as follows:

$$|\Omega_r(0)| = (1-2r)^2$$
$$|\Omega_r(1)| = 4r(1-6r)$$
$$|\Omega_r(2)| = 20r^2.$$

For any $x \in \Omega_r(i)$

$$v_r(x) \geq 2^{-i}\pi r^2.$$

For any $r > 0$ and a unit-area disk Ω, we partition Ω into three subregions $\Omega_r(0)$, $\Omega_r(1)$ and $\Omega_r(2)$ as illustrated in Fig. 1(b): $\Omega_r(0)$ is the disk of radius $\frac{1}{\sqrt{\pi}} - r$ centered at $\mathbf{o}$; $\Omega_r(1)$ is the annulus of radii $\frac{1}{\sqrt{\pi}} - r$ and $\sqrt{\frac{1}{\pi} - r^2}$ centered at $\mathbf{o}$; and $\Omega_r(2)$ is the annulus of radii $\sqrt{\frac{1}{\pi} - r^2}$ and $\frac{1}{\sqrt{\pi}}$ centered at $\mathbf{o}$. The areas of these three regions are

$$|\Omega_r(0)| = \left(1-\sqrt{\pi}r\right)^2$$
$$|\Omega_r(1)| = 2\pi r\left(\frac{1}{\sqrt{\pi}} - r\right)$$
$$|\Omega_r(2)| = \pi r^2.$$

For any $x \in \Omega_r(i)$

$$v_r(x) \geq 2^{-i}\pi r^2.$$

III. Critical Sensing Radius

This section is devoted to the proof of Theorem 1. Before going into details, we sketch the approach underlying the proof of Theorem 1. The equalities (3) and (4) will be derived from the inequalities (1) and (2) by a perturbation argument. The inequality (2) will be obtained from the inequality (1) by a de-Poissonization argument. The inequality (1) consists of the asymptotic upper bound and the asymptotic lower bound on $\Pr[C_{n,r_n}]$. The proof of the asymptotic upper bound on $\Pr[C_{n,r_n}]$ is based on the observation that the event $C_{n,r}$ implies that the $(k+1)$-*vacancy* $V_{n,r}$ defined by the Lebesgue measure (i.e., area) of the set of points in Ω which are not $(k+1)$-covered by the closed disks of radius r centered at the points in $\mathcal{P}_n(\Omega)$ is zero. Hence, $\Pr[C_{n,r}]$ is upper bounded by $\Pr[V_{n,r} = 0]$. Based on the Cauchy-Schwartz inequality, $\Pr[V_{n,r} = 0]$ can be further upper bounded in terms of the mean and variance of $V_{n,r}$. The proof of the asymptotic lower bound on $\Pr[C_{n,r_n}]$ is based on the following characterization of $(k+1)$-coverage by *open* disks: Let $L_{n,r}$ denote the number of (n,r)-crossing points where an (n,r)-*crossing point* is either an intersection point of $\partial\Omega$ and $\partial D_r(X)$ for some $X \in \mathcal{P}_n(\Omega)$, or an intersection point of $\partial D_r(X)$ and $\partial D_r(Y)$ for some $X, Y \in \mathcal{P}_n(\Omega)$ respectively, and $M_{n,r}$ denote the number of (n,r)-crossing points which are not $(k+1)$-covered by the *open* disks of radius r centered at the points in $\mathcal{P}_n(\Omega)$. Then, $C_{n,r}$ occurs if and only if $L_{n,r} > 0$ and $M_{n,r} = 0$ (see, e.g., [4]). Hence

$$\begin{aligned}\Pr[C_{n,r}] &= \Pr[L_{n,r} > 0, M_{n,r} = 0]\\ &= \Pr[L_{n,r} > 0] - \Pr[L_{n,r} > 0, M_{n,r} > 0]\\ &= \Pr[L_{n,r} > 0] - \Pr[M_{n,r} > 0]\\ &\geq \Pr[L_{n,r} > 0] - E[M_{n,r}]\end{aligned}$$

where the last inequality follows from the Markov inequality.

The remaining of this section proceeds as follows. Section III-A investigates the mean and variance of the $(k+1)$-vacancy $V_{n,r}$. Section III-B studies the mean number of crossing points which are not $(k+1)$-covered. Based on the results obtained in these two subsections, Section III-C give the proof of Theorem 1.

A. Mean and Variance of the $(k+1)$-Vacancy

The following lemma gives a general expression of the mean of the $(k+1)$-vacancy and an upper bound on the variance of the $(k+1)$-vacancy.

Lemma 6: For any n and any r

$$E[V_{n,r}] = \int_{\Omega} \phi_{n,r}(x)\,dx$$
$$Var[V_{n,r}] \leq \int_{\substack{x,y\in\Omega\\ \|x-y\|\leq 2r}} \phi_{n,r}(x)\,\phi_{n,r}(y\backslash x)\,dxdy.$$

Proof: For the simplicity of presentation, we suppress all subscripts. For any $x \in \Omega$, we use $U(x)$ to denote the event that the (closed or open) disk contains at most k points in $\mathcal{P}_n(\Omega)$. Note that $U(x)$ occurs if and only if x is *not* $(k+1)$-covered by (closed or open) disks of radius r centered at points in $\mathcal{P}_n(\Omega)$. Then

$$\Pr[U(x)] = \phi(x).$$

For any $x, y \in \Omega$, we use $U(y \backslash x)$ to denote the event that $(D(y) \setminus D(x)) \cap \Omega$ contains at most k points in $\mathcal{P}_n(\Omega)$. Then

$$\Pr[U(y \backslash x)] = \phi(y \backslash x).$$

For any event Q, we use $\mathbf{1}(Q)$ to denote the indicator of the event Q. Then

$$V = \int_\Omega \mathbf{1}(U(x))\, dx$$

and

$$V^2 = \left(\int_\Omega \mathbf{1}(U(x))\, dx\right)^2 = \int_{\Omega^2} \mathbf{1}(U(x))\,\mathbf{1}(U(y))\, dxdy.$$

By Fubini's theorem

$$\begin{aligned} E[V] &= \int_\Omega E[\mathbf{1}(U(x))]\, dx \\ &= \int_\Omega \Pr[U(x)]\, dx = \int_\Omega \phi(x)\, dx \end{aligned}$$

and

$$\begin{aligned} E\left[V^2\right] &= \int_{\Omega^2} E[\mathbf{1}(U(x))\,\mathbf{1}(U(y))]\, dxdy \\ &= \int_{\Omega^2} \Pr[U(x) \wedge U(y)]\, dxdy. \end{aligned}$$

Therefore,

$$\begin{aligned} Var[V] &= E\left[V^2\right] - E[V]^2 \\ &= \int_{\Omega^2} \Pr[U(x) \wedge U(y)]\, dxdy - \left(\int_\Omega \Pr[U(x)]\, dx\right)^2 \\ &= \int_{\Omega^2} (\Pr[U(x) \wedge U(y)] - \Pr[U(x)]\Pr[U(y)])\, dxdy \\ &= \int_{\substack{x,y\in\Omega \\ \|x-y\|\le 2r}} \begin{pmatrix} \Pr[U(x) \wedge U(y)] \\ -\Pr[U(x)]\Pr[U(y)] \end{pmatrix} dxdy \\ &\le \int_{\substack{x,y\in\Omega \\ \|x-y\|\le 2r}} \Pr[U(x) \wedge U(y)]\, dxdy. \end{aligned}$$

Clearly, the event $U(y)$ implies $U(y \backslash x)$, and the two events $U(x)$ and $U(y \backslash x)$ are independent. Thus,

$$\begin{aligned} \Pr[U(x) \wedge U(y)] &\le \Pr[U(x) \wedge U(y \backslash x)] \\ &= \Pr[U(x)]\Pr[U(y \backslash x)] = \phi(x)\,\phi(y \backslash x). \end{aligned}$$

So

$$Var[V] \le \int_{\substack{x,y\in\Omega \\ \|x-y\|\le 2r}} \phi_{n,r}(x)\,\phi_{n,r}(y \backslash x)\, dxdy. \qquad \square$$

Lemma 6 together with Lemma 5 implies the following lemma.

