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Fault tolerant deployment and topology control in wireless *ad hoc* networks

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Summary

We consider a large-scale of wireless *ad hoc* networks whose nodes are distributed randomly in a two-dimensional region Ω (more specifically, a unit square). Given n wireless nodes V, each with transmission range r_n, the wireless networks are often modeled by graph $G(V, r_n)$ in which two nodes are connected if and only if their Euclidean distance is no more than r_n. We first consider how to relate the transmission range with the number of nodes in a fixed area such that the resulted network can sustain k fault nodes in its neighborhood with high probability when all nodes have the same transmission range. We show that, for a unit-area square region Ω, the probability that the network $G(V, r_n)$ is k-connected is at least $e^{-e^{-\alpha}}$ when the transmission radius r_n satisfies $n\pi r_n^2 \geq \ln n + (2k-3)\ln\ln n - 2\ln(k-1)! + 2\alpha$ for $k > 1$ and n sufficiently large. This result also applies to mobile networks when the moving of wireless nodes always generates randomly distributed positions. We also conduct extensive simulations to study the practical transmission range to achieve certain probability the network being k-connectivity, when the number of nodes n is not large enough. The relation between the minimum node degree and the connectivity of graph $G(V, r)$ is also studied. Setting the transmission range of all nodes to r_n guarantees the k-connectivity with high probability, but some nodes may have excessive number of neighbors in the graph $G(V, r_n)$. We then present a localized method to construct a subgraph of the network topology $G(V, r_n)$ such that the resulting subgraph is still k-connected but with much fewer communication links maintained. We show that the constructed topology has only $O(k \cdot n)$ links and is a length spanner. Here a graph $H \subseteq G$ is spanner for graph G, if for any two nodes, the length of the shortest path connecting them in H is no more than a small constant factor of the length of the shortest path connecting them in G. Finally, we conduct some simulations to study the practical transmission range to achieve certain probability of k-connected when n is not large enough. Copyright © 2004 John Wiley & Sons, Ltd.

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1. Introduction

There are no wired infrastructures or cellular networks in ad hoc wireless network. Each mobile node‡ has an adjustable transmission range. Node v can receive the signal from node u if node v is within the transmission range of the sender u. Otherwise, two nodes communicate through multi-hop ad hoc wireless links by

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‡In this paper the term *node* often represents a network device, *vertex* is a graph term and *point* is a geometry term. We often interchange them if no confusion is caused.

using intermediate nodes to relay the message. Consequently, each node in the wireless network also acts as a router, forwarding data packets for other nodes. We consider that each wireless node has an omnidirectional antenna. This is attractive because a single signal transmission of a node can be received by all nodes within its vicinity which, we assume, is a disk centered at the node.

Wireless *ad hoc* networks are also called packet radio networks in the early 1970s. While many fundamental ideas existed about 20–30 years ago, in recent years we see tremendous research activity in wireless *ad hoc* networks due to its applications in various situations such as battlefield, emergency relief and so on. Mobile wireless networking enjoys a great advantage over the wired networking counterpart because it can be formed in a spontaneous way for various applications.

Hundreds of protocols [1–13] that take the unique characteristics of wireless *ad hoc* networks have been developed. Among them, energy efficiency, routing and MAC layer protocols have attracted most of the attention. One of the remaining fundamental and critical issues is to have faulttolerant network deployment without sacrificing the spectrum reusing property. In other words, the network should support multiple disjoint paths connecting every pair of nodes. Obviously, we can increase the transmission range of all nodes to increase the fault-tolerance of the network. However, increasing the transmission range will cause more signal interference (thus reduce the throughput) and increase the power consumption of every node. As power is a scarce resource in wireless networks, it is important to save the power consumption without losing the network connectivity. The universal minimum power used by all wireless nodes such that the induced network topology is connected is called the *critical power.*

Determining the critical power was studied by several researchers [14–16] recently when the wireless nodes are statically distributed. Both References [14] and [15] use the power assignment induced by the longest incident edge of the Euclidean minimum spanning tree over wireless nodes V. It was proved by Penrose [17] that, given a set of points uniformly and randomly distributed in a unit-area square, the longest edge of the minimum spanning tree asymptotically equals to the longest edge of the nearest neighbor graph. Since the nearest neighbor can be found locally, we can determine the critical power asymptotically using a localized method instead of constructing the minimum spanning tree if the wireless devices are randomly and uniformly distributed in a unit-area square.

Although determining the critical power for static wireless *ad hoc* networks is well studied, it remains to study the critical power for connectivity for mobile wireless networks. As the wireless nodes move around, it is impossible to have a unanimous critical power to guarantee the connectivity for all instances of the network configuration. Thus, we need to find a critical power, if possible, at which each node has to transmit to guarantee the connectivity of the network almost surely, i.e. with high probability sufficiently close to 1. For simplicity, we assume that the wireless devices are distributed in a unit square (or disk) according to some distribution function, e.g. uniform distribution or Poisson process. Additionally, we assume that the movement of wireless devices still keeps them in the same distribution (uniform or Poisson process). Gupta and Kumar [16] showed that there is a critical power almost surely when the wireless nodes are randomly and uniformly distributed in a unit-area disk. The result by Penrose [17] implies the same conclusion. Moreover, Penrose [17] gave the probability of the network to be connected if the transmission radius is set as a positive real number r and the number of nodes n goes to infinity.

Let $G(V, r)$ be the graph defined on V with edges $uv \in E$ if and only if $\|uv\| \leq r$. Here $\|uv\|$ is the Euclidean distance between nodes u and v. Let $\mathcal{G}_\Omega(\chi_n, r_n)$ be the set of graphs $G(V, r_n)$ for n nodes V that are uniformly and independently distributed in a two-dimensional region Ω, which could be a unit-area disk $\mathcal{D}$ or a unit square $\mathcal{C}$ with center at the origin. The problem considered by Gupta and Kumar [16] is then to determine the value of r_n such that a random graph in $\mathcal{G}_\mathcal{D}(\chi_n, r_n)$ is asymptotically connected with probability 1 as n goes to infinity. Let $\mathrm{P}_{\Omega,k}(\chi_n, r_n)$ be the probability that a graph in $\mathcal{G}_\Omega(\chi_n, r_n)$ is k-connected. Then Gupta and Kumar [16] showed that if $n\pi \cdot r_n^2 = \ln n + c(n)$, then $\mathrm{P}_{\Omega,1}(n, r_n) \to 1$ iff $c(n) \to +\infty$ as n goes to infinity. The result by Penrose [17] implies a stronger result: if $n\pi \cdot r_n^2 = \ln n + \alpha$, then $\mathrm{P}_1(n, r_n) = \mathrm{e}^{-\mathrm{e}^{-\alpha}}$ as n goes to infinity.

Fault tolerance is one of the central challenges in designing the wireless *ad hoc* networks. To make fault tolerance possible, first of all, the underlying network topology must have multiple disjoint paths to connect any two given wireless devices. Here the path could be vertex disjoint or edge disjoint. We use the vertex disjoint multiple paths in this paper considering the communication nature of the wireless networks. In this paper, we are interested in what is the condition of

r_n such that the underlying network topology $G(V, r_n)$ is k-connected almost surely when V is uniformly and randomly distributed over a two-dimensional domain Ω. For simplicity, we assume that the geometry domain Ω is a unit square $\mathcal{C}$. Gupta and Kumar [16] basically studied the connectivity problem for $k=1$ and Ω being a unit-area disk.

We show that given n points randomly distributed in a unit square $\mathcal{C}$, if the transmission range r_n satisfies $n\pi \cdot r_n^2 \geq \ln n + (2k-1)\ln\ln n - 2\ln k! + \alpha + 2\ln 8k/2^k\sqrt{\pi}$, then $G(V, r_n)$ is $(k+1)$-connected with probability at least $\mathrm{e}^{-\mathrm{e}^{-\alpha}}$ as n goes to infinity. Notice that, this result is analogous to the corresponding result for Bernoulli graphs $\mathcal{G}(n, p)$ (See Reference) [18]. A similar result was presented by Penrose [17,19] for the toroidal model instead of the Euclidean model. He showed that the hitting radius r_n such that the graph $G(V, r_n)$ is $(k+1)$-connected satisfies

$$\lim_{n\to\infty} \Pr(n\pi R_n^2 \leq \ln\, n + k\ln\ln n - \ln k! + \alpha) = \mathrm{e}^{-\mathrm{e}^{-\alpha}}$$

The toroidal metric is used to eliminate boundary effects.

Our theoretical value gives us insight on how to set the transmission radius to achieve the k-connectivity with certain probability for a network of n devices; or how many devices are needed to achieve the k-connectivity with certain probability when the transmission range of each device is a fixed value. This result also applies to mobile networks, when the moving of wireless nodes always generate randomly (or Poisson process) distributed node positions. Our result has applications in system design of large-scale wireless networks. For example, for setting up a sensor network monitoring a certain region, we should deploy how many sensors to have a multiple connected network, knowing each sensor can transmit to the farthest range r_0. Notice that our result holds only when the number of wireless devices n goes to infinity, which is difficult to deploy practically. We then conduct extensive simulations to study the transmission radius achieving k-connectivity with certain probability for practical settings. The relation between the minimum node degree and the connectivity of graph $G(V, r)$ is also studied here.

The remaining of the paper is organized as follows. In Section 2, we review some previous results studying the transition phenomena for wireless networks. Section 3 studies the critical transmission range for k-connectivity of the wireless *ad hoc* networks when the wireless nodes are randomly and uniformly distributed in a unit-area square $\mathcal{C}$. In Section 4, we present a localized method to control the network topology. The resulting topology cannot only sustain k node faults, but also approximates the original unit-disk graph well in terms of the energy consumption. Our experimental results presented in Section 5 will verify our theoretical results. We conclude our paper and discuss possible future research directions in Section 6.

