RANDOM GEOMETRIC ASPECTS IN MULTI-HOP WIRELESS NETWORKS

BY

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ABSTRACT

A randomly deployed large-scale wireless ad hoc network is naturally represented by an $r$-disk graph over a random point set. Relative neighborhood graph (RNG) has been widely used in localized topology control and geographic routing in wireless ad hoc and sensor networks. If all the nodes have the same transmission radii, the maximum edge length of the RNG is the smallest transmission radius for constructing the RNG by using only 1-hop neighbor information and it is referred to as the critical transmission radius of the RNG. The critical transmission radius of the RNG is the minimum requirement on the maximum transmission radius by those applications of RNG. Greedy forward routing (GFR) is attractive in wireless ad hoc and sensor networks due to its efficiency and scalability. GFR may discard a packet before it reaches its destination if the transmission radii of the nodes are small. To ensure that every packet can reach its destination, all nodes should have sufficiently large transmission radii. The critical transmission radius of a planar node set $V$ for greedy forward routing is the smallest transmission radius by $V$ which ensures successful delivery of any packet from any source node in $V$ to any destination node in $V$. In this thesis, we conduct probabilistic studies on the critical transmission radius for the RNG and GFR.
1.1 Random Geometric Graphs

A wireless ad hoc network is a collection of wireless devices (transceivers) located in a geographic region. Each node is equipped with an omnidirectional antenna and has limited transmission power. A communication session is established either through a single-hop radio transmission if the communication parties are close enough, or through relaying by intermediate devices otherwise. Since they have no need for a fixed infrastructure, wireless ad hoc networks can be flexibly deployed at low cost for varying missions such as decision making in the battlefield, emergency disaster relief and environmental monitoring.

In wireless ad hoc networks, due to the maximal transmission power, each node is associated with a maximal transmission radius. The network topology of a wireless ad hoc network is a graph in which two nodes have an edge between them if they are within each other’s transmission range. If all nodes have the same maximal transmission radius $r$, the induced network topology is exactly an $r$-disk graph in which two nodes have an edge if and only if the distance between them is at most $r$. The ad hoc wireless devices in many applications are randomly deployed in a large volume over the (finite) deployment region. Further, the wireless ad hoc devices in some other applications may be continuously in motion or be dynamically switched to on or off. Consequently, it is natural to represent the wireless network nodes by a random point process on a bounded region, and the induced $r$-disk graphs are called random geometric graphs.

Random graph have been studied as models of communication networks since 1950s. In the primitive random graph model proposed by Erdős and Rényi (1960) [7], each pair of devices has some probability (the same for all pairs of devices regardless of their separation) of being jointed by a channel. Such a model cannot represent accurately
a network of short-range nodes spread over a wide area because of the presence of local
correlation among the wireless links. Gilbert (1961) [10] proposed a simple random plane
geometric model for radio network in which the range of the network nodes is a parameter.
Gilbert’s model assumes that all the network nodes are represented by an infinite random
point process distributed on the entire plane and have the same maximum transmission
radius $r$. Two nodes have an edge between them if and only if their distance is at most
$r$ and the induced network topology is exactly an $r$-disk graph. More recently, a random
geometric model for wireless ad hoc networks which consist of finite radio nodes in a
bounded region, a bounded (or finite) variant of the Gilbert’s model has been adopted by
Gupta and Kumar (1998) [11] and other researchers. Under such a wireless ad hoc network
of finite number of nodes, the random point process representing the ad hoc devices is
usually assumed to be a uniform point process or a Poisson point process with density
$n$ over a unit-disk or a unit-square through proper scaling. With these assumptions, the
wireless ad hoc networks are exactly the $r$-disk graphs over a random point process (either
uniform or Poisson). Many recent progresses in random geometric graphs can be found
in the book by Penrose (2003) [27]. Throughout this thesis, we use $\mathcal{P}_n$ and $\mathcal{X}_n$ to denote a
Poisson point process with mean $n$ and a uniform $n$-point process over a unit-area disk
or a unit-square, respectively.

