Minimum CDS in Multihop Wireless Networks with Disparate Communication Ranges

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Abstract—Connected dominating set (CDS) has a wide range of applications in mutihop wireless networks. The Minimum CDS problem has been studied extensively in mutihop wireless networks with uniform communication ranges. However, in practice, the nodes may have different communication ranges either because of the heterogeneity of the nodes, or due to interference mitigation, or due to a chosen range assignment for energy conservation. In this paper, we present a greedy approximation algorithm for computing a Minimum CDS in multihop wireless networks with disparate communications ranges and prove that its approximation ratio is better than the best one known in the literature. Our analysis utilizes a tighter relation between the independence number and the connected domination number.

Index Terms—Connected dominating set, independent set, disk graph, wireless network, virtual backbone

1 Introduction

Connected dominating set (CDS) has a wide range of applications in multihop wireless networks (cf. a recent survey [2] and references therein). It plays a very important role in routing, broadcasting, and connectivity management in wireless ad hoc networks. Consider a multihop wireless network with undirected communication topology $G = (V, E)$. A CDS of G is a subset $U \subset V$ satisfying that each node in $V \setminus U$ is adjacent to at least one node in U and the subgraph of G induced by U is connected. A minimum CDS (MCDS) of G is a CDS of G with the smallest size. The problem of computing a MCDS in a multihop wireless networks with uniform communications ranges has been intensively studied in the literature. This problem is NP-hard [3], and a number of distributed algorithms for constructing a small CDS in wireless ad hoc networks have been proposed in [1], [5], [7], [8] among others.

However, in practice, the nodes may have different communication ranges either because of the heterogeneity of the nodes, or due to interference mitigation, or due to a chosen range assignment for energy conservation. In this paper, we assume all the nodes V lie in an euclidean plane, and each node v has a communication radius r_v. The communication topology of such a network is defined by a graph $G = (V, E)$ in which there is an edge between two nodes u and v if and only if they are within each other's communication range. By proper scaling, we assume that the smallest communication radius is one and the largest communication radius is R.

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MCDS in multihop wireless networks with disparate communication ranges have been studied in [6] and [9]. Thai et al. [6] applied the approximation algorithm given in [7] for MCDS in multihop wireless networks with uniform communication ranges to compute a CDS in a multihop wireless network with disparate communication ranges. The approximation bound of this algorithm involves the relation between the independence number α (the size of a maximum independent set) and connected domination number γ_c (the size of a minimum connected dominating set) of the communication topology. It was shown in [6] that

$$\alpha \le 10\lfloor \log_g R \rfloor \gamma_c,$$

where $g = \frac{1+\sqrt{5}}{2}$ is the golden ratio. With such a bound on α, an approximation bound

$$10\lfloor \log_g R \rfloor + 2 + \log(10\lfloor \log_g R \rfloor)$$

was derived in [6]. Xing et al. [9] targeted at obtaining a tighter approximation bound of the same approximation algorithm. They claimed (in [9, Theorem 3.1]) a tighter upper bound

$$\left(4\frac{5}{6} + 8\frac{2}{3}\lceil \log_g R \rceil\right)\gamma_c$$

on α. However, their proof of [9, Theorem 3.1] contains a critical error, which has no apparent fix. An explanation of this error and a counterexample are included in the supplementary material, which can be found on the Computer Society Digital Library at http://doi.ieeecomputersociety.org/10.1109/TMC.2012.58. Thus, the improved approximation bound based on the above bound of α in [9] becomes baseless.

In this paper, we first derive an improved upper bound on the number of independent nodes in the neighborhood of any node. For any $R \ge 1$, let

$$R^* = 5 + 8\lceil \log_g R \rceil.$$

We show that the number of independent nodes in the neighborhood of any node is at most R^*. Based on this upper bound, we then prove a tighter upper bound $(R^*-1)\gamma_c+1$ on α. Since the approximation bounds of the algorithms presented in [6] and [9] are directly derived from the upper bound of α, the approximation bounds of these two algorithms can be improved accordingly. We will adapt the two-phased greedy approximation algorithm presented in [8, Section 4] to multihop wireless networks with disparate communication ranges, and show that its approximation ratio is at most $R^* + \ln(R^*-2)+1$.

The remaining of this paper is organized as follows: In Section 2, we present an improved upper bound on the independence number α in terms of the connected domination number γ_c. In Section 3, we analyze the approximation bound of a two-phased greedy approximation algorithm for MCDS adapted from an algorithm originally proposed in [8] for computing MCDS with uniform communication radii. In Section 4, we summarize the paper and discuss future studies for potential improvements. Throughout this paper, $D(u,r)$ denotes the *closed* disk of radius r centered at u. The euclidean distance between two nodes u and v is denoted by $\|uv\|$. The cardinality of a finite set S is denoted by $|S|$.

2 Independence Number versus Connected Domination Number

In this section, we present an improved upper bound on the independence number α in terms of the connected domination number γ_c.

Theorem 1. $\alpha \le (R^*-1)\gamma_c+1$.

To prove Theorem 1, we need the following lemma which gives an upper bound on an independent set of nodes adjacent to an arbitrary node u.