2. Literature Review

Given an event Y, let $\Pr(Y)$ be the probability of Y. We denote the expected value of a random variable X by $\mathrm{E}[X]$, i.e. $\mathrm{E}[X] = \sum_x x \cdot \Pr(X = x)$ for a discrete variable. As standard, we write log for base-2 logarithm and ln for natural logarithm. We say a function $f(n) \to a$ if $\lim_{n\to\infty} f(n) = a$.

2.1. Point Process

A point set process is said to be a *uniform random point process*, denoted by χ_n, in a region Ω if it consists of n independent points each of which is uniformly and randomly distributed over Ω.

The standard probabilistic model of *homogeneous Poisson process* is characterized by the property that the number of nodes in a region is a random variable depending only on the area (or volume in higher dimensions) of the region. In other words,

- The probability that there are exactly k nodes appearing in any region Ψ of area A is $(\lambda A)^k/k! \cdot \mathrm{e}^{-\lambda A}$.
- For any region Ψ, the conditional distribution of nodes in Ψ given that exactly k nodes in the region is *joint uniform*.

Hereafter, we let $\mathcal{P}_n$ be a homogeneous Poisson process of intensity n on the unit square $\mathcal{C} = [-0.5, 0.5] \times [-0.5, 0.5]$.

2.2. Connectivity and Minimum Degree

A graph is called k-vertex connected (k-connected for simplicity) if, for each pair of vertices, there are k mutually vertex disjoint paths (except end-vertices) connecting them. Equivalently, a graph is k-connected if there is no a set of $k-1$ nodes whose removal will partition the network into at least two components. Thus, a k-connected wireless network can sustain the failure of $k-1$ nodes. A graph is called k-edge connected if, for each pair of vertices, there are k mutually edge disjoint paths connecting them. The *vertex connectivity*, denoted by $\kappa(G)$, of a graph G is the maximum k such that G is k vertex connected. The *edge connectivity*, denoted by $\xi(G)$, of a graph G is the maximum k such that G is k edge connected. The minimum degree of a graph G is denoted by $\delta(G)$ and the maximum degree of a graph G is denoted by $\Delta(G)$. Clearly, for any graph G, $\kappa(G) \leq \xi(G) \leq \delta(G) \leq \Delta(G)$. We will delete the symbol G in the above notations if it is clear from the context.

A graph property is called *monotone increasing* if G has such property then all graphs on the same vertex set containing G as a subgraph have this property. Let $\mathcal{Q}$ be any monotone increasing property of graphs, for example, the connectivity, the k-edge connectivity, the k-vertex connectivity, the minimum node degree at least k and so on. The *hitting radius* $\varrho(V, \mathcal{Q})$ is the infimum of all r such that graph $G(V,\ r)$ has property $\mathcal{Q}$. For example, $\varrho(V, \kappa \geq k)$ is the minimum radius r such that $G(V,\ r)$ is at least k vertex connected; $\varrho(V, \delta \geq k)$ is the minimum radius r at which the graph $G(V,\ r)$ has the minimum degree at least k. It is obvious that, for any V,

$$\varrho(V, \kappa \geq k) \geq \varrho(V, \delta \geq k)$$

Penrose [19] showed that these two hitting radii are asymptotically same for n points V randomly and uniformly distributed in a unit square and n goes to infinity.

2.3. Literature Review

The connectivity of random graphs, especially the geometric graphs and its variations, has been considered in the random graph theory literature [18], in the stochastic geometry literature [17,19–22] and in the wireless *ad hoc* network literature [16,23–29].

Let us first consider the connectivity problem. Given n nodes V randomly and independently distributed in a unit-area disk $\mathcal{D}$, Gupta and Kumar [16] showed that $G(V,\ r_n)$ is connected almost surely if $n\pi \cdot r_n^2 \geq \ln n + c(n)$ for any $c(n)$ with $c(n) \to \infty$ as n goes to infinity. Notice this bound is tight as they also proved that $\mathcal{G}_\Omega(\chi_n,\ r_n)$ is asymptotically disconnected with positive probability if $n\pi \cdot r_n^2 = \ln n + c(n)$ and $\lim\ \sup_n c(n) < +\infty$. In other words, the connectedness of the network has a transition phenomena when the transition range increases. The wireless network composed of randomly distributed mobile hosts will become connected almost abruptly.

Notice that they actually derived their results for a homogeneous Poisson process of points in $\mathcal{D}$ instead of the independent and uniform point process. They showed that the difference between them is negligible. Additionally, a similar result by Penrose [17,22] showed that the same result holds if the geometry domain in which the wireless nodes are distributed is a unit-area square $\mathcal{C}$ instead of the unit-area disk $\mathcal{D}$.

Independently, Penrose [17] showed that the longest edge M_n of the Euclidean minimum spanning tree (EMST) of n points randomly and uniformly distributed in a unit-area square $\mathcal{C}$ satisfies that

$$\lim_{n\to\infty} \Pr\ (n\pi M_n^2 - \ln n \leq \alpha) = \mathrm{e}^{-\mathrm{e}^{-\alpha}}$$

for any real number α. Remember that the longest edge of EMST is always the critical power [14,15]. Thus, the result in Reference [17] is actually stronger than that in Reference [16], since it will give the probability that the network is connected. For example, if we set $\alpha = \ln\ln n$, we have $\Pr\,(n\pi M_n^2 \leq \ln n + \ln\ln n) = \mathrm{e}^{-1/\ln n}$. It implies that the network is connected with probability at least $\mathrm{e}^{-1/\ln n}$ if the transmission radius of each node r_n satisfies $n\pi r_n^2 = \ln n + \ln\ln n$. Notice that $\mathrm{e}^{-1/\ln n} > 1 - 1/\ln n$ from $\mathrm{e}^{-x} > 1 - x$ for $x > 0$. By setting $\alpha = \ln n$, the probability that the graph $G(V,\ r_n)$ is connected is at least $\mathrm{e}^{-1/n} > 1 - 1/n$, where $n\pi r_n^2 = 2\ln n$. Notice that the above probability is only true when n goes to infinity. When n is a finite number, the probability of the graph being connected is smaller, i.e. we need transmission radius much larger than r_n to guarantee that the network of n randomly distributed points are connected almost surely. In this paper, we will present the first experimental study of the probability of the graph $G(V,\ r_n)$ being connected for finite number n. Notice that Bettstetter [23] also conducted simulations recently to study the k-connectivity, minimum degree being k and their relations. However, they used the toroidal model instead of the actual Euclidean model.

One closely related question to the critical transmission radius is the *coverage* problem. Consider disks of radius r are placed in a two-dimensional unit-area disk $\mathcal{D}$ with centers from a Poisson point process with intensity n, when these disks cover the unit-disk. A result shown by Hall [30] implies that if $n\pi \cdot r^2 = \ln n + \ln\ \ln n + c(n)$ and $c(n) \to \infty$, then the probability that there is a vacancy area in $\mathcal{D}$ is 0 as n goes to infinity; if $c(n) \to -\infty$, the probability that there is a vacancy in $\mathcal{D}$ is at least $1/20$. This implies that the hitting radius r_n such that $G(V,\ r_n)$ is connected satisfies $\pi \cdot r_n^2 \leq 4(\ln n + \ln\ln n + c(n)/n)$ for $c(n) \to +\infty$.

Another closely related problem is when will a Bernoulli graph be connected if we increase the probability of the links being chosen. Let $\mathcal{B}(n, p(n))$ be the set of graphs on n nodes in which each edge of the completed graph K_n is chosen independently with probability p(n). Then it has been shown that the probability that a graph in $\mathcal{B}(n, p(n))$ is connected goes to 1 if p(n)= $\ln n + c(n)/n$ for any $c(n) \to \infty$. Although their asymptotic expressions are the same with that by Gupta and Kumar [16], but we cannot apply this to the wireless model as in wireless networks, the existences of two edges are not independent, and we do not choose edges from the completed graph using Bernoulli model.

We then review the results concerning the k-connectivity of a random graph.

For general graphs, Bollobás and Thomason (see Theorem 7.5 of Reference [18]) proved that if $c(n) \to \infty$, $c(n) \leq \ln\ln\ln n$ and $p(n) = \ln n + (k-1) \ln\ln n - c(n)/n$, then graphs from $\mathcal{B}(n, p(n))$ almost surely have minimum degree k and thus almost surely are k-connected.

It was proved by Penrose [19] that, given any metric l_p with $2 \leq p \leq \infty$ and any positive integer k,

$$\lim_{n\to\infty} Pr\left(\varrho(\chi_n, \kappa \geq k) = \varrho(\chi_n, \delta \geq k)\right) = 1$$

The result is analogous to the well-known results in the graph theory [18] that graph becomes k vertex connected when it achieves the minimum degree k if we add the edges randomly and uniformly from $\binom{n}{2}!$ possibilities. The result by Penrose [19] says that a graph of $G(\chi_n, r)$ becomes k-connected almost surely at the moment it has minimum degree k by letting r go from 0 to ∞. However, this result does not imply that, to guarantee a graph over n points k-connected almost surely, we only have to connect every node to its k nearest neighbors. Let V be n points randomly and uniformly distributed in a unit square (or disk). Xue and Kumar [29] proved that, to guarantee a geometry graph over V connected, the number of nearest neighbors that every node has to connect is asymptotically $\Theta(\ln n)$. Dette and Henze [20] studied the maximum length of the graph by connecting every node to its k nearest neighbors asymptotically. We conjecture that, given n random points V over a unit-area square, to guarantee a geometry graph over V $(k+1)$-connected, the number of nearest neighbors that every node has to connect is asymptotically $\Theta(\ln n + (2k-1) \ln\ln n)$. We leave this as future work.