1.2 Critical Transmission Radius

Critical Transmission Radius for Relative Neighborhood Graph

Relative neighborhood graph (abbreviated by RNG) has been widely used in local-
ized topology control and geographic routing in wireless ad hoc and sensor networks. In
RNGs, two nodes have an edge between them if and only if there is no other node in the
intersection of two disks centered respectively at these two nodes and with the distance be-
tween them as their common radii. Assume all nodes have the same maximal transmission
radius \( r \). To construct the RNG by only 1-hop information, the transmission radius \( r \) should be large enough such that the RNG is a subgraph of the \( r \)-disk graph. Thus, the transmission radius should be not less than the maximum edge length of the RNG. On the other hand, for each node, if it can gather the information of all nodes that are not farther than its farthest neighbor in the RNG, it can decide all RNG edges incident to it. Therefore, the maximum edge length of the RNG is the smallest transmission radius for constructing the RNG by using only 1-hop neighbor information and it is referred to as the \textit{critical transmission radius} of the RNG. The maximum edge length of the RNG is the minimum requirement on the maximum transmission radius by those applications of RNG. In this thesis, we derive the precise asymptotic probability distribution of the maximum edge length of the RNG on a Poisson point process over a unit-area disk.

\textit{Critical Transmission Radius for Greedy Forward Routing}

Routing protocols in wireless ad hoc and sensor networks have been studied extensively in the past decade. The most relevant work includes various geographic routing algorithms. Greedy forward routing (abbreviated by GFR) is attractive in wireless ad hoc and sensor networks due to its efficiency and scalability. In GFR, each node discards a packet if none of its neighbors is closer to the destination of the packet than itself, or otherwise forwards the packet to the neighbor closest to the destination of the packet. Therefore, each node only need to maintain the locations of its one-hop neighbors and each packet should contain the location of the destination node. Thus, it can be implemented in a localized and memoryless manner. There are many variants of GFR. For example, in [32] and [40], the shortest projected distance to the destination on the straight line joining the current node and the destination node is considered as the greedy metrics. In [32], packets are allowed to be sent backward if there is no forwarding neighbor. In [40], only the nodes whose Voronoi cells intersect with the source-destination line segment are eligible. Some other variants of GFR can be found in [2], [4], [5], [8], [12], [14], [15], [17], [18], [19], [20], [29], [31].
However, GFR may discard a packet before it reaches its destination if the transmission radii of the nodes are small. To ensure that every packet can reach its destination, all nodes should have sufficiently large transmission radii. The *critical transmission radius* of a planar node set $V$ for greedy forward routing is the smallest transmission radius by $V$ which ensures successful delivery of any packet from any source node in $V$ to any destination node in $V$. The critical transmission radius of GFR is explicitly given by

$$\max_{u \in V} \min_{v \in V} \{ \|wu\| : \|wu\| \leq \|uv\|, w \in V \},$$

where $\|uv\|$ denotes the Euclidean distance between two nodes $u$ and $v$. In this thesis, we study the asymptotics of the critical transmission radius for GFR on a Poisson point process over a unit-area disk.

**1.3 Notation**

In what follows, $o$ is the origin of the Euclidean plane $\mathbb{R}^2$, $D$ is the unit-area (closed) disk centered at $o$, and $R_0$ the radius of $D$. We assume that $\mathcal{P}_n$ is the Poisson point process over $D$ with density $n$. We denote by $X_n = (X_1, \cdots, X_n)$ the uniform $n$-point process over $D$. An event is said to be *asymptotic almost sure* (abbreviated by a.a.s.) if it occurs with a probability converges to one as $n \to \infty$. An event is said to be *asymptotic almost rare* (abbreviated by a.a.r.) if it occurs with a probability converges to zero as $n \to \infty$. The symbols $O$, $\Omega$, $\Theta$, $o$, $\sim$, $\precsim$ and $\succeq$ always refer to the limit $n \to \infty$. To avoid trivialities, we tacitly assume $n$ to be sufficiently large if necessary. For simplicity of notation, the dependence of sets and random variables on $n$ will be frequently suppressed. For any finite set $S$, the cardinality of $S$ is denoted by $\text{card}(S)$. For any set $S$ and positive integer $k$, the $k$-fold Cartesian product of $S$ is denoted by $S^k$. The Euclidean norm of a point $x$ is denoted by $\|x\|$, and the Euclidean distance between two points $u$ and $v$ is denoted by $\|uv\|$. The Lebesgue measure (or area) of a measurable set $A \subset \mathbb{R}^2$ is denoted by $|A|$. The topological
boundary of a set $A \subset \mathbb{R}^2$ is denoted by $\partial A$. The open (respectively, closed) disk of radius $r$ centered at $x$ is denoted by $B(x, r)$ (respectively, $\overline{B}(x, r)$). All integrals considered will be Lebesgue integrals. For any finite planar set $V$, $K(V)$ denotes the complete (geometric) graph on $V$ which consists of line segments between all pairs of nodes in $V$.