Lemma 2. *Suppose that I is an independent set of nodes adjacent to a node u. Then, $|I| \le R^*$.*

Now, we prove Theorem 1 by using Lemma 2 which will be proved later. Let M be any maximum independent set of G, and OPT be any MCDS of G. Then, $|M| = \alpha$ and $|OPT| = \gamma_c$. Consider an arbitrary preorder traversal of $G[OPT]$ given by v_j with $1 \le j \le \gamma_c$. Let M_1 be the set of nodes in M that are adjacent to v_1. For any $2 \le j \le \gamma_c$, let M_j be the set of nodes in M that are adjacent to v_j but none of $v_1, v_2, \ldots, v_{j-1}$. Then, the γ_c sets M_j with $1 \le j \le \gamma_c$ form a partition of M. By Lemma 2, $|M_1| \le R^*$. For any $2 \le j \le \gamma_c$, there exists an index $1 \le j' \le j-1$ such that $v_{j'}$ is adjacent to v_j. Since $v_{j'}$ is not adjacent to any node in M_j, the set $\{v_{j'}\} \cup M_j$ is an independent set of nodes adjacent to v_j. Again by Lemma 2, we have

$$|M_j| + 1 \le R^*$$

and, consequently,

$$|M_j| \le R^* - 1.$$

Therefore,

$$\begin{aligned}|M| = \sum_{j=1}^{\gamma_c} |M_j| &\le R^* + (R^*-1)(\gamma_c - 1)\\ &= (R^*-1)\gamma_c + 1.\end{aligned}$$

This completes the proof of Theorem 1.

The rest of this section is devoted to the proof of Lemma 2. Consider an arbitrary node $u \in V$ and an independent set I of nodes adjacent to a node u. Let I_1 be the set of nodes in I lying in the closed disk of radius g centered at u, and for each $j \ge 2$ let

$$I_j = \{v \in I : g^{j-1} < \|uv\| \le g^j\}.$$

From [4], we have $|I_1| \le 12$. The following lemma on $|I_j|$ for $j \ge 2$ was proved in [9].

Lemma 3. *For any $j \ge 2$, $|I_j| \le 9$.*

We shall further prove the following lemma on $|I_j \cup I_{j+1}|$ for $j \ge 2$.

Lemma 4. *For any $j \ge 2$, $|I_j \cup I_{j+1}| \le 16$.*

These two lemmas together imply Lemma 2 immediately. If $\lceil \log_g R \rceil$ is odd, then

$$\begin{aligned}|I| &= \left| \bigcup_{j=1}^{\lceil \log_g R\rceil} I_j \right|\\ &= |I_1| + \sum_{i=1}^{(\lceil \log_g R\rceil - 1)/2} |I_{2i} \cup I_{2i+1}|\\ &\le 12 + 16 \cdot (\lceil \log_g R \rceil - 1)/2\\ &= 8\lceil \log_g R\rceil + 4 < R^*.\end{aligned}$$

If $\lceil \log_g R \rceil$ is even, then

$$\begin{aligned}|I| &= \left| \bigcup_{j=1}^{\lceil \log_g R\rceil} I_j \right|\\ &= |I_1| + |I_2| + \sum_{i=2}^{\lceil \log_g R\rceil/2 - 1} |I_{2i-1} \cup I_{2i}|\\ &\le 12 + 9 + 16(\lceil \log_g R \rceil/2 - 1)\\ &= 8\lceil \log_g R\rceil + 5 = R^*.\end{aligned}$$

So, Lemma 2 holds in both cases.

Next, we prove Lemma 4 by using a subtle angular argument. Fix a $j \ge 2$. We begin with the following two simple geometric lemmas.

Lemma 5. *Suppose that v and w are two distinct nodes in I_j satisfying that $\|uv\| \ge \|uw\|$. Then, $\angle wuv > 36^\circ$. In addition, for any $36^\circ \le \alpha < 60^\circ$,*

1. *If $\|uw\| \ge 2g^{j-1}\cos\alpha$, then $\angle wuv > \arccos \frac{g}{4\cos\alpha}$;*
2. *If $\|uv\| \le 2g^{j-1}\cos\alpha$, then $\angle wuv > \alpha$.*

Proof. Since v and w are two independent neighbors of u, we have

$$\|vw\| > \min\{r_v, r_w\} \ge \min\{\|uv\|, \|uw\|\} = \|uw\|.$$

Thus, v is outside the disk $D(w, \|uw\|)$. Since

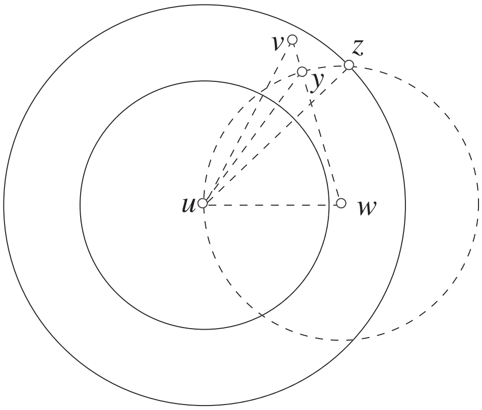

Fig. 1. v and w are two distinct nodes in I_j satisfying that $\|uv\| \geq \|uw\|$.

$$2\|uw\| > 2g^{j-1} > g^j,$$

the two circles $\partial D(u, g^j)$ and $\partial D(w, \|uw\|)$ intersect. Let z denote their intersection point which lies on the same side of line uw as v (see Fig. 1). Then,

$$\cos \angle wuz = \frac{\|uz\|}{2\|uw\|} < \frac{g^j}{2g^{j-1}} = \frac{g}{2} = \cos 36^\circ,$$

which implies $\angle wuz > 36^\circ$. Hence,

$$\angle wuv \geq \angle wuz > 36^\circ.$$

Clearly, $\angle wuv = 36^\circ$ if and only if $w \in \partial D(u, g^{j-1})$ and v is coincide with the point z.