Lemma 7: Let

$$r_n = \sqrt{\frac{\ln n + (2k+1)\ln\ln n + \xi_n}{\pi n}}.$$

with $\lim \xi_n = \xi$ for some $\xi \in \mathbb{R}$. Then,

$$E\left[n\left(n\pi r_n^2\right)V_{n,r_n}\right] \sim \begin{cases} e^{-\xi} + \frac{\sqrt{\pi}\eta}{2} e^{-\frac{\xi}{2}}, & \text{if } k = 0 \\ \frac{\sqrt{\pi}\eta}{2^{k+1}k!} e^{-\frac{\xi}{2}}, & \text{if } k \ge 1 \end{cases}$$

and

$$\begin{aligned} &Var\left[n\left(n\pi r_n^2\right)V_{n,r_n}\right] \\ &\quad\lesssim \begin{cases} 16e^{-\xi} + 32\sqrt{\pi}\eta e^{-\frac{\xi}{2}}, & \text{if } k = 0 \\ \frac{(k+1)(k+2)\sqrt{\pi}\eta}{2^{k-4}k!} e^{-\frac{\xi}{2}}, & \text{if } k \ge 1. \end{cases} \end{aligned}$$

B. Crossing Points

In this subsection, we prove the following lemma.

Lemma 8: Let

$$r_n = \sqrt{\frac{\ln n + (2k+1)\ln\ln n + \xi_n}{\pi n}}.$$

with $\lim \xi_n = \xi$ for some $\xi \in \mathbb{R}$. Then, $L_{n,r_n} > 0$ is asymptotically almost sure and $E[M_{n,r_n}] \lesssim \beta(\xi)$.

Proof: For simplicity, we suppress the subscripts. An (n,r)-cross pointing is said to be of the first (respectively, second) type if it is an intersection point between an intersection point between $\partial\Omega$ and $\partial D(X)$ for some $X \in \mathcal{P}_n(\Omega)$ (respectively, between $\partial D(X)$ and $\partial D(Y)$ for some $X, Y \in \mathcal{P}_n(\Omega)$). Let L' be the number of crossing points of the first type. Then

$$\begin{aligned} \Pr[L = 0] &\le \Pr[L' = 0] = e^{-n|\Omega \setminus \Omega(0)|} \\ &= e^{-\Theta(nr)} = o(1). \end{aligned}$$

Thus, $L > 0$ is asymptotically almost sure.

Let M' (respectively, M'') be the number of crossing points of the first (respectively, second) type which are not $(k+1)$-covered. By Lemma 7, $E[M_{n,r_n}] \lesssim \beta(\xi)$ would follow from

$$\begin{aligned} E[M'] &\sim \frac{1}{k!2^{k-1}\sqrt{\pi}} \eta e^{-\frac{\xi}{2}} \\ E[M''] &\le 4E\left[n\left(n\pi r^2\right)V\right]. \end{aligned}$$

We begin with the limit of $E[M']$. First consider the case that Ω is a square region. Let X be a random point and Q be the event that the right half-circle of $\partial D(X)$ and the upper side of Ω intersects at a point which is not $(k+1)$-covered. By symmetry,

$$E(M') = 8n\Pr(Q).$$

Note that for each z on the top side of Ω,

$$\frac{1}{4}\pi r^2 \le v(z) \le \frac{1}{2}\pi r^2,$$

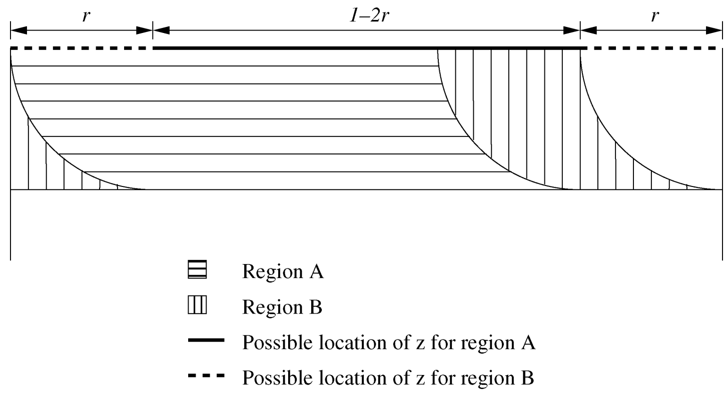


Fig. 2. Region A and B for the crosspoint in $\partial\Omega$.

and $v(z) = \frac{1}{2}\pi r^2$ if and only if z is at a distance of at least r from either vertical side of Ω. Let A be the set of points of x such that the right half-circle of $\partial D(x)$ and the upper side of Ω intersects at a point which is at a distance of at least r from both vertical sides of Ω, and B be the set of points of x such that the right half-circle of $\partial D(x)$ and the upper side of Ω intersects at a point which is at a distance of less r from either the left vertical side of Ω or the right vertical side of Ω (see Fig. 2). Then

$$|A| = r(1-2r), |B| = 2r^2 - \frac{1}{4}\pi r^2$$

and

$$\Pr(Q) = \int_{A\cup B} \Pr(Q \mid X = x)\, dx.$$

Since for any $x \in A$

$$\Pr(Q \mid X = x) = \sum_{i=0}^{k} \frac{\left(\frac{n\pi r^2}{2}\right)^i}{i!} e^{-\frac{n\pi r^2}{2}}$$

we have

$$\begin{aligned} n\int_A \Pr(Q \mid X = x)\, dx &= nr(1-2r)\sum_{i=0}^{k} \frac{\left(\frac{1}{2}n\pi r^2\right)^i}{i!} e^{-\frac{1}{2}n\pi r^2} \\ &\sim \frac{nr}{k!}\left(\frac{n\pi r^2}{2}\right)^k e^{-\frac{n\pi r^2}{2}} \\ &= \frac{1}{k!2^k\pi}\frac{1}{r}\left(n\pi r^2\right)^{k+1} e^{-\frac{n\pi r^2}{2}} \\ &\sim \frac{1}{k!2^k\sqrt{\pi}} e^{-\frac{\xi}{2}}. \end{aligned}$$

Since for any $x \in B$

$$\Pr(Q \mid X = x) \le \sum_{i=0}^{k} \frac{\left(\frac{1}{2}n\pi r^2\right)^i}{i!} e^{-\frac{1}{4}n\pi r^2}$$

we have

$$\begin{aligned} n\int_B \Pr(Q \mid X = x)\, dx &\le n\left(2r^2 - \frac{1}{4}\pi r^2\right)\sum_{i=0}^{k} \frac{\left(\frac{1}{2}n\pi r^2\right)^i}{i!} e^{-\frac{1}{4}n\pi r^2} \\ &= \left(\frac{2}{\pi} - \frac{1}{4}\right)\sum_{i=0}^{k} \frac{\left(n\pi r^2\right)^{i+1}}{2^i i!} e^{-\frac{1}{4}n\pi r^2} \\ &= o(1). \end{aligned}$$