1.4 Organization

The remaining of this thesis is organized as follows. In Chapter 2, we present several useful and important geometric inequalities which will be used in the proofs of the main results in this thesis. In Chapter 3, we derive the precise asymptotic distribution of the maximum edge length in the RNG of the Poisson point process over a unit-area disk. In Chapter 4, we obtain the precise asymptotic probability distribution for the critical transmission radius of GFR on the Poisson point process over a unit-area disk.
CHAPTER 2
GEOMETRIC INEQUALITIES

In this chapter, we shall present some useful and important geometric inequalities that will be used to prove the main theorems in this thesis. The results obtained in this chapter are purely geometric, with no probabilistic content. This Chapter is organized as follows. In Section 2.1, we present a partition of a unit-area disk. In Section 2.2, we give some geometric properties of a single lune or quasi-lune. In Section 2.3, we derive some geometric properties for which multiple lunes or quasi-lunes are involved. Lemma 8 is proved in Section 2.4. Lemma 9 is proved in Section 2.5.

2.1 Partition of A Unit-Area Disk

For \( x \in D \), let \( t(x) \) denote the distance between \( x \) and \( \partial D \), which is equal to \( \frac{1}{\sqrt{\pi}} - \|x\| \). For any \( 0 < \rho < \frac{1}{\sqrt{\pi}} \) (see Fig. 2.1), define

\[
D_\rho(0) = \{ x \in D : t(x) \geq \rho \},
\]

Figure 2.1. The partition of the unit-area disk \( D \).
\[ \mathbb{D}_\rho(1) = \left\{ x \in \mathbb{D} : \sqrt{\frac{1}{\pi} - \rho^2} \leq t(x) < \rho \right\}, \]
\[ \mathbb{D}_\rho(2) = \left\{ x \in \mathbb{D} : t(x) < \sqrt{\frac{1}{\pi} - \rho^2} \right\}. \]

For \( x \in \mathbb{D} \) and \( 0 < \rho < \frac{1}{\sqrt{\pi}} \), define \( \theta(x, \rho) \) as follows. If \( x \in \mathbb{D}_\rho(0) \), then \( \theta(x, \rho) = 2\pi \). If \( x \in \mathbb{D}_\rho(2) \), then \( \theta(x, \rho) = 0 \). If \( x \in \mathbb{D}_\rho(1) \), let \( u \) and \( v \) be the two intersection points of \( \partial B(x, \rho) \) and \( \partial \mathbb{D} \), and define \( \theta(x, \rho) = 2\pi - \angle u x v \) (see Fig. 2.2). We claim that \( \rho \theta(x, \rho) \leq 2\pi t(x) \). The claim holds trivially if \( x \in \mathbb{D}_\rho(0) \) or \( x \in \mathbb{D}_\rho(2) \). So, we consider the case that \( x \in \mathbb{D}_\rho(1) \). It's easy to see that \( \theta(x, \rho) \leq 4 \arcsin \frac{t(x)}{\rho} \). Using the equality \( \sin \alpha \geq \frac{2}{\pi} \alpha \) for any \( \alpha \in [0, \pi/2] \), we obtain

\[ \theta(x, \rho) \leq 4 \cdot \frac{\pi}{2} \cdot \frac{t(x)}{\rho} = \frac{2\pi t(x)}{\rho}. \]

Thus, \( \rho \theta(x, \rho) \leq 2\pi t(x) \).