Similarly, instead of considering χ_n, Penrose also considered a homogeneous Poisson point process with intensity n on the unit-area square $\mathcal{C}$. Penrose gave loose upper and lower bound on the hitting radius $r_n = \varrho(\mathcal{P}_n, \delta \geq k)$ as $\ln n/2^{d+1} \leq n r_n^r \leq d!2 \ln n$ for homogeneous Poisson point process on a d-dimensional unit cube, This result is too loose. More importantly, the parameter k does not appear in this estimation at all. In this paper, we derive an exact bound on r_n for two-dimensional n points V randomly and uniformly distributed in $\mathcal{C}$ such that the graph $G(V, r_n)$ is k-connected with high probability.

We also conduct experiments to study the probability that a graph has minimum degree k and has vertex connectivity k simultaneously. Surprisingly, we found that this probability is sufficiently close to 1, even n is at the scale of 100. This observation implies a simple method (by just computing the minimum vertex degree) to approximate the connectivity of a random geometry graph.

Penrose [17,19] also studied the k-connectivity problem for d-dimensional points distributed in a unit-area cube using the toroidal model instead of the Euclidean model as one way to eliminate the boundary effects. He [19] showed that the hitting radius r_n, such that the graph $G(V, r_n)$ is $(k+1)$-connected, satisfies

$$\lim_{n\to\infty} Pr\left(n\pi r_n^2 \leq \ln n + k \ln\ln n - \ln k! + \alpha\right) = e^{-e^{-\alpha}}$$

Dette and Henze [20] studied the largest length, denoted by $r_{n,k}$ here, of the kth nearest neighbor link for n points drawn independently and uniformly from the d-dimensional unit-length cube or the d-dimensional unit-volume sphere. They gave asymptotic result of this length accordingly as $k < d$, $k = d$ or $k < d$. For unit-volume cube, they use the norm l_∞ instead of the Euclidean norm l_2. For the unit-volume sphere, their result implies that, when $d = 2$ and $k > 2$,

$$\lim_{n\to\infty} Pr\left(n\pi r_{n,k}^2 \geq \ln n + (2k-3)\ln\ln n - 2\ln(k-1)! - 2(k-2)\ln 2 + \ln \pi + 2\alpha\right) = e^{-e^{-\alpha}}$$

Notice that Penrose [19] had shown that when the domain is a unit-area square, the probability that a random geometry graph $G(V, r_n)$ is k-connected and has minimum vertex degree k goes to 1 as n goes to infinity. Consequently, we can conjecture that the transmission radius r_n such that the graph $G(V, r_n)$

is k-connected with high probability satisfies $n\pi r_n^2 \simeq \ln n + (2k-3)\ln\ln n - 2\ln(k-1)! + 2\alpha$. We will prove this later.

3. Fault Tolerance by κ-Connectivity

In this section we concentrate on the hitting radius for the k-connectivity for n randomly and uniformly distributed points in a unit-area square $\mathcal{C}$. We build our result based on the result by Penrose [19].

For convenience, instead of the random point process χ_n, we consider a homogeneous Poisson point process of rate n, denoted by $\mathcal{P}_n$, on a unit-area square $\mathcal{C}$. Same as Reference [19], we let $\mathcal{E}(k, n, r)$ denote the expected number of points of $\mathcal{P}_n$ with degree k in a graph of $G(\mathcal{P}_n, r)$. Let $\mathcal{D}(\mathbf{x}, r)$ be the disk centered at $\mathbf{x}$ with radius r. Given a point $\mathbf{x}$, let $v_r(\mathbf{x})$ be the area of the intersection of $\mathcal{D}(\mathbf{x}, r)$ with the unit-area square $\mathcal{C}$. Additionally, let

$$\phi_{n,r,k}(\mathbf{x}) = (n \cdot v_r(\mathbf{x}))^k \cdot \frac{e^{-n \cdot v_r(\mathbf{x})}}{k!}$$

Here $\phi_{n,r,k}(\mathbf{x})$ is the probability that point $\mathbf{x}$ has degree k. Then, it was known [19] that

$$\mathcal{E}(k, n, r) = n \int_c \phi_{n,r,k}(\mathbf{x}) d\mathbf{x}.$$

Then Penrose [19] (Theorem 1.2) proved the following.

Theorem 1. Let α be any real number. Given any metric l_p on $\mathcal{C}$ with $1 < p \leq \infty$ and any integer $k \geq 0$, and r_n satisfying the following condition

$$\lim_{n\to\infty} \mathcal{E}(k, n, r_n) = e^{-\alpha}$$

then we have

$$\lim_{n\to\infty} \Pr\left(\varrho(\mathcal{P}_n, \delta \geq k+1) \leq r_n\right) = e^{-e^{-\alpha}}$$

Notice that the same theorem is true when the random point process $\mathcal{P}_n$ is used instead of the homogeneous Poisson point process. The remainder of this section is devoted to estimate the value r_n. Penrose [19] agreed that r_n *is not so easy to find because of the dominance of complicated boundary effects*. The estimated radius r_n also makes the graph $G(\mathcal{P}_n, r_n)$ k-connected with probability $e^{-e^{-\alpha}}$ when n goes to infinity, since Penrose [19] proved that it is almost surely that $\varrho(\chi_n, \kappa \geq k) = \varrho(\chi_n, \delta \geq k)$ and $\varrho(\mathcal{P}_n, \kappa \geq k) = \varrho(\mathcal{P}_n, \delta \geq k)$ as n goes to infinity.

3.1. Lower Bound

We first study the asymptotic lower bound for the hitting radius r_n for the $(k+1)$-connectivity.

Obviously, $v_r(\mathbf{x}) \leq \pi r^2$ for any point $\mathbf{x}$ inside the unit-area square $\mathcal{C}$. Since $\phi_{n,r,k}(\mathbf{x})$ is a monotone increasing function of $v_r(\mathbf{x})$, we have

$$\phi_{n,r,k}(\mathbf{x}) = (n \cdot v_r(\mathbf{x}))^k \frac{e^{-n \cdot v_r(\mathbf{x})}}{k!} < (n \cdot \pi r^2)^k \frac{e^{-n \cdot \pi r^2}}{k!}$$

We then bound $\mathcal{E}(k, n, r)$ as follows.

$$\mathcal{E}(k, n, r) = n \int_{\mathcal{C}} \phi_{n,r,k}(\mathbf{x}) d\mathbf{x} < n(n \cdot \pi r^2)^k \frac{e^{-n \cdot \pi r^2}}{k!}$$

Notice that if we use πr^2 for $v_r(\mathbf{x})$ instead of the actual area $v_r(\mathbf{x})$, the computed radius r is less than the actual required radius. This is because $v_r(\mathbf{x}) < \pi r^2$ for point $\mathbf{x}$ near the boundary of the square. Thus the probability that there is at least k neighbors within distance r of point $\mathbf{x}$ is increased when we use πr^2 for $v_r(\mathbf{x})$ for point $\mathbf{x}$ near the boundary. To remedy the approximated area πr^2, the actual value r should be larger than the computed one.

We estimate r when $v_r(\mathbf{x}) = \pi r^2$ is used as the area measurement. Let $y = \pi r^2$. From $\lim_{n\to\infty} \mathcal{E}(k, n, r_n) = e^{-\alpha}$, we have $e^{-\alpha} = \lim_{n\to\infty} n(n \cdot y)^k e^{-n \cdot y}/k!$. We will relax the condition by ignoring the condition of n going infinity. In other words, we consider that

$$e^{-\alpha} = n(n \cdot y)^k \frac{e^{-n \cdot y}}{k!}$$

It implies that, by taking ln on both sides,

$$-\alpha = \ln n + k \ln n + k \ln y - ny - \ln(k!)$$

Thus,

$$-k \ln y + ny = (k+1)\ln n - \ln(k!) + \alpha$$

Dividing both side by k, we have

$$\frac{n}{k} y - \ln y = \frac{k+1}{k} \ln n - \frac{1}{k} \ln(k!) + \frac{\alpha}{k}$$

Let $z = \frac{n}{k}y$. Then, $\ln y = \ln z + \ln k - \ln n$. Then

$$z - \ln z = \ln k - \ln n + \frac{k+1}{k}\ln n - \frac{1}{k}\ln(k!) + \frac{\alpha}{k}$$
$$= \frac{1}{k}\ln n + \ln k - \frac{1}{k}\ln(k!) + \frac{\alpha}{k}$$

Notice that if $z = \ln z + t$, then $z > t + \ln t$, where $t > 0$. Then, we have

$$z > \frac{1}{k}\ln n + \ln k - \frac{1}{k}\ln(k!) + \frac{\alpha}{k} + \ln \left(\frac{1}{k}\ln n + \ln k - \frac{1}{k}\ln(k!) + \frac{\alpha}{k}\right)$$
$$> \frac{1}{k}\ln n + \ln k - \frac{1}{k}\ln(k!) + \frac{\alpha}{k} + \ln\left(\frac{1}{k}\ln n\right)$$

Consequently, by substituting back $z = \frac{n}{k}\pi r^2$, we have

$$\frac{n}{k}\pi r^2 > \frac{\ln n}{k} + \ln k - \frac{1}{k}\ln(k!) + \frac{\alpha}{k} - \ln k + \ln\ln n$$

which implies that

$$n\pi r^2 > \ln n + k\ln\ln n - \ln k! + \alpha$$

Notice that the function $(n \cdot y)^k e^{-n \cdot y}/k!$ achieves the maximum value, when $y = k/n$. It is monotone decreasing for $y > k/n$ and monotone increasing for $y < k/n$. We always assume that k is a fixed constant throughout this article. Then we have the following theorem.