1) Suppose that $\|uw\| \geq 2g^{j-1}\cos\alpha$. We have

$$\cos \angle wuz = \frac{\|uz\|}{2\|uw\|} \leq \frac{g^j}{4g^{j-1}\cos\alpha} = \frac{g}{4\cos\alpha},$$

which implies $\angle wuz \geq \arccos\frac{g}{4\cos\alpha}$. Since v is outside the disk $D(w, \|uw\|)$, we have

$$\angle wuv > \angle wuz \geq \arccos\frac{g}{4\cos\alpha}.$$

2) Suppose that $\|uv\| \leq 2g^{j-1}\cos\alpha$. Let y be the intersection point of the line segment vw and $\partial D(w, \|uw\|)$. Then,

$$\|uy\| < \|uv\| \leq 2g^{j-1}\cos\alpha.$$

So,

$$\cos \angle wuy = \frac{\|uy\|}{2\|uw\|} < \frac{2g^{j-1}\cos\alpha}{2g^{j-1}} = \cos\alpha,$$

which implies $\angle wuy > \alpha$. Thus, we have

$$\angle wuv > \angle wuy > \alpha.$$

This completes the proof for lemma. □

Lemma 6. *Suppose that $w \in I_j$ and $v \in I_{j+1}$:*

1. *If $\|uw\| \geq 2g^{j-1}\cos\alpha$ for some $36^\circ \leq \alpha \leq \arccos\frac{g^2}{4}$, then $\angle wuv > \arccos\frac{g^2}{4\cos\alpha}$;*
2. *If $\|uv\| \leq 2g^j\cos\alpha$ for some $\arccos\frac{1}{g} \leq \alpha < 60^\circ$, then $\angle wuv > \arccos(g\cos\alpha)$.*

Proof. Since v and w are two independent neighbors of u and $\|uv\| > \|uw\|$, we have

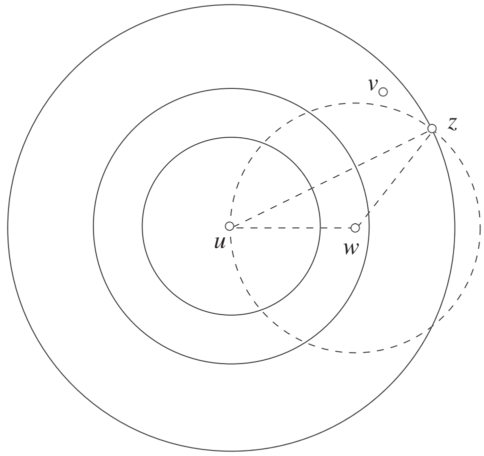

Fig. 2. Figure for Lemma 6(1).

$$\|vw\| > \min\{r_v, r_w\} \geq \min\{\|uv\|, \|uw\|\} = \|uw\|.$$

Thus, v is outside the disk $D(w, \|uw\|)$.

1) Since

$$\alpha \leq \arccos\frac{g^2}{4},$$

we have

$$2\|uw\| \geq 4g^{j-1}\cos\alpha \geq 4g^{j-1}\frac{g^2}{4} = g^{j+1}.$$

Thus, the two circles $\partial D(u, g^{j+1})$ and $\partial D(w, \|uw\|)$ intersect. Let z denote their intersection point which lies on the same side of line uw as v (see Fig. 2). Since

$$\|uw\| \geq 2g^{j-1}\cos\alpha,$$

we have

$$\cos \angle wuz = \frac{\|uz\|}{2\|uw\|} \leq \frac{g^{j+1}}{4g^{j-1}\cos\alpha} = \frac{g^2}{4\cos\alpha},$$

which implies that

$$\angle wuz \geq \arccos\frac{g^2}{4\cos\alpha}.$$

Thus,

$$\angle wuv > \angle wuz \geq \arccos\frac{g^2}{4\cos\alpha}.$$

2) Since

$$\alpha \geq \arccos\frac{1}{g},$$

we have

$$\|uv\| \leq 2g^j\cos\alpha \leq 2g^j\frac{1}{g} = 2g^{j-1} < 2\|uw\|.$$

Thus, the two circles $\partial D(u, \|uv\|)$ and $\partial D(w, \|uw\|)$ intersect. Let y denote their intersection point which lies on the same side of line uw as v (see Fig. 3). Since

$$\|uv\| \leq 2g^j\cos\alpha,$$

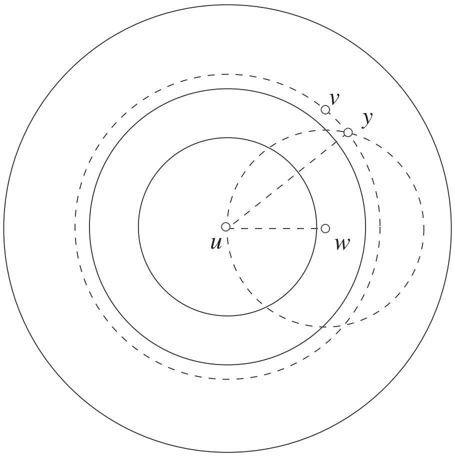

Fig. 3. Figure for Lemma 6(2).