2.2 Single Lune and Quasi-Lune

The lune of a line segment \( e = ab \), denoted by \( L(e) \) or \( L(a, b) \), is the intersection
of the two disks $B\left(a, \|ab\|\right)$ and $B\left(b, \|ab\|\right)$; $e$ is called the waist of the lune $L(e)$; the two intersection points $c$ and $d$ of $\partial B\left(a, \|ab\|\right)$ and $\partial B\left(b, \|ab\|\right)$ are called the vertices of the lune $L(e)$ (see Fig. 2.3a). It is easy to verify that

$$|L(a, b)| = \sigma \|ab\|^2,$$

where

$$\sigma = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}.$$ 

Thus, $\sigma$ is the area of the lune of two points with unit distance.

![Figure 2.3. Illustrations for lune $L(a, b)$ and quasi-lune $L_r(a, b)$.](image)

Given an ordered pair $(a, b) \in \mathbb{R}^2$. Suppose that $0 < r \leq \|ab\|$. Define

$$L_r(a, b) = B\left(a, r\right) \cap B\left(b, \|ab\|\right).$$

The set $L_r(a, b)$ is called a quasi-lune of the ordered pair $(a, b)$; the segment $ab$ is called the waist of $L_r(a, b)$; the two intersection points $c$ and $d$ of $\partial B\left(a, r\right)$ and $\partial B\left(b, \|ab\|\right)$ are called the vertices of the quasi-lune $L_r(a, b)$ (see Fig. 2.3b). The area of a quasi-lune is given in the following lemma.
Lemma 1 Let $0 < r \leq \|ab\|$. Then the area of $|L_r(a,b)|$ is given by

$$|L_r(a,b)| = 2 \|ab\|^2 \arcsin \frac{r}{2 \|ab\|} + r^2 \arccos \frac{r}{2 \|ab\|} - r^2 \sqrt{\left( \frac{\|ab\|}{r} \right)^2 - \frac{1}{4}}.$$  

Proof Let $c$ be the vertex of the quasi-lune $L_r(a,b)$ above segment $ab$ (see Fig. 2.4). Let $b'$ be the intersection point of the circle $\partial B(a,r)$ with the segment $ab$. Then $\|ab'\| = \|ac\| = r$. Let $\theta = \angle cab$. Then

$$\cos \theta = \frac{\|ac\|}{2 \|ab\|} = \frac{r}{2 \|ab\|}.$$  

Thus, the area of the sector $\angle cab'$ is

$$|\angle cab'| = \frac{1}{2} r^2 \theta = \frac{1}{2} r^2 \arccos \frac{r}{2 \|ab\|}.$$  

Figure 2.4. Area of the quasi-lune $L_r(a,b)$. 

Let $\varphi = \angle cba$. Since $\varphi + 2\theta = \pi$, we have

$$
\sin \varphi = \sin 2\theta = 2 \sin \theta \cos \theta = \frac{r}{2 \|ab\|^2} \sqrt{4 \|ab\|^2 - r^2}.
$$

Thus, the area of the triangle $\triangle cab$ is given by

$$
|\triangle cab| = \frac{1}{2} \|ab\|^2 \sin \varphi = \frac{1}{4} r \sqrt{4 \|ab\|^2 - r^2} = \frac{1}{2} r^2 \sqrt{\left(\frac{\|ab\|}{r}\right)^2 - \frac{1}{4}}.
$$

Since

$$
\sin \frac{\varphi}{2} = \cos \theta = \frac{r}{2 \|ab\|},
$$

we have $\varphi = 2 \arcsin \frac{r}{2 \|ab\|}$. Then, the area of the sector $\angle cba$ is

$$
|\angle cba| = \frac{1}{2} \|ab\|^2 \varphi = \frac{1}{2} \|ab\|^2 \cdot 2 \arcsin \frac{r}{2 \|ab\|} = \|ab\|^2 \arcsin \frac{r}{2 \|ab\|}.
$$

Note that $|L_r(a, b)| = 2 (|\angle cba| + |\angle cab'| - |\triangle cba|)$. Thus, the lemma holds.