Theorem 2. Given n wireless nodes V randomly and uniformly distributed in a unit-area square. If we want the graph $G(V, r_n)$ to be $(k+1)$-connected with probability at least $e^{-e^{-\alpha}}$, the transmission radius r_n satisfies

$$\pi r^2 > \ln n + k\ln\ln n - \ln k! + \alpha \qquad (1)$$

Notice that, for the toroidal model, Penrose [19] gave the same exact bound for r_n such that the graph is guaranteed to be $(k+1)$-connected asymptotically. Moreover, the result by Gupta and Kumar [16] and the result by Penrose [17] is just a special case when $k = 0$, if this bound is tight. Notice that, in our analysis, we implicitly assume that $k > 0$. Additionally, the lower bound of our analysis could be improved by considering a more tight area estimation for point $\mathbf{x}$ near the boundary of the square, but the analysis will be much more complicated.

3.2. Upper Bound

We showed that if we want the network $G(V, r_n)$ to be $(k+1)$-connected with probability at least $e^{-e^{-\alpha}}$, we have to set the transmission radius r_n satisfying inequality (1) for n points randomly and uniformly distributed in a unit-area square. In this section, we continue to study the upper bound of the transmission radius to achieve the same $(k+1)$-connectivity. The estimated upper bound is different from the lower bound even asymptotically. Again, we derive the upper bound from the equation $n\int_{\mathcal{C}} \phi_{n,r,k}(\mathbf{x})d\mathbf{x} = e^{-\alpha}$.

We partition the unit square to three regions: the region I is $[-0.5 + r, 0.5 - r] \times [-0.5 + r, 0.5 - r]$, the region III is four corners and the remaining is the region II (See Fig. 1). We compute the area $v_r(\mathbf{x})$ for point $\mathbf{x}$ located in these three regions separately. Obviously, for any $\mathbf{x}$ in region I, $v_r(\mathbf{x}) = \pi r^2$. For a point $\mathbf{x}$ in region II, assume its distance to the boundary of $\mathcal{C}$ is x, then the area

$$v_r(\mathbf{x}) = \pi r^2 - r^2\cos^{-1}\left(\frac{x}{r}\right) + x\sqrt{r^2 - x^2}$$

Here $0 \le x \le r$. Assume $x = r\cos\theta$, where $0 \le \theta \le \pi/2$. Then $v_r(\mathbf{x}) = r^2(\pi - \theta + \sin\theta\cos\theta)$. It is easy to show that

$$\frac{\pi r^2}{2}(1 + \cos\theta) \le r^2(\pi - \theta + \sin\theta\cos\theta)$$
$$\le \frac{\pi r^2}{2} + 2r^2\cos\theta$$

By substituting $x = r\cos\theta$, we bound $v_r(\mathbf{x})$ as follows

$$\frac{\pi r^2}{2} + \frac{\pi r}{2} \cdot x \le v_r(\mathbf{x}) \le \frac{\pi r^2}{2} + 2r \cdot x$$

Let r^* be the solution of $n\int_{\mathcal{C}} \phi_{n,r,k}(\mathbf{x})d\mathbf{x} = e^{-\alpha}$. Let Ω be any subregion of $\mathcal{C}$. Let $w(\mathbf{x})$ be any function such that $w(\mathbf{x}) \le v(\mathbf{x})$ and is monotone increasing of r. Let $\varphi_{n,r,k}(\mathbf{x}) = (n \cdot w(\mathbf{x}))^k \cdot e^{-n \cdot w(\mathbf{x})}/k!$. Thus, $\varphi_{n,r,k}(\mathbf{x}) \le \phi_{n,r,k}(\mathbf{x})$. Let r' be the solution of $n\int_{\Omega} \varphi_{n,r,k}$

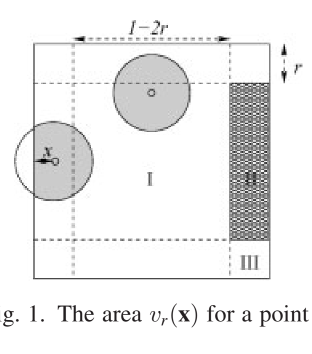

Fig. 1. The area $v_r(\mathbf{x})$ for a point $\mathbf{x}$.

$(\mathbf{x})d\mathbf{x} = e^{-\alpha}$. Then $r^* \leq r'$. This is because $w(x)$, $v_r(x)$ are monotone increasing functions of r, and $(ny)^k \cdot e^{-ny}/k!$ is monotone increasing function when $y \leq k/n$. Thus, to bound the transmission radius r from above so that the graph $G(V, r)$ is $(k+1)$-connected, we use the lower bound of $v_r(\mathbf{x})$ and we also only compute the integral for region I and II. Notice,

$$\int_{\mathcal{C}} (nv_r(\mathbf{x}))^k \cdot \frac{e^{-nv_r(\mathbf{x})}}{k!} d\mathbf{x}$$
$$> \int_I (nv_r(\mathbf{x}))^k \cdot \frac{e^{-nv_r(\mathbf{x})}}{k!} d\mathbf{x} + \int_{II} (nv_r(\mathbf{x}))^k \cdot \frac{e^{-nv_r(\mathbf{x})}}{k!} d\mathbf{x}$$

Obviously, for region I, we have

$$\int_I (nv_r(\mathbf{x}))^k \cdot \frac{e^{-nv_r(\mathbf{x})}}{k!} d\mathbf{x} = (n \cdot \pi r^2)^k \cdot \frac{e^{-n \cdot \pi r^2}}{k!} \cdot (1-2r)^2$$

The integral over region II is four times of the integral over the rectangle region near the boundary, where the length of the rectangle is $1 - 2r$ and the width is r. Assume that the distance of a point $\mathbf{x}$ to the boundary is x. Notice that $v_r(\mathbf{x}) > \pi r^2/2 + \pi r/2x$. Let $y = \pi r^2/2 + \pi r/2x$. We have

$$\int_{II} (n \cdot v_r(\mathbf{x}))^k \cdot \frac{e^{-n \cdot v_r(\mathbf{x})}}{k!} d\mathbf{x}$$
$$= 4(1-2r) \int_{x=0}^{r} (nv_r(x))^k \cdot \frac{e^{-nv_r(x)}}{k!} d\mathbf{x}$$
$$> \frac{8(1-2r)}{\pi \cdot k! \cdot r} \int_{y=\frac{\pi r^2}{2}}^{\pi r^2} (ny)^k e^{-ny} d\mathbf{y}$$
$$= \frac{8(1-2r)}{n\pi r} \left(e^{-t/2} \sum_{j=0}^{k} \frac{t^j}{j! 2^j} - e^{-t} \sum_{j=0}^{k} \frac{t^j}{j!} \right)$$

Here $t = n\pi r^2$. The last equation comes from $\int z^k e^{-z} dz = -e^{-z} k! \sum_{j=0}^{k} z^j/j!$. Then, the transmission radius $\varrho(\mathcal{P}_n, \kappa \geq k))$ is bounded from above by the solution of the following equation.

$$e^{-\alpha} = n \cdot t^k \frac{e^{-t}}{k!} \cdot (1-2r)^2 + \frac{8(1-2r)}{\pi r}$$
$$\left(e^{-t/2} \sum_{j=0}^{k} \frac{(t/2)^j}{j!} - e^{-t} \sum_{j=0}^{k} \frac{t^j}{j!} \right)$$
$$< n \cdot t^k \frac{e^{-t}}{k!} + \frac{8}{\pi r} k \cdot e^{-t/2} \frac{(t/2)^k}{k!}$$

The inequality comes from $e^{-t/2}\ (t/2)^j/j! < e^{-t/2}\ (t/2)^{j+1}/(j+1)!$ for $j < t/2$. Here we assume that $k < t/2$. Remember that here $t = n\pi r^2 \geq \ln n$ asymptotically from our lower bound analysis. The rest of the section is then devoted to approximate r using above inequality.