Therefore,

$$E(M') = 8n\Pr(Q) \sim \frac{\eta}{k!2^{k-1}\sqrt{\pi}} e^{-\frac{\xi}{2}}.$$

Now we consider the case that Ω is a disk region. Each node in $\Omega \setminus \Omega(0)$ produces two crossing points of the first type. Each node on $\partial\Omega(0)$ produces exactly one crossing pint of the first type. All other nodes does not produce any crossing point of the first type. By Lemma 4, a crossing point of the first type is not $(k+1)$-covered with an asymptotic probability $\frac{1}{k!}\left(\frac{n\pi r^2}{2}\right)^k e^{-\frac{n\pi r^2}{2}}$. Since $\partial\Omega(0)$ has zero measure, we have

$$\begin{aligned} &E(M') \\ &\sim 2n\left|\Omega \setminus \Omega(0)\right| \cdot \frac{1}{k!}\left(\frac{n\pi r^2}{2}\right)^k e^{-\frac{n\pi r^2}{2}} \\ &= 2n \cdot 2\sqrt{\pi} r\left(1 - \frac{\sqrt{\pi}}{2} r\right) \cdot \frac{1}{k!}\left(\frac{n\pi r^2}{2}\right)^k e^{-\frac{n\pi r^2}{2}} \\ &\sim \frac{2\sqrt{\pi}}{k!2^{k-1}\pi}\frac{1}{r}\left(n\pi r^2\right)^{k+1} e^{-\frac{n\pi r^2}{2}} \\ &\sim \frac{1}{k!2^{k-1}\sqrt{\pi}}\eta e^{-\frac{\xi}{2}}. \end{aligned}$$

Next we derive the asymptotic upper bound on $E[M'']$. Fix an ordered pair of random nodes X and Y. Let Z be the intersecting point of the two circles $\partial D(X)$ and $\partial D(Y)$ with $\overrightarrow{XY} \times \overrightarrow{XZ} \ge 0$. Let Q denote the event that Z lies inside Ω and Z is not $(k+1)$-covered. Then by symmetry,

$$E[M''] = n(n-1)\Pr[Q].$$

Since X and Y are with uniform distribution over Ω, the probability that Z lies in a sufficiently small circular disk of area dz centered at a point $z \in \Omega$ is equal to the Lebesgue measure of (X, Y) such that $Z \in dz$. Thus, by Lemma 3, the probability that Z lies in a sufficiently small circular disk of area dz centered at a point $z \in \Omega$ is equal to $4\pi r^2 dz$ if $dz \subset \Omega(0)$, and at most $4\pi r^2 dz$ due to the boundary effect otherwise. Therefore

$$\begin{aligned}\Pr[Q] &= \int_{z\in\Omega} \Pr(Q \mid Z = z)\Pr(Z \in dz) \\ &\leq \int_{z\in\Omega} \phi_{n,r}(z)\, 4\pi r^2 dz = 4\pi r^2 E[V].\end{aligned}$$

Hence,

$$E[M''] \leq n(n-1)\cdot 4\pi r^2 E[V] \leq 4E\left[n\left(n\pi r^2\right)V\right]. \quad \square$$

C. Proof of Theorem 1

We first prove the inequality (1). For simplicity, we suppress the subscripts. By Lemma 7, it is straightforward to show that

$$\frac{E\left[n\left(n\pi r^2\right)V\right]^2}{Var\left[n\left(n\pi r^2\right)V\right]} \gtrsim \alpha(\xi).$$

By the Cauchy–Schwartz inequality

$$\begin{aligned}\Pr[V=0] &\leq \frac{Var[V]}{E[V^2]} = \frac{1}{1+\frac{E[V]^2}{Var[V]}} \\ &= \frac{1}{1+\frac{E[n(n\pi r^2)V]^2}{Var[n(n\pi r^2)V]}} \lesssim \frac{1}{1+\alpha(\xi)}.\end{aligned}$$

Thus,

$$\Pr[C] \leq \Pr[V=0] \lesssim \frac{1}{1+\alpha(\xi)}.$$

On the other hand, by Lemma 8

$$\Pr[C] \geq \Pr[L_{n,r} > 0] - E[M_{n,r}] \gtrsim 1 - \beta(\xi).$$

So, the inequality (1) holds.

Now we derive the inequality (2) from the inequality (1) using a de-Poissonization argument. By coupling Poisson point processes with uniform point processes, we obtain the following relations:

$$\begin{aligned} C_{n-n^{3/4},r_n} &\subseteq C'_{n,r_n} \cup \left\{Po\left(n-n^{3/4}\right) > n\right\}, \\ C'_{n,r_n} &\subseteq C_{n+n^{3/4},r_n} \cup \left\{Po\left(n+n^{3/4}\right) < n\right\}.\end{aligned}$$

By Chebyshev's inequality

$$\begin{aligned}\Pr\left(Po\left(n-n^{3/4}\right) > n\right) &= o(1) \\ \Pr\left(Po\left(n+n^{3/4}\right) < n\right) &= o(1).\end{aligned}$$

Thus,

$$\begin{aligned}\Pr\left[C'_{n,r_n}\right] &\geq \Pr\left[C_{n-n^{3/4},r_n}\right] - o(1) \\ \Pr\left[C'_{n,r_n}\right] &\leq \Pr\left[C_{n+n^{3/4},r_n}\right] + o(1).\end{aligned}$$

Note that

$$\begin{aligned}\left(n \pm n^{3/4}\right)\pi r_n^2 &= n\pi r_n^2 + o(1) \\ \ln\left(n \pm n^{3/4}\right) &= \ln n + o(1) \\ \ln\ln\left(n \pm n^{3/4}\right) &= \ln\ln n + o(1).\end{aligned}$$

So

$$\begin{aligned}&\left(n\pm n^{3/4}\right)\pi r_n^2 - \left(\ln\left(n\pm n^{3/4}\right) + (2k+1)\ln\ln\left(n\pm n^{3/4}\right)\right) \\ &=, n\pi r_n^2 - (\ln n + (2k+1)\ln\ln n) + o(1) \\ &= \xi + o(1).\end{aligned}$$

By the inequality (1),

$$\begin{aligned}\Pr\left[C_{n-n^{3/4},r_n}\right] &\gtrsim 1-\beta(\xi) \\ \Pr\left[C_{n+n^{3/4},r_n}\right] &\lesssim \frac{1}{1+\alpha(\xi)}.\end{aligned}$$

Hence,

$$\begin{aligned}\Pr\left[C'_{n,r_n}\right] &\gtrsim 1-\beta(\xi) \\ \Pr\left[C'_{n,r_n}\right] &\lesssim \frac{1}{1+\alpha(\xi)}.\end{aligned}$$

So, the inequality (2) holds.

Finally, we prove the equalities (3) and (4) from the inequalities (1) and (2) using a perturbation argument. We only provide the proof for the event C_{n,r_n}, since the proof for the event C'_{n,r_n} is exactly the same. We first prove the equality (3). Assume that $\lim \xi_n = \infty$. Note that $\lim_{\xi\to\infty}\alpha(\xi) = 0$. For any arbitrarily small $\epsilon > 0$, let ξ' such that $\beta(\xi') = \epsilon$, and set

$$r'_n = \sqrt{\frac{\ln n + (2k+1)\ln\ln n + \xi'}{\pi n}}.$$

Since $\lim \xi_n = \infty$, for sufficiently large n, $\xi_n > \xi'$, and thus $r_n \geq r'_n$, which further implies that $C_{n,r_n} \supseteq C_{n,r'_n}$. Consequently

$$\Pr[C_{n,r_n}] \geq \Pr\left[C_{n,r'_n}\right] \gtrsim 1 - \beta(\xi') = 1-\epsilon.$$

Since ϵ can be arbitrarily small, the equality (3) holds. Next, we prove the equality (4). Assume that $\lim \xi_n = -\infty$. Note that $\lim_{\xi\to-\infty}\beta(\xi) = \infty$. For any arbitrarily small $\epsilon > 0$, let ξ'' such that $\alpha(\xi'') = \frac{1-\epsilon}{\epsilon}$, and set

$$r''_n = \sqrt{\frac{\ln n + (2k+1)\ln\ln n + \xi''}{\pi n}}$$

Since $\lim \xi_n = -\infty$, for sufficiently large n, $\xi_n < \xi''$, and thus $r_n \leq r''_n$, which further implies that $C_{n,r_n} \subseteq C_{n,r''_n}$. Consequently

$$\Pr[C_{n,r_n}] \leq \Pr\left[C_{n,r''_n}\right] \lesssim \frac{1}{1+\alpha(\xi'')} = \epsilon.$$

Since ϵ can be arbitrarily small, the equality (4) holds. This completes the proof of Theorem 1.