We remark that the area formula for $|L_r(a, b)|$ in Lemma 1 also holds when $\|ab\| < r \leq 2 \|ab\|$. The proof for Lemma 1 still works, but $b'$ denotes the intersection point of the circle $\partial B(a, r)$ with the ray $ab$.

If a line segment $e \subset \mathbb{D}$ and the midpoint of $e$ is apart from $\partial \mathbb{D}$ by at least $\frac{\sqrt{3}}{2} \|e\|$. 
then $L(e) \subset \mathbb{D}$ and $|L(e) \cap \mathbb{D}| = |L(e)|$. The next lemma gives a lower bound on $|L(e) \cap \mathbb{D}|$ if otherwise.

![Figure 2.5](image)

**Figure 2.5.** $L(e) \cap \mathbb{D}$ contains the triangle $abd$ and the half lune $abu$.

**Lemma 2** Consider a line segment $e \subset \mathbb{D}$ with midpoint $z$. If $t(z) \leq \frac{\sqrt{3}}{2} \|e\|$, then

$$|L(e) \cap \mathbb{D}| \geq \frac{1}{2} |L(e)| + \frac{\|e\|}{2} t(z).$$

**Proof** Let $a$ and $b$ be the two endpoints of $e$, and $c$ and $d$ be the two vertices of $L(e)$ with $c$ being farther away from the center $o$ of $\mathbb{D}$ (see Figure 2.5). Then, the half lune $abd$ is fully contained in $\mathbb{D}$. If $c \in \partial \mathbb{D}$, then the triangle $abc$ is contained in $\mathbb{D}$ and its area is $\frac{\|c\|}{2} \|cz\| \geq \frac{\|e\|}{2} t(z)$. So, the lemma holds if $c \in \partial \mathbb{D}$. Now assume that $c \notin \partial \mathbb{D}$. Let $u$ be the intersection point of $cz$ and $\partial \mathbb{D}$. Then, the triangle $abu$ is fully contained in $\mathbb{D}$ and its area is $\frac{\|u\|}{2} \|uz\| \geq \frac{\|e\|}{2} t(z)$. So, the lemma also holds if $c \notin \partial \mathbb{D}$.

For quasi-lunes, we have similar results as lunes. If $ab \subset \mathbb{D}$, $\|ab\| < 1/\sqrt{\pi}$ and the midpoint of $ab$ is apart from $\partial \mathbb{D}$ by at least $\frac{\sqrt{3}}{2} \|ab\|$, then $L_r(a, b) \subset L(a, b) \subset \mathbb{D}$. Similar to Lemma 2 for lunes, the next lemma gives a lower bound on $|L_r(a, b) \cap \mathbb{D}|$ if otherwise.
Figure 2.6. $L_r (a, b) \cap \mathbb{D}$ contains the triangle $ab'u$ and the lower half of $L_r (a, b)$.

Lemma 3 Consider a line segment $ab \subset \mathbb{D}$ with $\|ab\| < 1/\sqrt{\pi}$. Let $z$ be the midpoint of $ab$. If $t (z) \leq \frac{\sqrt{3}}{2} \|ab\|$, then for any $\frac{2}{3} \|ab\| < r \leq \|ab\|$, we have

$$|L_r (a, b) \cap \mathbb{D}| \geq \frac{1}{2} |L_r (a, b)| + \frac{r}{4} t (z).$$

Proof Let $c$ and $d$ be the two vertices of $L (a, b)$ with $c$ being farther away from the center of $\mathbb{D}$ (see Figure 2.6). Then, the half lune $abd$ of $L (a, b)$ is fully contained in $\mathbb{D}$. Thus, the half quasi-lune of $L_r (a, b)$ which is on the same side of $ab$ as $d$ is also fully contained in $\mathbb{D}$. Let $w$ be the intersection point of $cz$ and $\partial L_r (a, b)$. Then,

$$\|wz\| = \sqrt{r^2 - \left(\frac{1}{2} \|ab\|\right)^2} \geq \sqrt{\left(\frac{2}{3} \|ab\|\right)^2 - \left(\frac{1}{2} \|ab\|\right)^2} = \frac{\sqrt{7}}{6} \|ab\|.$$
Let $b'$ be the point on the segment $ab$ satisfying that $\|ab\| = r$. If $w \in \mathbb{D}$, then the triangle $ab'w$ is contained in $\mathbb{D}$ and its area is $\frac{r}{2} \| wz \| \geq \frac{r}{4} t(z)$. So, the lemma holds if $w \in \mathbb{D}$. Now assume that $w \notin \mathbb{D}$. Let $u$ be the intersection point of $cz$ and $\partial \mathbb{D}$. Then, the triangle $ab'u$ is contained in $\mathbb{D}$ and its area is $\frac{r}{2} \| uz \| \geq \frac{r}{4} t(z)$. So, the lemma also holds if $w \notin \mathbb{D}$.