Let $A = n \cdot t^k\ e^{-t}/k!$ and $B = 8/\pi r\, k \cdot e^{-t/2}\ (t/2)^k/k!$. Thus, $B/A = 8ke^{t/2}/2^k n\pi r = 8k/\sqrt{\pi}\ e^{t/2}/\sqrt{nt}$. Then, by taking ln on both sides of the inequality, we have

$$-\alpha < \ln A + \ln\left(1 + \frac{B}{A}\right)$$
$$= \ln n + k \ln t - t - \ln k! + \ln\left(1 + \frac{8k}{2^k\sqrt{\pi}} \frac{e^{t/2}}{\sqrt{nt}}\right)$$

Thus, we have

$$t < \ln n + k \ln t - \ln k! + \alpha + ln\left(1 + \frac{8ke^{t/2}}{2^k\sqrt{\pi n t}}\right) \tag{2}$$

Notice that $\ln(1+x) < x$ for any $1 > x > 0$ and $\ln(1+x) \simeq \ln x$ for x sufficiently larger than one. We solve inequality (2) by recursion as follows. First, let $t_1 = \ln n - \ln k! + \alpha$ as the initial solution. It is easy to show that $B/A = 8k/\sqrt{\pi}\ e^{t_1/2}/\sqrt{nt_1} \ll 1$. Thus, we can estimate the solution by substituting t_1 to inequality (2)

$$t_2 < \ln n + k \ln t_1 - \ln k! + \alpha + \ln\left(1 + \frac{8k}{2^k\sqrt{\pi}} \frac{e^{t_1/2}}{\sqrt{nt_1}}\right)$$

When n is large enough, we have $t_2 \simeq \ln n + k \ln \ln n - \ln k! + \alpha$. In this situation, however, we have $B/A = (8k/2^k\sqrt{\pi})\ e^{t_2/2}/\sqrt{nt_2} = 8k/2^k\sqrt{\pi}\ \sqrt{(\ln n)^k e^{\alpha}/}\ \sqrt{k! \cdot t_2}$ goes to infinity when n goes to infinity. Thus, by substituting $t_2 = \ln n + k \ln \ln n - \ln k! + \alpha$ to inequality (2), we have the third estimation of the solution as follows

$$t_3 < \ln n + k \ln t_2 - \ln k! + \alpha + \ln\left(1 + \frac{8ke^{t_2/2}}{2^k\sqrt{\pi n t_2}}\right)$$
$$\simeq \ln n + k \ln t_2 - \ln k! + \alpha + \ln \frac{8ke^{t_2/2}}{2^k\sqrt{\pi n t_2}}$$
$$= \ln n + k \ln t_2 - \ln k! + \alpha + \ln \frac{8k}{2^k\sqrt{\pi}}$$
$$+ \frac{1}{2}(k \ln \ln n + \alpha - \ln k! - \ln t_2)$$
$$= \ln n - \frac{3}{2} \ln k! + \frac{3}{2}\alpha + \frac{1}{2} k \ln \ln n$$
$$+ \left(k - \frac{1}{2}\right) \ln t_2 + \ln \frac{8k}{2^k\sqrt{\pi}}$$

Notice that

$$\begin{aligned}\ln t_2 &= \ln(\ln n + k \ln \ln n - \ln k! + \alpha)\\ &= \ln \ln n + \ln\left(1 + \frac{k \ln \ln n - \ln k! + \alpha}{\ln n}\right)\\ &< \ln \ln n + \frac{k \ln \ln n - \ln k! + \alpha}{\ln n}\end{aligned}$$

Thus, we have the third estimation t_3 as

$$\begin{aligned}t_3 &\simeq \ln n - \frac{3}{2}\ln k! + \frac{3}{2}\alpha + \frac{1}{2}k \ln \ln n\\ &\quad + \left(k - \frac{1}{2}\right)\ln \ln n + \ln\frac{8k}{2^k\sqrt{\pi}}\\ &= \ln n + \frac{1}{2}(3k-1)\ln \ln n - \frac{3}{2}\ln k!\\ &\quad + \frac{3}{2}\alpha + \ln\frac{8k}{2^k\sqrt{\pi}}\end{aligned}$$

We can continue to substitute t_3 to get a more accurate solution t_4 and so on. It is easy to show that the final solution of t is bounded by solution of the following equality, when n goes to infinity

$$\begin{aligned}t &= \ln n + k \ln t - \ln k! + \alpha\\ &\quad + \ln\left(\frac{8k}{2^k\sqrt{\pi}}\frac{e^{t/2}}{\sqrt{n \cdot t}}\right)\\ &= \ln n + k \ln t - \ln k! + \alpha\\ &\quad + \ln\frac{8k}{2^k\sqrt{\pi}} + \frac{t}{2} - \frac{1}{2}\ln n - \frac{1}{2}\ln t\end{aligned}$$

This implies that

$$t = \ln n - 2\, ln\, k! + 2\alpha + 2 \ln\frac{8k}{\sqrt{\pi}} + (2k-1)\ln t$$

Thus we can bound t by the following approximation, when n goes to infinity

$$\begin{aligned}t &= \ln n + (2k-1)\ln \ln n - 2 \ln k! + 2\alpha\\ &\quad + 2 \ln\frac{8k}{2^k\sqrt{\pi}}\end{aligned}$$

Consequently, we have the following.

Theorem 3. Given n wireless nodes V randomly and uniformly distributed in a unit-area square. If we set the transmission radius r_n to satisfy that

$$\begin{aligned}n\pi r^2 &> \ln n + (2k-1)\ln \ln n - 2 \ln k!\\ &\quad + 2\alpha + 2 \ln\frac{8k}{2^k\sqrt{\pi}}\end{aligned}$$

then the graph $G(V, r_n)$ is $(k+1)$-connected with probability at least $e^{-e^{-\alpha}}$, when n goes to infinity.

Obviously, if $\alpha \to \infty$ then $e^{-e^{-\alpha}\to 1}$. For example, if we set $\alpha = \ln \ln n$, i.e. want the graph $G(V, r_n)$ to be $(k+1)$-connected with probability at least $e^{-1/\ln n} > 1 - 1/(\ln n)$, we have to set the transmission radius r_n that satisfies

$$n\pi r^2 > \ln n + (2k+1)\ln \ln n - 2\ln k! + 2 \ln\frac{8k}{2^k\sqrt{\pi}}$$

If we want the graph $G(V, r_n)$ to be $(k+1)$-connected with probability at least $e^{-1/n} > 1 - (1/n)$, we have to set the transmission radius r_n satisfying

$$n\pi r^2 > 3\ln n + (2k-1)\ln \ln n - 2\ln k! + 2\ln\frac{8k}{2^k\sqrt{\pi}}$$

Additionally, if $\alpha \to -\infty$, then $e^{-e^{-\alpha}\to 0}$. Then it implies that the graph $G(V, r_n)$ will be $(k+1)$-connected with very low probability if this bound of the hitting radius is tight.

Notice that the above analysis of the asymptotic upper bound of the transmission radius can also be used to derive a tighter lower bound on the transmission radius. We use the fact that $\pi r^2/2 + \pi r/2 \cdot x \leq v_r(\mathbf{x})$ to derive the upper bound of the transmission radius. To analyze the lower bound, we have to use the fact that $v_r(\mathbf{x}) \leq \pi r^2/2 + 2r \cdot x$ to estimate the area $v_r(\mathbf{x})$ for point $\mathbf{x}$ near the boundary. In addition, we have to compute the integral in all three regions. To simplify the analysis, for point $\mathbf{x}$ in region III, we also use $v_r(\mathbf{x}) \leq \pi r^2/2 + 2r \cdot x$ to estimate the area $v_r(\mathbf{x})$. Then similar to the above analysis of upper bound, the lower bound on t is at least the solution of the following equation

$$e^{-\alpha} = n \cdot t^k \frac{e^{-t}}{k!} \cdot (1 \cdot 2r)^2 + \frac{2}{r}\left(e^{-\frac{t}{2}}\sum_{j=0}^{k}\frac{t^j}{2^j j!} - e^{-t}\sum_{j=0}^{k}\frac{t^j}{j!}\right)$$

By tedious computing, we can compute the asymptotic lower bound as

$$t > \ln n + (2k-1)\ln\ln n - 2\ln k! + 2\alpha$$

Remark Although we have computed the lower and upper bounds for the transmission range r_n such that the graph $G(V, r_n)$ is $(k+1)$-connected with probability at least $e^{-e^{-\alpha}}$, these bounds hold only when n goes to infinity and k is assumed to be a constant. When n is a practical finite number (especially when n

is comparable with $k!$), our bounds do not hold any-more. This observation is witnessed by our experimental results.

4. Topology Control For Fault Tolerance

In this section, we study how to control the network topology given a n nodes network that is already k fault tolerant. After selecting the hitting radius for the k-connectivity, we can model the network topology as a *unit disk graph* (UDG) by scaling the radius to one unit. A unit-disk graph is the graph in which two nodes are connected if their distance is not more than one unit.

Due to the nodes' limited resource in wireless *ad hoc* networks, the scalability is crucial for network operations. One effective approach is to maintain only a linear number of links using a localized construction method. However, this sparseness of the constructed network topology should not compromise on the fault tolerance and compromise too much on the power consumptions for both unicast and broadcast/multicast communications. We are interested in constructing a sparse network topology efficiently for a set of static wireless nodes such that every unicast route in the constructed network topology is power efficient, in addition to be k fault tolerant. Here a route is *power efficient* for unicasting if its power consumption is no more than a constant factor of the minimum power needed to connect the source and the destination. A network topology is said to be power efficient if there is a power efficient route to connect any two nodes in this topology.

In the most common power-attenuation model, the signal power falls as $1/r^{\beta}$, where r is the distance from the transmitter antenna and β is a constant between 2 and 5 dependent on the wireless transmission environment. This is called *path loss*. For simplicity, we only consider the path loss of the signal. Thus, the power needed to support a link uv is $||uv||^{\beta}$, where $||uv||$ is the Euclidean distance between u and v.

Lukovszki [31,36] gave a method to construct a spanner that can sustain k-nodes or links failures for complete graph. Our topology control method is based on this method and the following Yao structure [32]. The *Yao graph* over a (directed) graph G with an integer parameter $p \geq 6$, denoted by $\overrightarrow{YG}_p(G)$, is defined as follows. At each node u, any p equal-separated rays originated at u define p equal cones. In each cone, choose the shortest (directed) edge $uv \in G$, if there is any, and add a directed link $\overrightarrow{uv}$. Ties are broken arbitrarily. Let $YG_p(G)$ be the undirected graph by ignoring the direction of each link in $\overrightarrow{YG}_p(G)$. See the following Figure 2 for an illustration of selecting edges incident on u in the Yao graph.