we have

$$\cos \angle wuy = \frac{\|yu\|}{2\|wu\|} < \frac{2g^j \cos\alpha}{2g^{j-1}} = g\cos\alpha,$$

which implies that

$$\angle wuy > \arccos(g\cos\alpha).$$

Thus,

$$\angle wuv > \angle wuy > \arccos(g\cos\alpha).$$

This completes the proof for lemma. □

We remark that Lemma 3 follows from Lemma 5 immediately. We further apply Lemma 5 to derive some necessary conditions for $|I_j| = 9$ below.

Lemma 7. *Suppose that I_j consists of nine nodes $v_1, v_2, \ldots, v_9$ sorted in the increasing order of the distances from u. Then,*

1. $\|uv_1\| \leq 2g^{j-1}\cos 58.6^\circ$ *and* $\|uv_9\| \geq 2g^{j-1}\cos 39^\circ$;
2. $\|uv_2\| \leq 2g^{j-1}\cos 58.2^\circ$ *and* $\|uv_8\| \geq 2g^{j-1}\cos 39.8^\circ$;
3. $\|uv_3\| \leq 2g^{j-1}\cos 56.29^\circ$ *and* $\|uv_7\| \geq 2g^{j-1}\cos 43.2^\circ$.

Proof. We will use the following fact multiple times in this proof: Suppose that I' is a subset of five nodes in I_j. Then, among five consecutive sectors centered at u formed by the five nodes in I', at least one of them does not contain any other node in I_j. This is because $|I_j \setminus I'| = 4 < 5$ and hence at least one of those five sectors does not contain any node in $I_j \setminus I'$.

1) We prove the first part of lemma by contradiction. Assume to the contrary that either

$$\|uv_1\| > 2g^{j-1}\cos 58.6^\circ$$

or

$$\|uv_9\| < 2g^{j-1}\cos 39^\circ.$$

We first claim that the angle separation of any two nodes in I_j at u is greater than 39 degree. Indeed, if

$$\|uv_1\| > 2g^{j-1}\cos 58.6^\circ,$$

then

$$\|uv_i\| > 2g^{j-1}\cos 58.6^\circ$$

for all $1 \leq i \leq 9$, and hence the claim holds by Lemma 5(1). If

$$\|uv_9\| < 2g^{j-1}\cos 39^\circ,$$

then

$$\|uv_i\| < 2g^{j-1}\cos 39^\circ$$

for all $1 \leq i \leq 9$, and hence the claim holds by Lemma 5(2). So, our claim is true. We proceed in two cases.

Case 1: $\|uv_5\| \geq 2g^{j-1}\cos 50^\circ$. Let v_i and v_k be the two nodes in $\{v_5, v_6, \ldots, v_9\}$ such that the sector $\measuredangle v_i u v_k$ centered at u does not contain any other node in I_j. Then, by Lemma 5(1), $\angle v_i u v_k > 51^\circ$. So, the total of the nine consecutive angles at u formed by the nodes in I_j is greater than

$$51^\circ + 8 \cdot 39^\circ = 363^\circ > 360^\circ,$$

which is also a contradiction.

Case 2: $\|uv_5\| < 2g^{j-1}\cos 50^\circ$. Let v_i and v_k be the two nodes in $\{v_1, v_2, \ldots, v_5\}$ such that the sector $\measuredangle v_i u v_k$ centered at u does not contain any other node in I_j. Then, by Lemma 5(2), $\angle v_i u v_k > 50^\circ$. So, the total of the nine consecutive angles at u formed by the nodes in I_j is greater than

$$50^\circ + 8 \cdot 39^\circ = 362^\circ > 360^\circ,$$

which is a contradiction.

In either case, we have reached a contradiction. Therefore, the first part of the lemma holds.

2) We prove the second part of the lemma by contradiction. Assume to the contrary that either

$$\|uv_2\| > 2g^{j-1}\cos 58.2^\circ$$

or

$$\|uv_8\| < 2g^{j-1}\cos 39.8^\circ.$$

We first claim that there exists a node $v_a \in I_j$ such that the angle separation of any two nodes in $I_j \setminus \{v_a\}$ at u is greater than 39.8 degree. Indeed, if

$$\|uv_2\| > 2g^{j-1}\cos 58.2^\circ,$$

then

$$\|uv_i\| > 2g^{j-1}\cos 58.2^\circ$$

for all $2 \leq i \leq 9$, and hence the claim holds for $a = 1$ by Lemma 5(1). If

$$\|uv_8\| < 2g^{j-1}\cos 39.8^\circ,$$

then

$$\|uv_i\| < 2g^{j-1}\cos 39.8^\circ$$

for all $1 \leq i \leq 8$, and hence the claim holds for $a = 9$ by Lemma 5(2). So, our claim is true. We remark that the angle separation between v_a and any other node is still greater than 36 degree. We proceed in two cases.