Let $b'$ be the point on the segment $ab$ satisfying that $\|ab\| = r$. Then,

$$L(a,b') \subseteq L_r(a,b) \subseteq L(a,b).$$

The following Lemma 4 gives upper bounds of the area differences between the lune $L(a,b)$ and a quasi-lune $L_r(a,b)$ or a smaller lune $L(a,b')$ contained in it.

Figure 2.7. The area differences between the lune $L(a,b)$ and a quasi-lune $L_r(a,b)$ or a smaller lune $L(a,b')$ contained in it.
Lemma 4 Assume $r < \|ab\| \leq (1 + \varepsilon)r$ for some $\varepsilon > 0$. Let $a'$ and $b'$ be the two points on the segment $ab$ satisfying that $\|ab'\| = \|a'b\| = r$.

(1) Assume $\varepsilon < 0.1$, then $|L(a, b) \setminus L_r(a, b)| \leq \frac{\pi}{2.7} (\|ab\|^2 - r^2)$;

(2) Assume $\varepsilon < 0.001$, then $|L(a, b) \setminus L_r(a, b)| \leq 2.2\varepsilon(r + r\varepsilon)^2$; in addition, if $\|aa'\| = \|b'b\| \leq \varepsilon r$, then $|L(a, b) \setminus L_r(a', b')| \leq 8\varepsilon(r + r\varepsilon)^2$.

Proof (1) Let $c'$ be a vertex of $L_r(a, b)$ (see Fig. 2.7). Since $1 + \varepsilon < 1.1$, we have

$$\angle c'ab = \arccos \frac{r}{2\|ab\|} \leq \arccos \frac{1}{2.2} \leq \frac{\pi}{2.7}.$$ 

So,

$$|L(a, b) \setminus L_r(a, b)| \leq (\|ab\|^2 - r^2) \cdot \angle c'ab \leq \frac{\pi}{2.7} (\|ab\|^2 - r^2).$$

(2) Since $\varepsilon < 0.001$, by (1) we have

$$\angle c'ab = \arccos \frac{r}{2\|ab\|} \leq \arccos \frac{1}{2.002}.$$ 

$$|L(a, b) \setminus L_r(a, b)| \leq (\|ab\|^2 - r^2) \angle c'ab < 2.2\varepsilon(r + r\varepsilon)^2.$$ 

Since $\|a'b'\| \geq r - 2r\varepsilon$, we have

$$|L(a, b) \setminus L(a', b')| = \frac{\pi}{\beta_0} (\|ab\|^2 - \|a'b'\|^2) \leq \frac{\pi}{\beta_0} r^2(1 + \varepsilon)^2 - \frac{\pi}{\beta_0} r^2(1 - 2\varepsilon)^2 < 8\varepsilon(r + r\varepsilon)^2.$$
Similarly, if \( ||aa'|| = ||bb'|| \leq 2.07\epsilon r \), then

\[ |L(a, b) \setminus L(a', b')| < 12.7\epsilon(r + \epsilon)^2. \]

This completes the proof of the lemma.

The next lemma gives the bounds of some useful angles which are related to a lune and a quasi-lune.

Figure 2.8. Some useful angles which are related to a lune and a quasi-lune.