X.-Y. Li, P.-J. Wan and O. Frieder [34–35] had proposed to use the Yao structure on the unit-disk graph for topology control without sacrificing too much on the energy conservation. Some researchers used a similar construction named θ-graph [36]. The difference is that, in each cone, it chooses the edge which has the shortest projection on the axis of the cone instead of the shortest edge. Here the axis of a cone is the angular bisector of the cone. For more detail, please refer to Reference [36]. It is obvious that the Yao structure does not sustain k faults in a neighborhood of any node, since each node only has at most p neighbors and one neighbor selected in each cone at most. However, we can modify the Yao structure as follows such that the structure is k-fault tolerant.

Each node u defines any p equal-separated rays originated at u, thus defines p equal cones, where $p > 6$. In each cone, node u chooses the $k+1$ closest nodes in that cone, if there is any, and add directed links from u to these nodes. Ties are broken arbitrarily. Let $Y_{p,k+1}$ be the final topology formed by all nodes.

Theorem 4. The structure $Y_{p,k+1}$ can sustain k nodes faults if original unit-disk graph is k node faults tolerant.

Proof. For simplicity, assume that all k fault nodes $v_1, v_2, \ldots, v_k$ are neighbors of a node u. We show that the remaining graph of $Y_{p,k+1}$ (removed of nodes $v_1, v_2, \ldots, v_k$ and all links incident on them) is still connected.

Notice that the original unit-disk graph is k node faults tolerant. Thus, the degree of each node is at least $k+1$. Additionally, with the k fault nodes $v_1, v_2, \ldots, v_k$ removed, there is still a path in UDG to connect any pair of remaining nodes. Assume that the path uses node u and have a link uw. We will prove

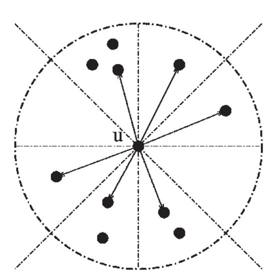

Fig. 2. The narrow regions are defined by eight equal cones. The closest node in each cone is a neighbor of u.

by induction that there is a path in the remaining graph to connect u and w.

If uw has the smallest distance among all pair of nodes, then uw must be in $Y_{p,k+1}$, thus in the remaining graph.

Assume the statement is true for node pair whose distance is the rth shortest. Consider uw with the $(r+1)$th shortest length.

If w is one of the $k+1$ closest nodes to u in some cone, then link uw remains in the remaining graph. Otherwise, for the cone in which node w resides, there must have other $k+1$ nodes that are closer to u than w and they are connected by u in $Y_{p,k+1}$. Since we only have k failure nodes, at least one of the links of $Y_{p,k+1}$ in that cone will survive, say link ux. It is easy to show that $||xw|| < ||uw|| < 1$. Then link uw can be replaced by link ux and a path from x to w by induction. This finishes the proof.

Notice that for the case where the nodes removed are not all neighbors of the same node, the induction proof also holds. Induction is based on all pair of nodes.

Our techniques of constructing k-connected subgraph of UDG (assuming UDG is already k-connected here) can be applied to a more general graph G if there is an embedding, denoted by $E(G)$, of G in the plane such that there is an edge in $E(G)$ if and only if their distance is not more than one unit. Notice that here an embedding of G in the plane is to assign each vertex a two-dimensional position.

We then show that the above structure approximates the original unit-disk graph well. More specifically, we will show that it is a spanner even with k fault nodes. Let $\Pi_G(u, v)$ be the shortest path connecting u and v in a weighted graph G, and $||\Pi_G(u, v)||$ be the length of $\Pi_G(u, v)$. Then a graph G is a t-spanner of a graph H if $V(G) = V(H)$ and, for any two nodes u and v of $V(H)$, $||\Pi_G(u, v)|| \leq t \, ||\Pi_H(u, v)||$. With H understood, we also call t the *length stretch factor* of the spanner G.

Let $\varrho G(u, v)$ be the path found by a unicasting routing method ϱ from node u to v in a weighted graph G, and $||\varrho G(u, v)||$ be the length of the path. The spanning ratio achieved by a routing method is ϱ defined as $\max_{u,v} ||\varrho G(u, v)||/||uv||$. Notice that the spanning ratio achieved by a specific routing method could be much larger than the spanning ratio of the underlying structure. Nonetheless, a structure with a small spanning ratio is necessary for some routing method to possibly perform well.

Theorem 5. The structure $Y_{p,k+1}$ is a length spanner even with k nodes faults.

Proof. To prove the length spanner property, it is easy to show that we only have to prove each pair of nodes u and w with $||uw|| \leq 1$ is approximated by a path with length no more than a constant factor, say β, of $||uw||$. The proof is similar to Theorem 4: we prove it by induction on the length of $||uw||$. Follow the proof of Theorem 4, we only have to show that

$$||ux|| + \beta||xw|| \leq \beta||uw||$$

for any node x with $||ux|| < ||uw||$ and x lies in the same cone as w does. Obviously, we need to set

$$\beta = \max_{\forall x, ||ux|| < ||uw||} \frac{||ux||}{||uw|| - ||xw||}$$

Notice that $\alpha = \angle wux < 2\pi/p$. Then a simple geometry reveals that $\beta = \max \cos\theta / \cos(\theta + \alpha)$, where $\theta = 1/2 \angle uwx \leq \pi - \alpha/2$. The minimum value for β is $1/1 - 2\sin(\pi/p)$. In other words, the spanning ratio of the remaining structure is at most β.

Due to limited power and resource of wireless nodes, wireless topologies always prefer to have bounded node degree, such that every wireless nodes only keep constant neighbors. The node degree of the structure $Y_{p,k+1}$ is at most $p(k+1)$, where $p \geq 6$. Recently, Bahramgiri *et al.* [37] showed how to decide the minimum transmission range of each node such that the resulted directed communication graph is k-connected. We can prove that their resulted graph is also a length spanner even with k nodes faults (the proof is omitted here since it is similar to ours). However, their method does not bound the node degree. Figure 3(a) shows an example in which node u can have as many as neighbors even after applying their method. Then we give a careful enhancement of their protocol to bound the node degree.

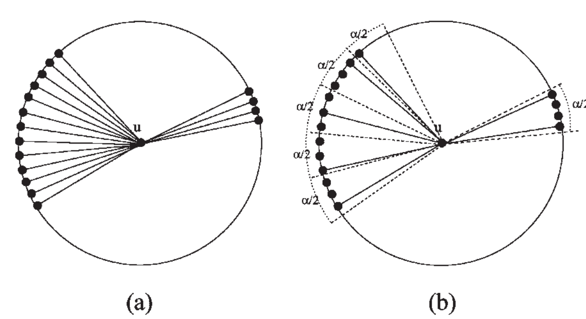

Fig. 3. (a) node u does not have bounded degree in graph generated by Bahramgiri's protocol; (b) new method to bound node degree for Bahramgiri's protocol.

In Bahramgiri's method, they increase the power step by step until there is no gap greater than α between the successive neighbors or the power reaches the maximum power. They proved that if $\alpha \leq 2\pi/3k$. then the resulted graph is k-connected. After applying their method, we can remove some links by the following method. For a node u, we divide its transmission range into $4\pi/\alpha$ equal cones (each cone have an angle $\alpha/2$). We select only one neighbor in each cone c if there is any, delete all other links. However, if for a cone c, one of its adjacent cones, say b, does not have any neighbors of u, we select the boundary neighbor v such that vu forms the smallest angle with cone b; if both adjacent cones of c are empty, we select *two* neighbors in c (close to the two boundary of cone c respectively); if c does not have empty adjacent cones, we can select any one of the neighbors. See Figure 3(b) for illustration. Since the gap between any two successive remaining neighbors is still not greater than α (except the empty cones), it is easy to show that the constructed graph is still k-connected if $\alpha \leq 2\pi/3k$. The node degree is bounded by $2\pi/\frac{\alpha}{2} = 4\pi/\alpha$. When $\alpha = 2\pi/3k$, the node degree is bounded by $6k$, which is almost the same as ours.

5. Experiments

We had analyzed the theoretical condition for the transmission radius r_n such that the graph $G(V, r_n)$ is $(k+1)$-connected with high probability. To confirm our theoretical analysis, we conduct simulations to see what is the practical value of r_n such that the wireless network $G(V, r_n)$ is $(k+1)$-connected with high probability. Notice that Bettstetter [23] also conducted simulations recently to study the k-connectivity, minimum degree being k and their relations. No explicit expression of r is given in Reference [23].

5.1. System Model

The geometry domain, in which the wireless nodes are distributed, is a unit square $\mathcal{C} = [-0.5, 0.5] \times [-0.5, 0.5]$. As shown by previous results, we know that the random point process $\mathcal{X}_n$ and the homogeneous Poisson point process $\mathcal{P}_n$ will have the same connectivity behavior asymptotically. For the simplicity of conducting simulations, we choose n points that are randomly and uniformly distributed in $\mathcal{C}$. For each randomly generated point set V and a transmission radius r, we construct the graph $G(V, r)$ in a centralized manner. To speed up the construction of $G(V, r)$, we partition the points into grids of size r. Thus, a point p can only connect with points from at most nine grids: one grid containing p and eight adjacent grids.

5.2. Computing the Connectivity

One of the major steps in conducting the simulations is to compute the connectivity of an induced unit-disk graph $G(V, r_n)$. It is easy to test whether a graph is connected by simply checking if a spanning tree contains all n nodes. To test whether the graph $G(V, r_n)$ is k-connected, we use the following observation: it is k-connected if and only if the minimum cut is at least k, which is equivalent to that the flow between any pair of nodes is at least k. So, given the graph $G(V, r_n)$, we compute the maximum flow between any pair of nodes by assigning each edge a weight one. A simpler method by using BFS to compute how many disjoint paths connecting a node v to a node u. The time complexity of this approach is $O(n^2 m)$, where m is the number of edges in $G(V, r)$ that could be as large as n^2. For unit-capacity flow, there is an $O(\min(m, n^{3/2}) m^{1/2})$ time complexity algorithm [38].