IV. Critical Number of Nodes

In this section, we will derive Theorem 2 from Theorem 1 using a scaling argument. First, we scale the points of $\mathcal{P}_n(\sqrt{s}\Omega)$ (respectively, $\mathcal{X}_n(\sqrt{s}\Omega)$) by a factor of $1/\sqrt{s}$. The resulting points form a Poisson (respectively, uniform) point process with mean n (respectively, a uniform n-point process) on Ω. Second, we scale the sensing radius from $1/\sqrt{\pi}$ to $1/\sqrt{\pi s}$. Then, $K_{s,n}$ (respectively, $K'_{s,n}$) occurs if and only if $C_{n,1/\sqrt{\pi s}}$ (respectively, $C'_{n,1/\sqrt{\pi s}}$)occurs. This implies that

$$\Pr[K_{s,n}] = \Pr\left[C_{n,1/\sqrt{\pi s}}\right]$$

$$\Pr\left[K'_{s,n}\right] = \Pr\left[C'_{n,1/\sqrt{\pi s}}\right].$$

Thus, we only need to obtain the asymptotic bounds on $\Pr\left[C_{n,1/\sqrt{\pi s}}\right]$ and $\Pr\left[C'_{n,1/\sqrt{\pi s}}\right]$ for $n = \mu(s)\, s$ where $\mu(s)$ is given in Theorem 2. For this purpose, we prove the following technical lemma.

Lemma 9: Let $\xi(s)$ be such that $\lim_{s\to\infty} \xi(s) = \xi$ for some $\xi \in [-\infty, +\infty]$ and $\xi(s) \geq -\ln s - l \ln\ln s$ for some $l \geq 1$. Let

$$\mu(s) = \ln s + l \ln\ln s + \xi(s).$$

Then

$$\lim_{s\to\infty} \left(\mu(s) - \ln(\mu(s)\, s) - (l-1)\ln\ln(\mu(s)\, s)\right) = \xi.$$

Proof: Let $\mu = \mu(s)$. Note that

$$\begin{aligned}
&\mu - \ln(\mu s) - (l-1)\ln\ln(\mu s)\\
&\quad= \xi(s) + \ln s + l\ln\ln s - \ln(\mu s) - (l-1)\ln\ln(\mu s)\\
&\quad= \xi(s) + l\ln\ln s - \ln\mu - (l-1)\ln\ln(\mu s)\\
&\quad= \xi(s) - \ln\frac{\mu}{\ln s} - (l-1)\ln\left(1 + \frac{\ln\mu}{\ln s}\right).
\end{aligned}$$

If $|\xi(s)| = o(\ln s)$, then

$$\lim_{s\to\infty} \frac{\mu}{\ln s} = 1,\ \lim_{s\to\infty} \frac{\ln\mu}{\ln s} = 0.$$

So

$$\lim_{s\to\infty} \left(\mu - \ln(\mu s) - (l-1)\ln\ln(\mu s)\right) = \lim_{s\to\infty} \xi(s) = \xi.$$

Now we assume that $|\xi(s)| \neq o(\ln s)$. Then $\lim_{s\to\infty} |\xi(s)| = \infty$. We first consider the case that $\lim_{s\to\infty} \xi(s) = \infty$. For sufficiently large s

$$\begin{aligned}
&\mu - \ln(\mu s) - (l-1)\ln\ln(\mu s)\\
&\quad= \xi(s) - \ln\left(1 + \frac{\ln\ln s + \xi(s)}{\ln s}\right) - (l-1)\ln\left(1 + \frac{\ln\mu}{\ln s}\right)\\
&\quad\geq \xi(s) - \frac{\ln\ln s + \xi(s)}{\ln s} - (l-1)\frac{\ln\mu}{\ln s}\\
&\quad\geq \xi(s) - \frac{\ln\ln s + \xi(s)}{\ln s}\\
&\qquad - (l-1)\frac{\ln(\ln s + \ln\ln s) + \ln\xi(s)}{\ln s}\\
&\quad= \xi(s)\left(1 - \frac{l}{\ln s}\right) - \frac{\ln\ln s + (l-1)\ln(\ln s + \ln\ln s)}{\ln s}.
\end{aligned}$$

Thus,

$$\lim_{s\to\infty} \left(\mu - \ln(\mu s) - (l-1)\ln\ln(\mu s)\right) \geq \lim_{s\to\infty} \xi(s) = \infty.$$

So, the lemma holds in this case.

Next, we consider the case that $\lim_{s\to\infty} \xi(s) = -\infty$. Since $|\xi(s)| \neq o(\ln s)$ and

$$\xi(s) \geq -\ln s - l\ln\ln s$$

we have $|\xi(s)| = \Theta(\ln s)$. Thus,

$$\frac{\mu}{\ln s} = \Theta(1),\ \frac{\ln\mu}{\ln s} = o(1).$$

So, we have

$$\begin{aligned}
&\mu - \ln(\mu s) - (l-1)\ln\ln(\mu s)\\
&\quad= \xi(s) - \ln\frac{\mu}{\ln s} - (l-1)\ln\left(1 + \frac{\ln\mu}{\ln s}\right)\\
&\quad= \xi(s) - \Theta(1) - o(1).
\end{aligned}$$

Therefore

$$\lim_{s\to\infty} \left(\mu - \ln(\mu s) - (l-1)\ln\ln(\mu s)\right) = -\infty.$$

□

Next, we proceed on deriving the asymptotic bounds on $\Pr\left[C_{\mu(s)s,1/\sqrt{\pi s}}\right]$ and $\Pr\left[C'_{\mu(s)s,1/\sqrt{\pi s}}\right]$ where $\mu(s)$ is given in Theorem 2. Clearly, if $\lim_{s\to\infty} \mu s < \infty$, which happens only when $\lim_{s\to\infty} \xi(s) = -\infty$, then

$$\lim_{s\to\infty} \Pr\left[C_{\mu(s)s,1/\sqrt{\pi s}}\right] = \lim_{s\to\infty} \Pr\left[C_{\mu(s)s,1/\sqrt{\pi s}}\right] = 0.$$

So assume that $\lim_{s\to\infty} \mu(s)\, s = \infty$. By Lemma 9

$$\begin{aligned}
&\mu(s)\, s \cdot \pi \left(1/\sqrt{\pi s}\right)^2\\
&\qquad - \ln(\mu(s)\, s) - (2k+1)\ln\ln(\mu(s)\, s)\\
&\quad= \mu(s) - \ln(\mu(s)\, s) - (2k+1)\ln\ln(\mu(s)\, s)\\
&\quad\to \xi \text{ (as } s \to \infty).
\end{aligned}$$

Thus, by Theorem 1, if $\xi \neq \pm\infty$, then

$$\begin{aligned}
1 - \beta(\xi) &\leq \lim_{s\to\infty} \Pr\left[C_{\mu(s)s,1/\sqrt{\pi s}}\right] \leq \frac{1}{1+\alpha(\xi)}\\
1 - \beta(\xi) &\leq \lim_{s\to\infty} \Pr\left[C'_{\mu(s)s,1/\sqrt{\pi s}}\right] \leq \frac{1}{1+\alpha(\xi)};
\end{aligned}$$

if $\xi = \infty$, then

$$\lim_{s\to\infty} \Pr\left[C_{\mu(s)s,1/\sqrt{\pi s}}\right] = \lim_{s\to\infty} \Pr\left[C'_{\mu(s)s,1/\sqrt{\pi s}}\right] = 1$$

and if $\xi = -\infty$, then

$$\lim_{s\to\infty} \Pr\left[C_{\mu(s)s,1/\sqrt{\pi s}}\right] = \lim_{s\to\infty} \Pr\left[C'_{\mu(s)s,1/\sqrt{\pi s}}\right] = 0.$$

This completes the proof of Theorem 2.