**Lemma 5** Let \( p \in \partial B(a, r) \setminus B(b, r) \), \( h \) is on the ray \( ab \) and \( u, v \in \partial B(a, ||ab||) \). Assume \( r < ||ab|| \leq (1 + \epsilon)r \) for some \( \epsilon < 0.001 \). (see Fig. 2.8)

1. If \( ||pb|| \in [r, (1 + \epsilon)r] \), then \( \max\{\angle pab, \angle pb'a\} < 60.1^\circ \) and \( \angle pba > 59.9^\circ \);
2. If \( ||ph|| \in [r, (1 + \epsilon)r] \), then \( \anglepha > 59.8^\circ \);
3. If \( ||uv|| < 0.03||ab|| \), then \( \angleuva > 89^\circ \);
(4) If $\|ab\| \geq r(1 - 2.07\varepsilon)$, then $\angle uab > 59.5^\circ$.

**Proof** (1) Let $\varphi = \angle pab$, $\psi = \angle pba$, $\phi = \angle pb'a$. Clearly, $\psi < 60^\circ$. By the law of sine we have

$$\frac{\sin \varphi}{\sin \psi} = \frac{||pb||}{||pa||} \leq \frac{(1 + \varepsilon)r}{r} = 1 + \varepsilon < 1.001,$$

$$\sin \angle b'pb = \frac{||b'b||}{||pb||} \sin \phi \leq \frac{\varepsilon r}{r} < 0.001.$$

Then $\varphi < 60.1^\circ$ since

$$\sin \varphi < 1.001 \sin \psi < 1.001 \sin 60^\circ < \sin 60.1^\circ.$$

Also, $\phi = \psi + \angle b'pb < 60.1^\circ$. Similarly, we have $\psi > 59.9^\circ$.

(2) From part (1) we have $\angle pah > 59.9^\circ$. Then

$$\sin \angle pha = \frac{||pa||}{||ph||} \sin \angle pah > \frac{r}{R} \sin 59.9^\circ > \frac{1}{1.001} \sin 59.9^\circ > \sin 59.8^\circ,$$

which implies that $\angle pha > 59.8^\circ$.

(3) $\angle uva = \arccos \frac{||uv||}{2||ab||} > \arccos 0.015 > 89^\circ$.

(4) $\angle uab \geq 2 \arcsin \frac{||ab||}{2||ab||} \geq 2 \arcsin 0.498 > 59.5^\circ$. 
This completes the proof of Lemma 5.

The following lemma derives two useful asymptotic equalities which will be used in Chapter 4.

![Figure 2.9. Lunes and quasi-lunes for two asymptotic equalities.](image)

**Lemma 6** Given a constant \( c \) and a sequence of numbers \( (c_n : n \geq 1) \) satisfying that \( c_n \to \infty \) but \( c_n = o(\ln \ln n) \), let \( \sigma = \frac{2\pi}{3} - \frac{\sqrt{3}}{2} \), \( r_n = \sqrt{\frac{\ln n + c}{\sigma n}} \) and \( R_n = \sqrt{\frac{\ln n + c_n}{\sigma n}} \). If \( r_n < \|ab\| \leq R_n \), then

\[
\begin{align*}
|L(a,b) \setminus L_{r_n}(a,b)| &= \frac{\pi}{3} \left( \|ab\|^2 - r_n^2 \right) + o \left( \frac{1}{n} \right), \\
|L(a,b) \setminus (L_{r_n}(a,b) \cup L_{r_n}(b,a))| &= o \left( \frac{1}{n} \right).
\end{align*}
\]

**Proof** Let \( u \) be a vertex of \( L(a,b) \), \( v \) be a vertex of \( L_r(a,b) \) and \( w \) be a vertex of \( L_r(b,a) \) satisfying that \( u, v \) and \( w \) lies on the same side of \( ab \) (see Fig. 2.9). Let \( \alpha = \angle uvw \). Then,
\( \angle uaw = \angle ubv = 2\alpha \), and hence \( \angle vaw = 3\alpha \). We first show that

\[
\alpha \leq \frac{\pi}{2\sqrt{3}} \left( 1 - \frac{r_n}{R_n} \right).
\]

Since

\[
\angle vab = \arccos \frac{\|av\|}{2\|ab\|} \leq \arccos \frac{r_n}{2R_n},
\]

we have,

\[
\alpha = \angle vab - \angle uab \leq \arccos \frac{r_n}{2R_n} - \frac{\pi}{3}.
\]