Case 1: $\|uv_5\| \geq 2g^{j-1}\cos 50°$. Let v_i and v_k be the two nodes in $\{v_5, v_6, \ldots, v_9\}$ such that the sector $\measuredangle v_i u v_k$ centered at u does not contain any other node in I_j. Then, by Lemma 5(1), $\angle v_i u v_k > 51°$. Let k be the number of consecutive angles at u formed by the nodes in I_j other than $\angle v_i u v_k$ with v_a on the boundary. Then, $k \leq 2$. So, the total of the nine consecutive angles at u formed by the nodes in I_j is greater than

$$\begin{aligned}51° + (8-k)\cdot 39.8° + k\cdot 36° &= 51° + 8\cdot 39.8° - k\cdot 3.8° \\ &\geq 51° + 8\cdot 39.8° - 2\cdot 3.8° \\ &= 361.8° > 360°,\end{aligned}$$

which is also a contradiction.

Case 2: $\|uv_5\| < 2g^{j-1}\cos 50°$. Let v_i and v_k be the two nodes in $\{v_1, v_2, \ldots, v_5\}$ such that the sector $\measuredangle v_i u v_k$ centered at u does not contain any other node in I_j. Then, by Lemma 5(2), $\angle v_i u v_k > 50°$. Let k be the number of consecutive angles at u formed by the nodes in I_j other than $\angle v_i u v_k$ with v_a on the boundary. Then, $k \leq 2$. So, the total of the nine consecutive angles at u formed by the nodes in I_j is greater than

$$\begin{aligned}50° + (8-k)\cdot 39.8° + k\cdot 36° &= 50° + 8\cdot 39.8° - k\cdot 3.8° \\ &\geq 50° + 8\cdot 39.8° - 2\cdot 3.8° \\ &= 360.8° > 360°,\end{aligned}$$

which is a contradiction.

In either case, we have reached a contradiction. Therefore, the first part of the lemma holds.

3) We prove the third part of the lemma by contradiction. Assume to the contrary that either

$$\|uv_3\| > 2g^{j-1}\cos 56.29°$$

or

$$\|uv_7\| < 2g^{j-1}\cos 43.2°.$$

We claim that there exist two nodes $v_a, v_b \in I_j$ such that $\angle v_a u v_b > 58.2°$ and the angle separation at u of any two nodes in $I' = I_j \setminus \{v_a, v_b\}$ is greater than 43.2 degree. Indeed, if

$$\|uv_3\| > 2g^{j-1}\cos 56.29°,$$

then

$$\|uv_i\| > 2g^{j-1}\cos 56.29°$$

for all $3 \leq i \leq 9$ and hence the angle separation at u of any two nodes in $I_j \setminus \{v_1, v_2\}$ is greater than 43.2 degree by Lemma 5(1). By the second part of this lemma, we have

$$\|uv_2\| \leq 2g^{j-1}\cos 58.2°,$$

which implies $\angle v_1 u v_2 > 58.2°$ by Lemma 5(2). Thus, the claim holds with $a = 1$ and $b = 2$. Similarly, if

$$\|uv_7\| > 2g^{j-1}\cos 43.2°,$$

then

$$\|uv_i\| > 2g^{j-1}\cos 43.2°$$

for all $1 \leq i \leq 7$ and hence the angle separation at u of any two nodes in $I_j \setminus \{v_8, v_9\}$ is greater than 43.2 degree by Lemma 5(2). By the second part of this lemma, we have

$$\|uv_8\| \geq 2g^{j-1}\cos 39.8°,$$

which implies that $\angle v_8 u v_9 > 58.2°$ by Lemma 5(1). Thus, the claim holds with $a = 8$ and $b = 9$. So, our claim is true. We proceed in two cases.

Case 1. The sector $\measuredangle v_a u v_b$ centered at u does not contain any node in I'. Then, among the nine consecutive angles at u formed by the nodes in I_j, $\angle v_a u v_b$ is greater than 58.2 degree, the two other angles with v_a and v_b on the boundary, respectively, are each greater than 36 degree, and the rest six angles are all greater than 43.2 degree. So, the total of these nine angles is greater than

$$58.2° + 2\cdot 36° + 6\cdot 43.2° = 389.4° > 360°,$$

which is a contradiction.

Case 2. The sector $\measuredangle v_a u v_b$ centered at u contains at least one node in I'. Then, among the nine consecutive angles at u formed by the nodes in I_j, the four angles with v_a and v_b on the boundary, respectively, are each greater than 36 degree, and the rest five angles are all greater than 43.2 degree. So, the total of these nine angles is greater than

$$4\cdot 36° + 5\cdot 43.2° = 360°,$$

which is also a contradiction.