5.3. Experimental Results

5.3.1. Transition phenomena

A graph property of $G(V, r)$ is said to satisfy a transition phenomena if there is a radius r_0 such that the graph $G(V, r)$ almost surely does not have this property when $r < r_0$ and the graph $G(V, r)$ almost surely has this property when $r > r_0$. It was already shown that the property that $G(V, r)$ has the minimum node degree k satisfies a transition phenomena; additionally, the graph $G(V, r)$ is k-connected satisfies a transition phenomena.

Our simulations shown in Figures 4 and 5 confirm the theoretical results. We found that the transition becomes faster when the number of nodes increases. For testing the transition phenomena of the connectivity, we test $n = 50$ and $n = 100$, two cases. We test $0.1 \leq r \leq 0.9$ using interval 0.02, i.e. we test total 40 different transmission radii. Given a transmission radius r and number of nodes n, we generate 500 sets of random n points in $\mathcal{C}$. We compute the connectivity of each graph $G(V, r)$ and summarize how many is k-connected for $k = 1, 2, 3$ and 4. For testing the transition phenomena of the min-degree, we test $n = 100$, 200, 300 and 400. Other settings are same as the test for connectivity transition.

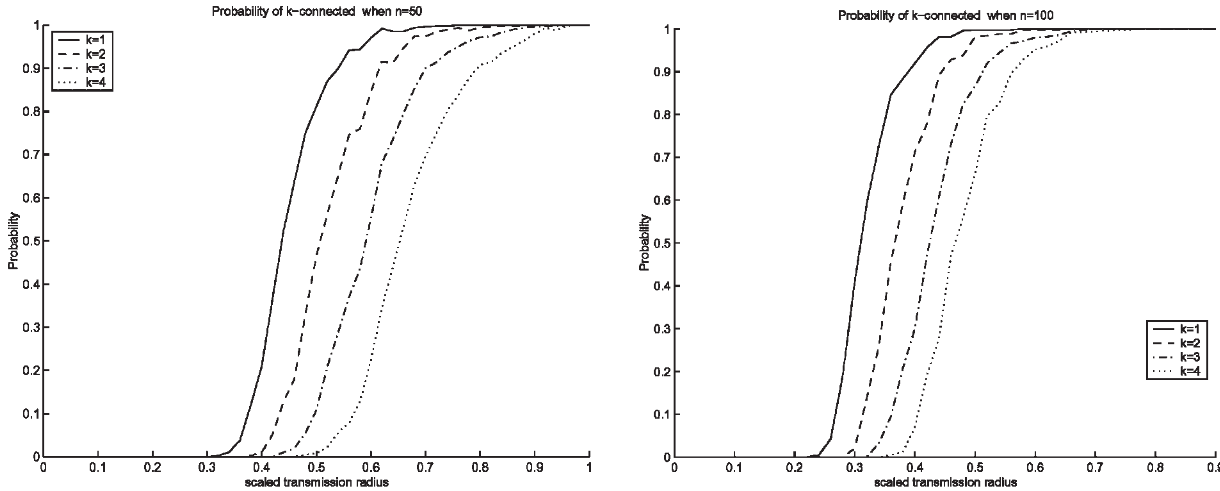

Fig. 4. Transition phenomena of a graph being *k*-connected.

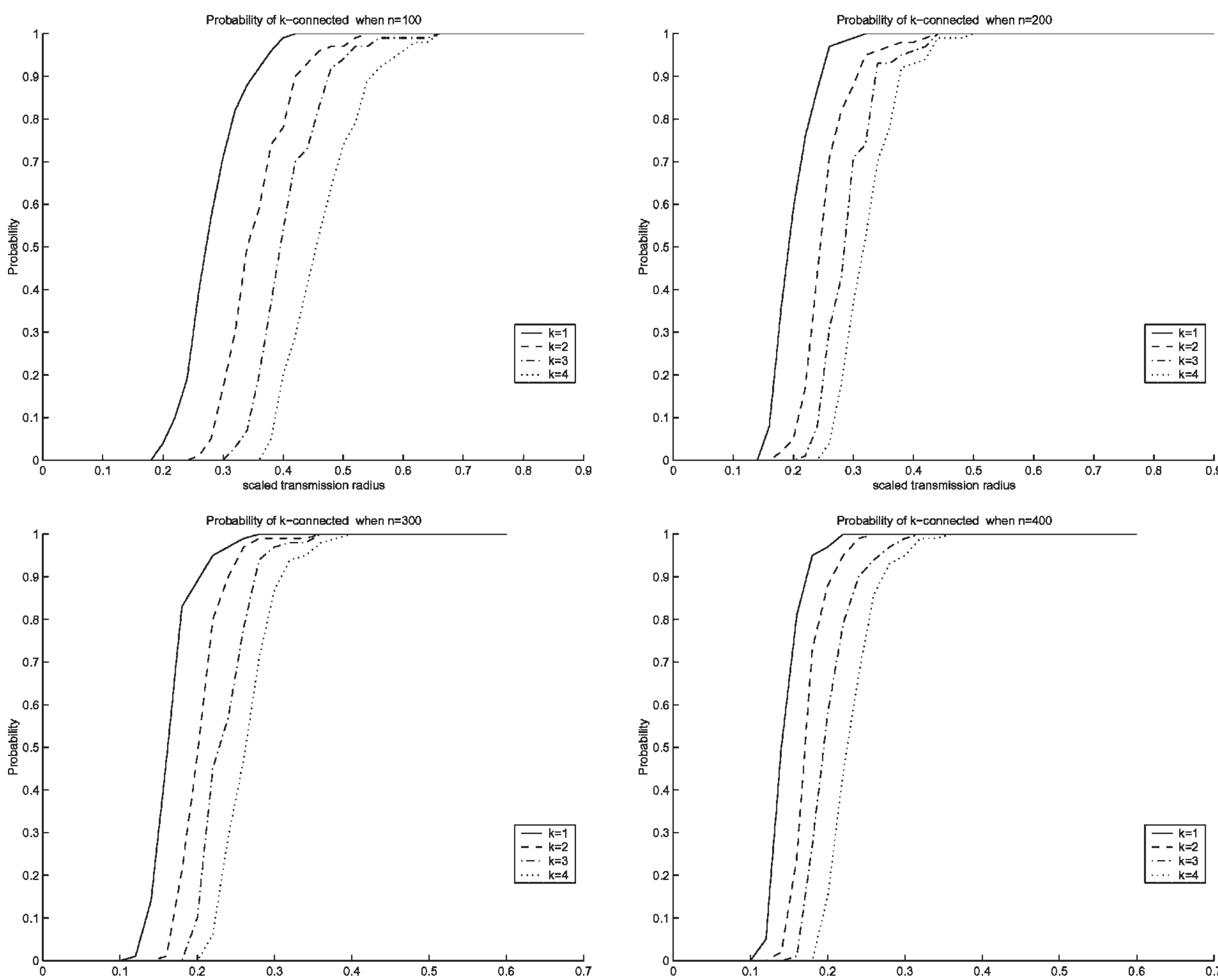

Fig. 5. Transition phenomena of a graph has minimum degree *k*.

5.3.2. Connectivity and minimum degree

Penrose [19] showed that the hitting radius for k-connectivity and the hitting radius for achieving minimum degree k are asymptotically same for points randomly and uniformly distributed in a unit-area square as n goes to infinity. We conduct extensive simulations on various number of points $n = 50, 100, 200, 300, 400$ and 500. Given n, k and α, we select r according to the bound given in Theorem 3. Here the connectivity $k = 1, 2$ and $\alpha \in \{0, \ln \ln n, \ln n\}$. Thus, there are total 36 cases. For each case, we generate 500 random point sets. Our simulations, illustrated by Figure 6, show that the probability that $G(V, r)$ is k-connected when its minimum degree is k is already sufficiently close to one when n is at the order of 50, especially when α is set as $\ln n$. This surprising result implies a fast method to approximate the connectivity of a graph by simply counting the minimum node degree.

5.3.3. Connectivity for small point set

Theoretically, we derived an asymptotic bound of the transmission range r_n for n points randomly and uniformly distributed in a unit-area square such that the graph $G(V, r_n)$ is k-connected with certain probability. We have to admit that the result holds only when n is large enough compared with $k!$. We first conduct simulations to measure the gap between the theoretical probability of graph $G(V, r)$ being k-connected and the actual statistical probability of it being k-connected for various radius r. Typically, we set $n\pi r^2 = \ln n + (2k-1)\ln \ln n - 2\ln k! + 2\alpha + 2\ln 8k/(2^k\sqrt{\pi})$. Then test all 54 cases of $n = 50, 100, 200, 300, 400$ and 500, $k = 1, 2, 3$ and 4, $\alpha = 0$, $\ln \ln n$ and ln. The corresponding theoretical k-connectivity probabilities for them are $1/e, 1 - 1/\ln n$ and $1 - 1/n$ when $\alpha = 0$, $\ln \ln n$ and ln respectively. The probability is computed over 500 different random point sets. Figure 7 illustrates our simulation results.

It is not surprising that the probability found by simulations is much lower than the theoretical analysis (denoted by the upper blue curves). Notice that the theoretical range r is not always monotone increasing of k when n is a small value. This is the reason some curves cross each other in our figures.

Figure 8 illustrates our simulation results for the probability that $G(V, r)$ has minimum degree k compared with the theoretical analysis. Notice, as expected, the probability gap for min-degree is smaller than that for the k-connectivity.