V. Conclusion

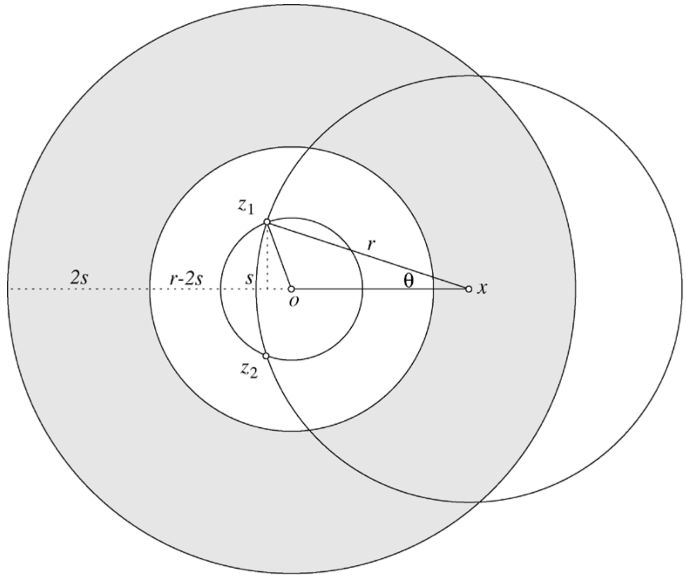
In this paper, we address the asymptotic $(k+1)$-coverage of a square or disk region by a Poisson or uniform point process. A major technical challenge is the handling of the boundary effect. As indicated by our analyzes, for $k \geq 1$ the boundary effect completely dominates the probability of $(k+1)$-coverage, while for $k = 0$ the boundary effect still affects significantly the probability of $(k+1)$-coverage. For the purpose of comparison between with and without the boundary effect, let us consider the asymptotic $(k+1)$-coverage of a square by Poisson point process with unit-area coverage range. With boundary effect, the asymptotic $(k+1)$-coverage requires that the sensor density n/s should grow with the area s at least according to

$$n/s = \ln s + 2(k+1)\ln\ln s + \xi(s)$$

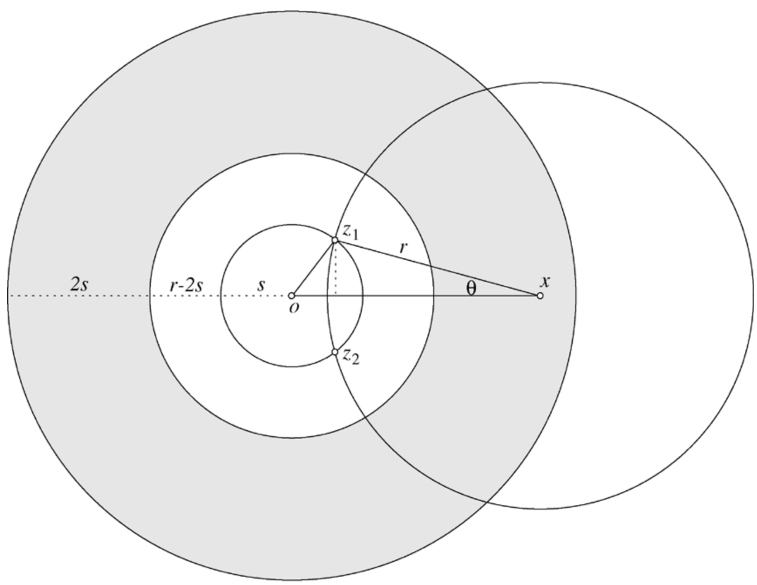
with $\lim_{s\to\infty} \xi(s) = \infty$. Without the boundary effect, the asymptotic $(k+1)$-coverage only requires that the sensor density n/s grows with the area s according to

$$n/s = \ln s + (k+2)\ln\ln s + \xi(s)$$

with $\lim_{s\to\infty} \xi(s) = \infty$ [6].



(a)



(b)

Fig. 3. The shaded annulus is the set A of all possible points x. (a) $\|x\| \in \left[r-s, \sqrt{r^2-s^2}\right]$ and (b) $\|x\| \in \left[\sqrt{r^2-s^2}, r+s\right]$.

Another interesting observation is the similar asymptotics of the k-coverage and the $(k+1)$-connectivity. To illustrate this similarity, let's consider a Poisson point process $\mathcal{P}_n$ on the unit-area square or disk. The asymptotic $(k+1)$-connectivity requires that the communication radius should grow with n at least according to $\ln n + (2k-1)\ln\ln n + \xi_n$ with $\lim_{n\to\infty} \xi_n = \infty$ [3]. Similarly, the asymptotic k-connectivity requires that the sensing radius should also grow with n at least according to $\ln n + (2k-1)\ln\ln n + \xi_n$ with $\lim_{n\to\infty} \xi_n = \infty$. It would be interesting to discover the exact correlation between the k-coverage and the $(k+1)$-connectivity.

APPENDIX

In the Appendix, we are going to give the proofs of Lemma 3, 4, and 5.

Proof (Lemma 3): Without loss of generality, we assume that S is centered at the origin o. Let A be the annulus centered at o with radii $r-s$ and $r+s$ (see Fig. 3). Then A is exactly the set of points x such that $\|x-z\| = r$ for some point $z \in S$. Now fix a $x \in A$. Let $\widehat{z_1 z_2}$ denote the arc of the boundary circle of the disk $D_r(x)$ which lies inside the disk S with $\overrightarrow{xz_1} \times \overrightarrow{xz_2} \geq 0$ (see Fig. 3). Note that the angle $\angle z_1 x z_2$ only depends on $\|x\|$. Let $\theta = \theta(\|x\|)$ be the half of the angle $\angle z_1 x z_2$. For $\|x\| \in \left[r-s, \sqrt{r^2-s^2}\right]$, θ increases with $\|x\|$ and

$$\|x\| = \sqrt{r^2 - (r\sin\theta)^2} - \sqrt{s^2 - (r\sin\theta)^2}$$

(see Fig. 3(a)). For $\|x\| \in \left[\sqrt{r^2-s^2}, r+s\right]$, θ decreases with $\|x\|$ and

$$\|x\| = \sqrt{r^2 - (r\sin\theta)^2} + \sqrt{s^2 - (r\sin\theta)^2}$$

(see Fig. 3(b)).