In either case, we have reached a contradiction. Therefore, the first part of the lemma holds. □

Now, are ready to prove Lemma 4. Assume to the contrary that $|I_j \cup I_{j+1}| = l \geq 17$. Let

$$I_j \cup I_{j+1} = \{v_i : 1 \leq i \leq l\},$$

where $v_1, v_2, \ldots, v_l$ are sorted in the increasing order of the distances from the node u. By Lemma 3, we have

$$\max\{|I_j|, |I_{j+1}|\} \leq 9.$$

Since $l \geq 17$, we must have

$$\begin{aligned}\max\{|I_j|, |I_{j+1}|\} &= 9, \\ \min\{|I_j|, |I_{j+1}|\} &\geq 8.\end{aligned}$$

We consider two cases:

Case 1: $|I_j| = 9$. Then, $|I_{j+1}| \geq 8$. By Lemma 7, we have

$$\|uv_7\| \geq 2g^{j-1}\cos 43.2°.$$

Let $J = \{v_7, v_8, v_9\}$. By Lemma 5(1), the angle separation between any two nodes in J at u is greater than 56.29 degree. We further consider two subcases:

Subcase 1.1. There exist two nodes $v_a, v_b \in J$ such that the sector $\measuredangle v_a u v_b$ centered at u does not contain any node in I_{j+1} (see Fig. 4). Let v_i and v_k be the two nodes in I_{j+1} such that the sector $\measuredangle v_i u v_k$ contains v_a and v_b but does not contain any other node in I_{j+1}, and v_i, v_a, v_b, and v_k are in the clockwise direction with respect to u. By Lemma 6(1),

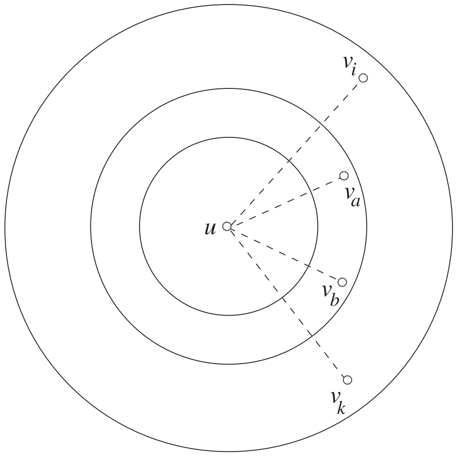

Fig. 4. Figure for Subcase 1.1: a sector $\measuredangle v_a u v_b$ for $a, b \in \{7, 8, 9\}$ does not contain any node in I_{j+1}.

$$\min\{\angle v_k u v_b, \angle v_a u v_i\} > 26^\circ.$$

Thus,

$$\begin{aligned} &\angle v_k u v_b + \angle v_b u v_a + \angle v_a u v_i \\ &\quad > 2 \cdot 26^\circ + 56.29^\circ \\ &\quad = 108.29^\circ. \end{aligned}$$

Hence, the total of the $|I_{j+1}|$ consecutive angles at u formed by the nodes in I_{j+1} is greater than

$$\begin{aligned} &108.29^\circ + (|I_{j+1}| - 1) \cdot 36^\circ \\ &\quad \geq 108.29^\circ + 7 \cdot 36^\circ = 360.\ 29^\circ \\ &\quad > 360^\circ, \end{aligned}$$

which is a contradiction.

Subcase 1.2. For any two nodes $v_a, v_b \in J$, the sector $\measuredangle v_a u v_b$ centered at u contains at least one node in I_{j+1} (see Fig. 5). For each $a = 7, 8$, and 9, let $v'_a, v''_a \in I_{j+1}$ satisfying that v_a is the only node contained in the sector $\measuredangle v'_a u v''_a$ centered at u among all the nodes in $I_{j+1} \cup J$. Then, by Lemmas 6(1) and 7, we have

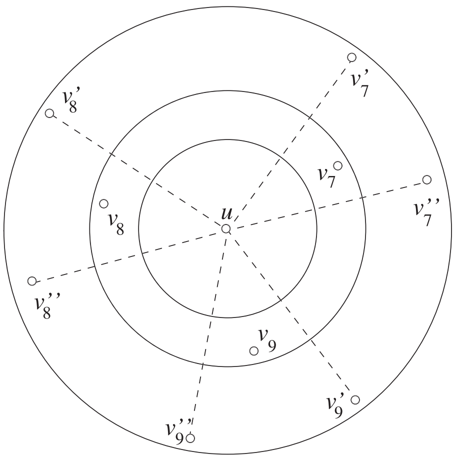

Fig. 5. Figure for Subcase 1.2: every sector $\measuredangle v_a u v_b$ for each $a = 7$, 8, and 9 contains a node in I_{j+1}.

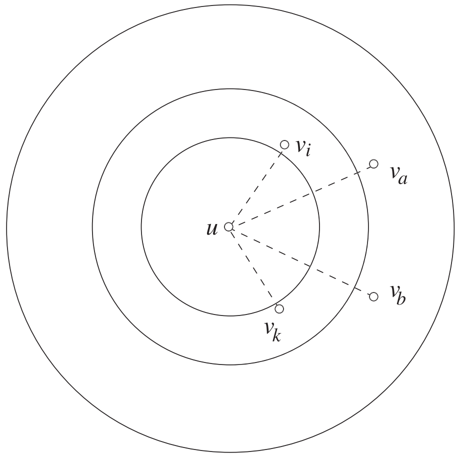

Fig. 6. Figure for Subcase 2.1: a sector $\measuredangle v_a u v_b$ for $a, b \in \{9, 10, 11\}$ does not contain any node in I_j.

$$\begin{aligned} &\min\{\angle v'_7 u v_7, \angle v_7 u v''_7\} > 26^\circ, \\ &\min\{\angle v'_8 u v_8, \angle v_8 u v''_8\} > 31.5^\circ, \\ &\min\{\angle v'_9 u v_9, \angle v_9 u v''_9\} > 32.5^\circ. \end{aligned}$$

Thus,

$$\begin{aligned} &\angle v'_7 u v''_7 + \angle v'_8 u v''_8 + \angle v'_9 u v''_9 \\ &\quad > 2 \cdot (26^\circ + 31.5^\circ + 32.5^\circ) \\ &\quad = 180^\circ. \end{aligned}$$

Hence, the total of the $|I_{j+1}|$ consecutive angles at u formed by the nodes in I_{j+1} is greater than

$$\begin{aligned} &180^\circ + (|I_{j+1}| - 3) \cdot 36^\circ \\ &\quad \geq 180^\circ + 5 \cdot 36^\circ = 360^\circ, \end{aligned}$$

which is a contradiction.