5.3.4. Practical transmission ranges for k-connectivity

Since the asymptotic bound of the transmission range r_n for n points randomly and uniformly distributed in a unit-area square such that the graph $G(V, r_n)$ is k-connected with certain probability holds only when n is large enough compared with $k!$, we need to study what is the actual transmission range required to achieve the k-connectivity with certain probability. It is possible to analyze more accurately what is the theoretical requirement for r_n when n is not large enough. However, the analysis is much more complicated as we cannot omit some ‘constant’ terms in any

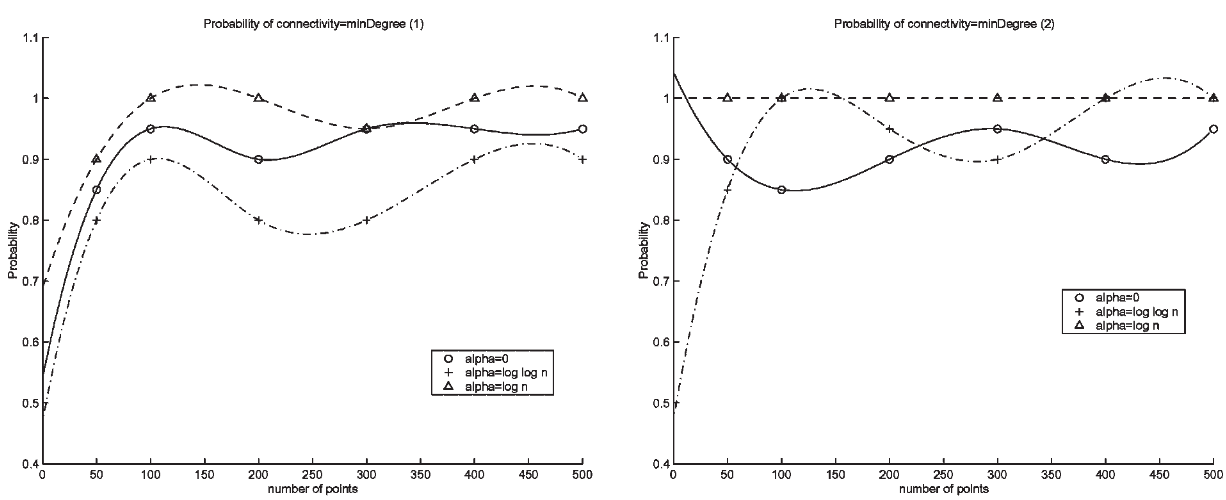

Fig. 6. The probability of a graph with minimal degree k is k-connected. Upper figure is for $k = 1$ and lower figure is for $k = 2$.

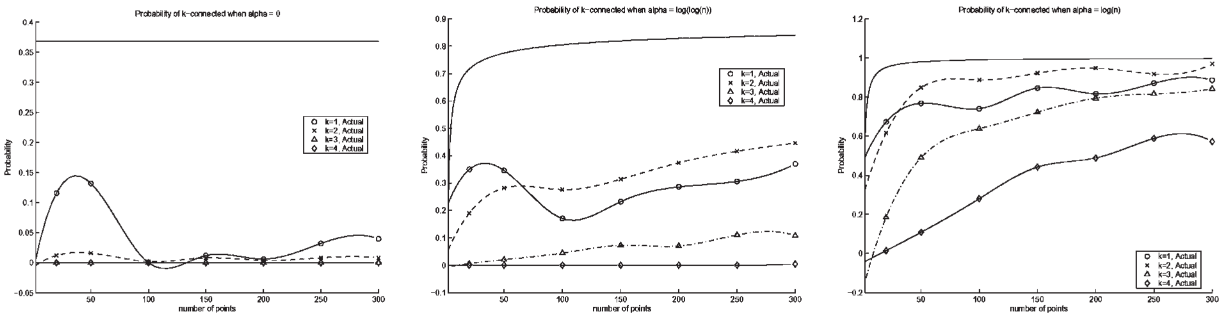

Fig. 7. Probability $G(V, r)$ is k-connected if r is set theoretically.

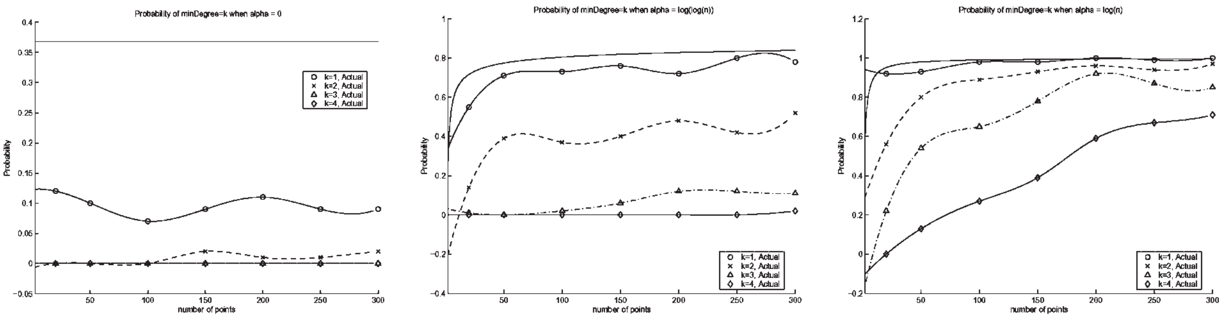

Fig. 8. Probability $G(V, r)$ has minimum degree k if r is set theoretically.

formula anymore. We leave this tight analysis as possible future work. Alternatively, we conduct simulations to find that practical transmission ranges when n is not large enough (see Figure 9). It is not surprising that the actual required range is larger than the theoretical bound. However, we found that the actual transmission range takes a similar decreasing pattern as the theoretical result when n goes to infinity.

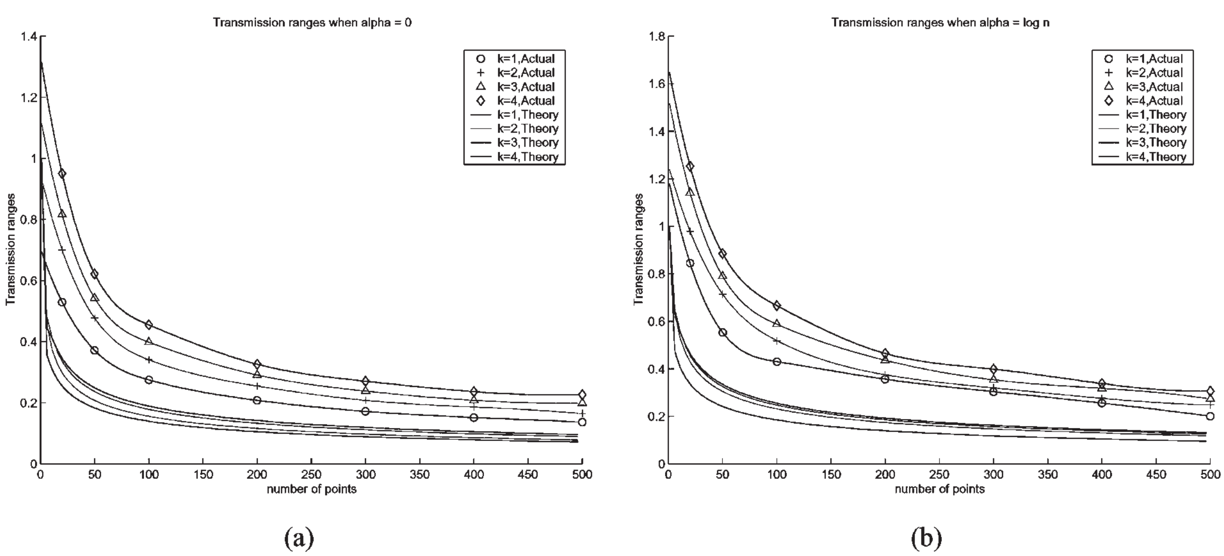

Fig. 9. (a) Practical range that $G(V, r)$ is k-connected with probability $1/e$; (b) practical range that $G(V, r)$ is k-connected with probability $1 - n$.

6. Conclusion

We consider a large-scale of wireless *ad hoc* networks whose nodes are distributed in a two-dimensional unit square region. As fault-tolerance is imperative for wireless networks, we showed that, to make the graph $G(V, r_n)$ $(k+1)$-connected almost surely, the transmission range r_n should satisfy $n\pi \cdot r_n^2 \geq \ln n + (2k-1) \ln \ln n - 2\ln(k-1)! + c(n)$ for any $c(n)$ with $c(n) \to \infty$ as n goes to infinity. Our result holds also in mobile networks when the movement of nodes are also random. We also conducted extensive simulations to study the relations between the minimum node degree and the connectivity of the induced unit-disk graphs. Practical transmission ranges were also studied by simulations when n is not a large integer. We found that, although it is different from the theoretical analysis when n is small, it has the same decreasing pattern as our theoretical analysis. We leave an accurate theoretical analysis of the transmission range to achieve k-connectivity, minimum degree k when number of nodes n is small.

We also presented a localized method to control the network topology given a k-faults tolerant deployment of wireless nodes such that the resulting topology is still fault tolerant but with much fewer communication links maintained. We showed that the constructed topology has only linear number $O(k \cdot n)$ of links and is a length spanner.

We assumed that the wireless nodes are generated by random point process, or Poisson point process. In practical applications, the wireless nodes could have some other estimated distributions such as the inhomogeneous Poisson point process. This is much more complicated than the cases studied by known previous results. We leave this as possible future work.

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