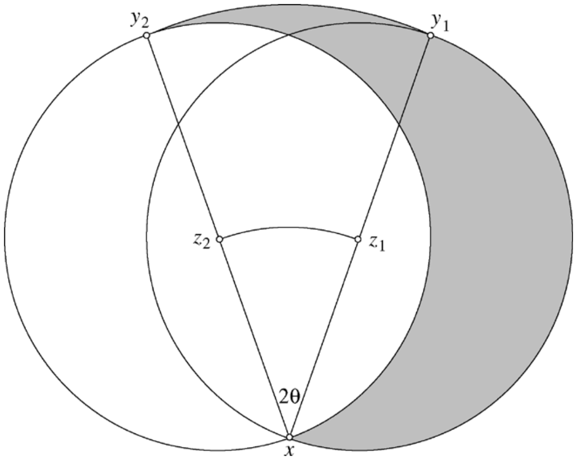


Fig. 4. The shaded region is set $B(x)$ of all possible points y for a fixed x.

Let y_1 (respectively, y_2) be such that xy_1 (respectively, xy_2) is a diameter of the disk $D_r(z_1)$ (respectively, $D_r(z_2)$) (see Fig. 4). Let $\measuredangle xy_1y_2$ denote the sector of the disk $D_{2r}(x)$ subtended by y_1 and y_2, and let

$$B(x)=(\measuredangle xy_1y_2 \setminus (D_r(z_1)\cup D_r(z_2)))\cup(D_r(z_1)\setminus D_r(z_2))$$

which is the shaded region illustrated in Fig. 4. Then

$$\mathcal{F}=\{(x,y): x\in A, y\in B(x)\}.$$

By a simple geometric argument, we can show that $B(x)$ and the sector $\measuredangle xy_1y_2$ have the same area, i.e.

$$|B(x)|=(2r)^2\cdot\theta(\|x\|).$$

Thus,

$$\begin{aligned}|\mathcal{F}|&=\int_A dx\int_{B(x)}dy=4r^2\int_A\theta(\|x\|)\,dx\\&=4\pi r^2\int_{r-s}^{r+s}2t\cdot\theta(t)\,dt.\end{aligned}$$

Using integration by parts on the integral yields

$$\begin{aligned}&\int_{r-s}^{r+s}2t\cdot\theta(t)\,dt\\
&\quad=\int_{r-s}^{r+s}\theta(t)\,dt^2\\
&\quad=t^2\theta(t)\big|_{r-s}^{r+s}-\int_{r-s}^{r+s}t^2d\theta(t)=-\int_{r-s}^{r+s}t^2d\theta(t)\\
&\quad=-\int_{r-s}^{\sqrt{r^2-s^2}}t^2d\theta(t)-\int_{\sqrt{r^2-s^2}}^{r+s}t^2d\theta(t)\\
&\quad=-\int_0^{\arcsin\frac{s}{r}}\left(\sqrt{r^2-(r\sin\theta)^2}-\sqrt{s^2-(r\sin\theta)^2}\right)^2d\theta\\
&\qquad-\int_{\arcsin\frac{s}{r}}^0\left(\sqrt{r^2-(r\sin\theta)^2}+\sqrt{s^2-(r\sin\theta)^2}\right)^2d\theta\\
&\quad=\int_0^{\arcsin\frac{s}{r}}4r\cos\theta\sqrt{s^2-(r\sin\theta)^2}d\theta\\
&\quad=4\int_0^{\arcsin\frac{s}{r}}\sqrt{s^2-(r\sin\theta)^2}d(r\sin\theta)\\
&\quad=4\int_0^s\sqrt{s^2-x^2}dx=\pi s^2.\end{aligned}$$

Hence, the lemma follows. □

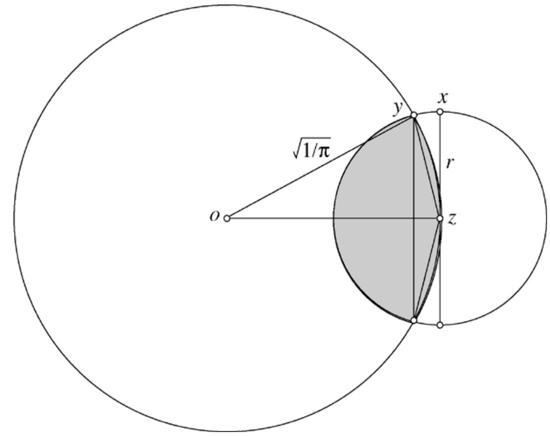


Fig. 5. The shaded region is $D_r(z)\cap\Omega$.

Proof (Lemma 4): For the simplicity of presentation, we suppress all subscripts. Let xz be a radius of $D(z)$ which is perpendicular to oz, and y be an intersection point between $\partial\Omega$ and $\partial D(z)$ which lies to the same side of oz as x (see Fig. 5). Clearly, $v(z)<\frac{1}{2}\pi r^2$. As $\sum_{i=0}^k\frac{\lambda^i}{i!}e^{-\lambda}$ decreases with λ

$$\phi(z)\geq\sum_{i=0}^k\frac{\left(\frac{n\pi r^2}{2}\right)^i}{i!}e^{-\frac{n\pi r^2}{2}}>\frac{\left(\frac{n\pi r^2}{2}\right)^k}{k!}e^{-\frac{n\pi r^2}{2}}.$$

On the other hand, the angle $\angle ozy=\arccos\frac{\sqrt{\pi}r}{2}$, hence

$$\angle xzy=\frac{\pi}{2}-\arccos\frac{\sqrt{\pi}r}{2}=\arcsin\frac{\sqrt{\pi}r}{2}.$$

Thus, the area of the sector $\measuredangle xzy$ is $\frac{1}{2}r^2\arcsin\frac{\sqrt{\pi}r}{2}$. This implies that

$$v(z)>\frac{1}{2}\pi r^2-r^2\arcsin\frac{\sqrt{\pi}r}{2}.$$

Since

$$nr^2\arcsin\frac{\sqrt{\pi}r}{2}\sim\frac{\sqrt{\pi}nr^3}{2}=o(1)$$

we have

$$e^{nr^2\arcsin\frac{\sqrt{\pi}r}{2}}\sim 1.$$

Thus,

$$\begin{aligned}\phi(z)&\leq\sum_{i=0}^k\frac{\left(\frac{n\pi r^2}{2}\right)^i}{i!}e^{-\frac{n\pi r^2}{2}+nr^2\arcsin\frac{\sqrt{\pi}r}{2}}\\
&=\left(\sum_{i=0}^k\frac{\left(\frac{n\pi r^2}{2}\right)^i}{i!}e^{-\frac{n\pi r^2}{2}}\right)e^{nr^2\arcsin\frac{\sqrt{\pi}r}{2}}\\
&\sim\sum_{i=0}^k\frac{\left(\frac{n\pi r^2}{2}\right)^i}{i!}e^{-\frac{n\pi r^2}{2}}\sim\frac{\left(\frac{n\pi r^2}{2}\right)^k}{k!}e^{-\frac{n\pi r^2}{2}}.\end{aligned}$$

So the lemma follows. □

Lemma 5 will be established by Lemmas 10–12. These lemmas are presented in a format more general than required for the proof of Lemma 5 for being applicable to other probabilistic studies beyond this paper.