Case 2: $|I_j| = 8$. Then, $|I_{j+1}| = 9$. By Lemma 7, we have

$$\|uv_{11}\| \leq 2g^j \cos 56.29^\circ.$$

Let $J = \{v_9, v_{10}, v_{11}\}$. By Lemma 5(2), the angle separation between any two nodes in J at u is greater than 56.29 degrees. We further consider two subcases:

Subcase 2.1. There exist two nodes $v_a, v_b \in J$ such that the sector $\measuredangle v_a u v_b$ centered at u does not contain any node in I_j (see Fig. 6). Let v_i and v_k be the two nodes in I_j such that the sector $\measuredangle v_i u v_k$ contains v_a and v_b but does not contain any other node in I_j, and v_i, v_a, v_b, and v_k are in the clockwise direction with respect to u. By Lemma 6(2),

$$\min\{\angle v_k u v_b, \angle v_a u v_i\} > 26^\circ.$$

Thus,

$$\begin{aligned} &\angle v_k u v_b + \angle v_b u v_a + \angle v_a u v_i \\ &\quad > 2 \cdot 26^\circ + 56.29^\circ \\ &\quad = 108.29^\circ. \end{aligned}$$

Hence, the total of the eight consecutive angles at u formed by the nodes in I_j is greater than

$$108.29^\circ + 7 \cdot 36^\circ = 360.\ 29^\circ > 360^\circ,$$

which is a contradiction.

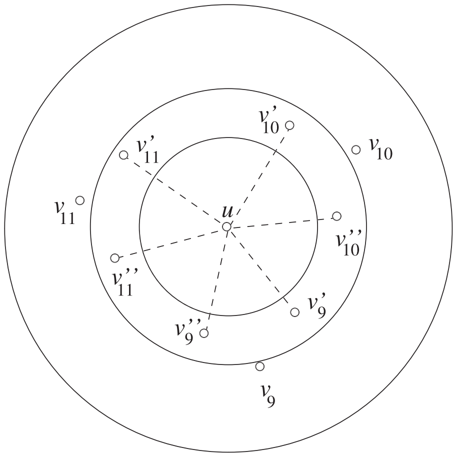

Fig. 7. Figure for Subcase 2.2: every sector $\measuredangle v_a u v_b$ for each $a = 9$, 10, and 11 contains a node in I_j.

Subcase 2.2. For any two nodes $v_a, v_b \in J$, the sector $\measuredangle v_a u v_b$ centered at u contains at least one node in I_j (see Fig. 7). For each $a = 9$, 10, and 11, let $v'_a, v''_a \in I_j$ satisfying that v_a is the only node contained in the sector $\measuredangle v'_a u v''_a$ centered at u among all the nodes in $I_j \cup J$. Then, by Lemmas 6(2) and 7, we have

$$\begin{aligned}\min\{\angle v'_9 u v_9, \angle v_9 u v''_9\} &> 32.5^\circ,\\ \min\{\angle v'_{10} u v_{10}, \angle v_{10} u v''_{10}\} &> 31.5^\circ,\\ \min\{\angle v'_{11} u v_{11}, \angle v_{11} u v''_{11}\} &> 26^\circ.\end{aligned}$$

Thus,

$$\begin{aligned}&\angle v'_9 u v''_9 + \angle v'_{10} u v''_{10} + \angle v'_{11} u v''_{11}\\ &\quad > 2 \cdot (32.5^\circ + +31.5^\circ + 26^\circ)\\ &\quad = 180^\circ.\end{aligned}$$

Hence, the total of the eight consecutive angles at u formed by the nodes in I_j is greater than

$$180^\circ + 5 \cdot 36^\circ = 360^\circ,$$

which is a contradiction.

Thus, in every case, we have reached a contradiction. So, we must have $|I_j \cup I_{j+1}| \leq 16$. This completes the proof of Lemma 4.

3 Greedy Approximation Algorithm for MCDS

In this section, we present a greedy algorithm adapted from the two-phased greedy approximation algorithm originally proposed in [8] for computing a CDS in a multihop wireless network with uniform communication ranges to multihop wireless networks with disparate communication ranges.

The greedy algorithm consists of two phases. The first phase selects a maximal independent set (MIS) I of G. Specifically, we construct an arbitrary rooted spanning tree T of G, and select an MIS I of G in the first-fit manner in the breadth-first-search ordering in T. The second phase selects a set C of connectors to interconnect I. For any subset $U \subseteq V \setminus I$, $f(U)$ denotes the number of connected components in $G[I \cup U]$. For any $U \subseteq V \setminus I$ and any $w \in V \setminus I$, the *gain* of w with respect to U is defined to be $f(U) - f(U \cup \{w\})$. The second phase greedily selects C iteratively as follows: Initially C is empty. While $f(C) > 1$, choose a node $w \in V \setminus (I \cup C)$ with *maximum* gain with respect to C and add w to C. When $f(C) = 1$, then $I \cup C$ is a CDS. Let C be the output of the second phase. Then, $I \cup C$ is the output CDS.