Lemma 10: Let

$$n\pi r_n^2 = \ln n + \ell \ln\ln n + \xi_n$$

with $\ell \in \mathbb{R}$ and $\lim \xi_n = \xi$ for some $\xi \in \mathbb{R}$. Then for any integer $k \geq 0$ and any integer m

$$n\left(n\pi r_n^2\right)^{\ell-k} \int_{\Omega_{r_n}(0)} \phi_{n,r_n}(x)\, dx \sim \frac{e^{-\xi}}{k!}$$
$$n\left(n\pi r_n^2\right)^{\frac{\ell+1}{2}-k} \int_{\Omega_{r_n}(1)} \phi_{n,r_n}(x)\, dx \sim \frac{\sqrt{\pi}\eta}{2^{k+1}k!} e^{-\frac{\xi}{2}}$$
$$n\left(n\pi r_n^2\right)^{m} \int_{\Omega_{r_n}(2)} \phi_{n,r_n}(x)\, dx = o(1).$$

Proof: For the simplicity of presentation, we suppress all subscripts. By a slight modification of the proofs in [3, Theorems 2 and 3], we can show that for any integer $k \geq 0$ and any integer m

$$n\left(n\pi r^2\right)^{\ell-k} \int_{\Omega(0)} \frac{\left(nv(x)\right)^k e^{-nv(x)}}{k!} dx \sim \frac{e^{-\xi}}{k!},$$
$$n\left(n\pi r^2\right)^{\frac{\ell+1}{2}-k} \int_{\Omega(1)} \frac{\left(nv(x)\right)^k e^{-nv(x)}}{k!} dx \sim \frac{\sqrt{\pi}\eta}{2^{k+1}k!} e^{-\frac{\xi}{2}}$$
$$n\left(n\pi r^2\right)^{m} \int_{\Omega(2)} \frac{\left(nv(x)\right)^k e^{-nv(x)}}{k!} dx = o(1).$$

Due to the space limit, we omit the proof details here. Consequently, Lemma 10 would follow if we can show that for any $S \subseteq \Omega$

$$\int_S \phi(x)\, dx \sim \int_S \frac{\left(nv(x)\right)^k}{k!} e^{-nv(x)} dx.$$

Next, we fix an $S \subseteq \Omega$ and prove the above asymptotic equality. Clearly

$$\int_S \phi(x)\, dx \geq \int_S \frac{\left(nv(x)\right)^k}{k!} e^{-nv(x)} dx.$$

Since $v(x) \geq \frac{\pi r^2}{4}$, we have

$$\phi(x) = \left(1 + \sum_{i=1}^{k} \frac{\frac{k!}{(k-i)!}}{\left(nv(x)\right)^i}\right) \frac{\left(nv(x)\right)^k}{k!} e^{-nv(x)}$$
$$\leq \left(1 + \sum_{i=1}^{k} \frac{\frac{k!}{(k-i)!}}{\left(\frac{n\pi r^2}{4}\right)^i}\right) \frac{\left(nv(x)\right)^k}{k!} e^{-nv(x)}.$$

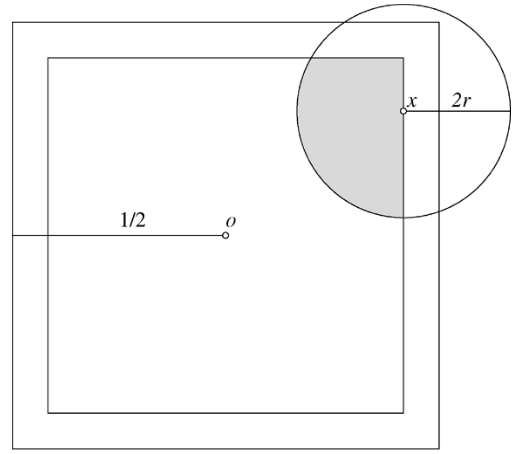
Thus,

$$\int_S \phi(x)\, dx \leq \left(1 + \sum_{i=1}^{k} \frac{\frac{k!}{(k-i)!}}{\left(\frac{n\pi r^2}{4}\right)^i}\right) \int_S \frac{\left(nv(x)\right)^k}{k!} e^{-nv(x)} dx.$$

So

$$\int_S \phi(x)\, dx \sim \int_S \frac{\left(nv(x)\right)^k}{k!} e^{-nv(x)} dx. \qquad \square$$

For any $x \in \Omega$, let Ω_r^x denote the set of points $y \in \Omega$ satisfying that $\|y - x\| \leq 2r$ and $dist(y, \partial\Omega) \geq dist(x, \partial\Omega)$ (see Fig. 6).



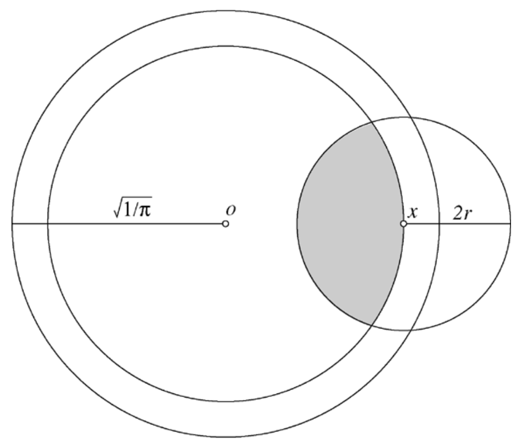


Fig. 6. The shaded region is Ω_r^x.

Lemma 11: For any $x \in \Omega_r(0)$ and any $y \in \Omega_r^x$

$$v_r(y \backslash x) \geq \frac{\pi}{2} r \|x - y\|.$$

If Ω is the unit-area square, then for any $x \in \Omega_r(1)$ and any $y \in \Omega_r^x$

$$v_r(y \backslash x) \geq \frac{\pi}{4} r \|x - y\|.$$

Suppose that $\epsilon > 0$ is an arbitrarily small constant, and Ω is the unit-area disk. Then when r is sufficiently small, for any $x \in \Omega_r(1) \cup \Omega_r(2)$ and any $y \in \Omega_r^x$,

$$v_r(y \backslash x) \geq \left(\frac{\pi}{4} - \epsilon\right) r \|x - y\|.$$

Proof: For the simplicity of presentation, we suppress all subscripts. It was proved in the proof of Lemma 2 in [5] that if $\|x - y\| \leq 2r$, then

$$|D(y) \setminus D(x)| \geq \frac{\pi}{2} r \|x - y\|.$$

Thus, if $x \in \Omega(0)$ and $y \in \Omega^x$,

$$v(y \backslash x) = |D(y) \setminus D(x)| \geq \frac{\pi}{2} r \|x - y\|.$$

Now, we assume that Ω is the unit-area square, $x \in \Omega(1)$ and $y \in \Omega^x$. Then, for the same value of $\|x - y\|$, $v(y \backslash x)$ achieves its minimum when both x and y are in $\partial\Omega$, with the minimum equal to

$$\frac{1}{2}\left|D(y)\setminus D(x)\right| \geq \frac{\pi}{4} r\left\|x-y\right\|.$$

So, the lemma also holds in this case. Finally, we assume that Ω is the unit-area disk. Let ϵ any arbitrarily small positive constant. It was proved in the proof of Lemma 2 in [5] that for any $x \in \Omega(1) \cup \Omega(2)$ and any $y \in \Omega^x$

$$v(y\backslash x) \geq \left(\frac{\pi}{4} - \frac{4r}{\frac{1}{\sqrt{\pi}} + \sqrt{\frac{1}{\pi} - (2r)^2}}\right) r\left\|x-y\right\|.$$

Note that

$$\lim_{r\to 0} \frac{4r}{\frac{1}{\sqrt{\pi}} + \sqrt{\frac{1}{\pi} - (2r)^2}} = 0.$$

Thus, for sufficiently small r

$$\frac{4r}{\frac{1}{\sqrt{\pi}} + \sqrt{\frac{1}{\pi} - (2r)^2}} \leq \epsilon$$

and thus

$$v(y\backslash x) \geq \left(\frac{\pi}{4} - \epsilon\right) r\left\|x-y\right\|.$$