The correctness of the second phase follows from the following bound on the gain established in [8].

Lemma 8. *Suppose that there are $f(U) > 1$ for some $U \subseteq V \setminus I$. Then, there exists a $w \in V \setminus (I \cup U)$ whose gain with respect to U is at least*

$$\max\{1, \lceil f(U)/\gamma_c \rceil - 1\}.$$

We apply the above lemma to derive the following upper bound on $|C|$.

Lemma 9. $|C| \leq (\ln(R^* - 2) + 2)\gamma_c$.

Proof. For each $1 \leq i \leq |C|$, we denote by C_i the sequence of the first i nodes in C. We also set $C_0 = \emptyset$. Let k be the first (smallest) nonnegative integer such that

$$f(C_k) < 2\gamma_c + 2.$$

We claim that

$$|C \setminus C_k| \leq 2\gamma_c - 1.$$

By Lemma 8, each node in $C \setminus C_k$ has gain at least one. If $f(C_k) \leq 2\gamma_c$, then

$$|C \setminus C_k| \leq f(C_k) - 1 \leq 2\gamma_c - 1.$$

If $f(C_k) = 2\gamma_c + 1$, then the first node in $C \setminus C_k$ has gain at least two with respect to C_k by Lemma 8, and hence

$$2 + (|C \setminus C_k| - 1) \leq f(C_k) - 1 = 2\gamma_c,$$

which also implies that

$$|C \setminus C_k| \leq 2\gamma_c - 1.$$

Thus, the claim holds.

The previous claim implies that

$$\begin{aligned}|C| &= k + |C \setminus C_k|\\ &\leq k + 2\gamma_c - 1\\ &= (k-1) + 2\gamma_c.\end{aligned}$$

Thus, it is sufficient to show that

$$k - 1 \leq \gamma_c \ln(\Delta - 2).$$

This inequality holds trivially if $k \leq 1$. So, we assume that $k > 1$. For each $0 \leq i \leq k$, let

$$\ell_i = f(C_i) - \gamma_c.$$

Then,

$$|I| - \gamma_c = \ell_0 > \ell_1 > \cdots > \ell_k \geq \gamma_c + 2.$$

By Lemma 8, for each $0 \leq i \leq k$,

$$\begin{aligned}\ell_{i-1} - \ell_i &= f(C_{i-1}) - f(C_i) \\ &\geq \frac{f(C_{i-1})}{\gamma_c} - 1 \\ &= \frac{\ell_{i-1}}{\gamma_c},\end{aligned}$$

and hence

$$\frac{\ell_{i-1} - \ell_i}{\ell_{i-1}} \geq \frac{1}{\gamma_c}.$$

Therefore,

$$\begin{aligned}\frac{k}{\gamma_c} &\leq \sum_{i=1}^{k} \frac{\ell_{i-1} - \ell_i}{\ell_{i-1}} \\ &\leq \ln \frac{\ell_0}{\ell_k} \\ &\leq \ln \frac{|I| - \gamma_c}{\gamma_c + 2}.\end{aligned}$$

By Theorem 1,

$$\begin{aligned}\frac{|I| - \gamma_c}{\gamma_c + 2} &\leq \frac{(R^* - 1)\gamma_c + 1 - \gamma_c}{\gamma_c + 2} \\ &= \frac{(R^* - 2)\gamma_c + 1}{\gamma_c + 2} \\ &\leq R^* - 2.\end{aligned}$$

Thus,

$$\frac{k}{\gamma_c} \leq \ln(R^* - 2),$$

which implies

$$k \leq \gamma_c \ln(R^* - 2).$$

This completes the proof of the lemma. □

From Theorem 1 and Lemma 9, we obtain the following bound on the size of the CDS output by the greedy algorithm.

Theorem 10. $|I \cup C| \leq (R^* + \ln(R^* - 2) + 1)\gamma_c + 1$.

4 Discussion

The relation between the independence number α and the connected domination number γ_c plays a key role in deriving the approximation bounds of various two-phased greedy approximation algorithms adapted for MCDS of multihop wireless networks with disparate communication ranges [6], [8], [9]. In this paper, we first proved that $\alpha \leq (R^* - 1)\gamma_c + 1$, where $R^* = 5 + 8\lceil \log_g R \rceil$ for any $R \geq 1$. From this relation, we then derived an approximation bound $R^* + \ln(R^* - 2) + 1$ of the two-phased greedy approximation algorithm adapted from [8]. This approximation bound is better than the known ones obtained in [6] and [9].

Tighter relation between α and γ_c may be derived with more sophisticated analyses. A possible approach of obtaining tighter relation between α and γ_c is to develop a tighter bound on the number of independent nodes that can be packed in the neighborhood of a pair of adjacent nodes. An attempt along this approach has been made in [9], but the argument in [9] contains a critical error. However, we do believe that this approach is very promising to achieve tighter relation between α and γ_c.

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