So the lemma hold in this case as well. □

Lemma 12: Suppose that $\epsilon > 0$ is an arbitrarily small constant. Then when r is sufficiently small, for any n

$$\begin{aligned}\int_{\substack{x,y\in\Omega\\ \|x-y\|\leq 2r}} &\phi_{n,r}(x)\,\phi_{n,r}(y\backslash x)\,dxdy\\ &\leq \frac{8(k+1)(k+2)}{n(n\pi r^2)}\int_{\Omega(0)}\phi_{n,r}(x)\,dx\\ &\quad+\frac{(32+\epsilon)(k+1)(k+2)}{n(n\pi r^2)}\int_{\Omega(1)}\phi_{n,r}(x)\,dx\\ &\quad+8\pi r^2\int_{\Omega(2)}\phi_{n,r}(x)\,dx.\end{aligned}$$

Proof: For the simplicity of presentation, we suppress all subscripts. By symmetry

$$\int_{\substack{x,y\in\Omega\\ \|x-y\|\leq 2r}} \phi(x)\,\phi(y\backslash x)\,dxdy = 2\int_{\Omega}\phi(x)\,dx\int_{\Omega^x}\phi(y\backslash x)\,dy.$$

So, we only need to prove that

$$\begin{aligned}\int_{\Omega}\phi(x)\,dx&\int_{\Omega^x}\phi(y\backslash x)\,dy\\ &\leq \frac{4(k+1)(k+2)}{n(n\pi r^2)}\int_{\Omega(0)}\phi_{n,r}(x)\,dx\\ &\quad+\frac{(16+\epsilon/2)(k+1)(k+2)}{n(n\pi r^2)}\int_{\Omega(1)}\phi_{n,r}(x)\,dx\\ &\quad+4\pi r^2\int_{\Omega(2)}\phi_{n,r}(x)\,dx.\end{aligned}$$

We first show that if there is a constant $c > 0$ such that $v(y\backslash x) \geq cr\left\|x-y\right\|$ for any $y \in \Omega^x$, then

$$n\left(n\pi r^2\right)\int_{\Omega^x}\phi(y\backslash x)\,dy \leq \left(\frac{\pi}{c}\right)^2 \frac{(k+1)(k+2)}{n(n\pi r^2)}.$$

As $\sum_{i=0}^{k}\frac{\lambda^i}{i!}e^{-\lambda}$ decreases with λ, we have

$$\begin{aligned}\int_{\Omega^x}&\phi(y\backslash x)\,dy\\ &\leq \int_{\Omega^x}\sum_{i=0}^{k}\frac{(cnr\|x-y\|)^i}{i!}e^{-cnr\|x-y\|}dy\\ &\leq \int_{\mathbb{R}^2}\sum_{i=0}^{k}\frac{(cnr\|x-y\|)^i}{i!}e^{-cnr\|x-y\|}dy\\ &= \int_{t=0}^{\infty}\sum_{i=0}^{k}\frac{(cnrt)^i}{i!}e^{-cnrt}2\pi t dt\\ &= \left(\frac{\pi}{c}\right)^2\frac{2}{n(n\pi r^2)}\sum_{i=0}^{k}\int_{t=0}^{\infty}\frac{(cnrt)^{i+1}}{i!}e^{-cnrt}d(cnrt)\\ &= \left(\frac{\pi}{c}\right)^2\frac{2}{n(n\pi r^2)}\sum_{i=0}^{k}\int_{t=0}^{\infty}\frac{t^{i+1}}{i!}e^{-t}dt\\ &= \left(\frac{\pi}{c}\right)^2\frac{2}{n(n\pi r^2)}\sum_{i=0}^{k}(i+1)\int_{t=0}^{\infty}\frac{t^{i+1}}{(i+1)!}e^{-t}dt\\ &= \left(\frac{\pi}{c}\right)^2\frac{2}{n(n\pi r^2)}\sum_{i=0}^{k}(i+1)\\ &= \left(\frac{\pi}{c}\right)^2\frac{(k+1)(k+2)}{n(n\pi r^2)}.\end{aligned}$$

Now by Lemma 11

$$\int_{\Omega(0)}\phi(x)\,dx\int_{\Omega^x}\phi(y\backslash x)\,dy \leq \frac{4(k+1)(k+2)}{n(n\pi r^2)}\int_{\Omega(0)}\phi(x)\,dx.$$

If Ω is a unit-area square, then by Lemma 11

$$\int_{\Omega(1)}\phi(x)\,dx\int_{\Omega^x}\phi(y\backslash x)\,dy \leq \frac{16(k+1)(k+2)}{n(n\pi r^2)}\int_{\Omega(1)}\phi(x)\,dx.$$

Now we assume that Ω is a unit-area disk. Let ϵ be any arbitrarily small positive constant. Let ϵ' be another constant satisfying that

$$\left(\frac{\pi}{\frac{\pi}{4}-\epsilon'}\right)^2 = 16 + \frac{\epsilon}{2}.$$

Then by Lemma 11, when r is sufficiently small, for any n, any $x \in \Omega(1)$ and any $y \in \Omega^x$

$$v(y\backslash x) \geq \left(\frac{\pi}{4} - \epsilon'\right) r\left\|x-y\right\|.$$

Thus, when r is sufficiently small, for any n

$$\int_{\Omega(1)}\phi(x)\,dx\int_{\Omega^x}\phi(y\backslash x)\,dy \leq \frac{\left(16+\frac{\epsilon}{2}\right)(k+1)(k+2)}{n(n\pi r^2)}\int_{\Omega(1)}\phi(x)\,dx.$$

Finally

$$\int_{\Omega(2)}\phi(x)\,dx\int_{\Omega^x}\phi(y\backslash x)\,dy \leq \int_{\Omega(2)}\phi(x)\,dx\int_{\Omega^x}dy \leq 4\pi r^2\int_{\Omega(2)}\phi(x)\,dx.$$

Hence, the lemma follows. □

Now we are ready to prove Lemma 5.

Proof (Lemma 5): Let $r = r_n$ be as given in Lemma 5. By Lemma 10,

$$n\left(n\pi r^2\right)\int_{\Omega_r(1)}\phi_{n,r}(x)\,dx \sim \frac{\sqrt{\pi}\eta}{2^{k+1}k!}e^{-\frac{\xi}{2}}$$

$$n\left(n\pi r^2\right)^m\int_{\Omega_r(2)}\phi_{n,r}(x)\,dx = o(1)$$

and if $k = 0$

$$n\left(n\pi r^2\right)\int_{\Omega_r(0)}\phi_{n,r}(x)\,dx \sim e^{-\xi}$$

otherwise

$$n\left(n\pi r^2\right)\int_{\Omega_r(0)}\phi_{n,r}(x)\,dx = o(1).$$

Thus, the equality (5) follows immediately. Together with these (asymptotic) equalities, Lemma 12 implies that if $k = 0$

$$n^2\left(n\pi r^2\right)^2\int_{\substack{x,y\in\Omega\\ \|x-y\|\le 2r}}\phi_{n,r}(x)\,\phi_{n,r}(y\backslash x)\,dxdy \lesssim 16e^{-\xi} + (32+\epsilon)\sqrt{\pi}\eta e^{-\frac{\xi}{2}}$$

otherwise

$$n^2\left(n\pi r^2\right)^2\int_{\substack{x,y\in\Omega\\ \|x-y\|\le 2r}}\phi_{n,r}(x)\,\phi_{n,r}(y\backslash x)\,dxdy \lesssim (32+\epsilon)(k+1)(k+2)\frac{\sqrt{\pi}\eta}{2^{k+1}k!}e^{-\frac{\xi}{2}}.$$

Let $\epsilon \to 0$, we obtain the inequality (6). □

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