Thus,

\[
\sin \alpha \leq \sin \left( \arccos \frac{r_n}{2R_n} - \frac{\pi}{3} \right)
= \sqrt{4R_n^2 - r_n^2 - \sqrt{3} r_n}
= \frac{4R_n}{R_n^2 - r_n^2}
= \frac{R_n}{\sqrt{4R_n^2 - r_n^2 + \sqrt{3} r_n}}
\leq \frac{R_n}{\sqrt{4R_n^2 - R_n^2 + \sqrt{3} r_n}}
= \frac{1}{\sqrt{3}} \left( 1 - \frac{r_n}{R_n} \right).
\]

Hence,

\[
\alpha \leq \frac{\pi}{2} \sin \alpha \leq \frac{\pi}{2\sqrt{3}} \left( 1 - \frac{r_n}{R_n} \right).
\]

Now, we show that

\[
\left( 1 - \frac{r_n}{R_n} \right) \left( R_n^2 - r_n^2 \right) = o \left( \frac{1}{n} \right).
\]
Indeed,

\[
n \left(1 - \frac{r_n}{R_n}\right) \left(R_n^2 - r_n^2\right) = \frac{n (R_n^2 - r_n^2)^2}{R_n (R_n + r_n)} = O\left(1\right) \frac{(n\pi (R_n^2 - r_n^2))^2}{n\pi R_n^2} = O\left(1\right) \frac{(c_n - c)^2}{\ln n + \xi_n} = O\left(1\right) \frac{c_n^2}{\ln n} = o\left(1\right).
\]

Finally, the first equality in the lemma follows from

\[
|L (a, b) \setminus L_{r_n} (a, b)| \geq \frac{\pi}{3} (\|ab\|^2 - r_n^2),
\]

and

\[
|L (a, b) \setminus L_{r_n} (a, b)| \leq \left(\frac{\pi}{3} + \alpha\right) (\|ab\|^2 - r_n^2) = \frac{\pi}{3} (\|ab\|^2 - r_n^2) + \alpha (\|ab\|^2 - r_n^2) \leq \frac{\pi}{3} (\|ab\|^2 - r_n^2) + \frac{\pi}{2\sqrt{3}} \left(1 - \frac{r_n}{R_n}\right) (R_n^2 - r_n^2) = \frac{\pi}{3} (\|ab\|^2 - r_n^2) + o\left(1 \frac{n}{n}\right).
\]

The second equality in the lemma follows from

\[
|L (a, b) \setminus (L_{r_n} (a, b) \cup L_{r_n} (b, a))| \leq 3\alpha (R_n^2 - r_n^2)
\]
\[
\leq \frac{\sqrt{3}\pi}{2} \left( 1 - \frac{r_n}{R_n} \right) \left( R_n^2 - r_n^2 \right) \\
= o \left( \frac{1}{n} \right).
\]

This completes the proof of the lemma.

### 2.3 Multiple Compatible Lunes and Quasi-Lunes

In this section, we establish some important geometric inequalities for multiple compatible lunes and quasi-lunes. First we consider the case in which only two lunes (or quasi-lunes) are involved. Two line segments (or the corresponding two lunes) are said to be compatible if the two endpoints of either segment is not contained in the lune of the other segment.

We start with a lemma which was proved in [34] (Lemma 2 of this paper).

**Lemma 7** Suppose that \(e_1\) and \(e_2\) are two compatible segments satisfying that \(\|e_1\|, \|e_2\| \in \left[ \frac{1}{2} R, R \right]\) for some \(R > 0\). Let \(z_1\) and \(z_2\) be respectfully the midpoints of \(e_1\) and \(e_2\) satisfying that \(\|z_1z_2\| \leq \sqrt{3} R\). Then

\[
| L(e_2) \setminus L(e_1) | \geq 0.039 R \|z_1z_2\|. 
\]

In addition, if the segment \(e_2\) intersects with both sides of \(L(e_1)\), then

\[
| H \setminus L(e_1) | \geq 0.039 R \|z_1z_2\|, 
\]

where \(H\) is the half lune of \(L(e_2)\) divided by the segment \(e_2\) which is on the different side of \(e_2\) from \(z_1\).
The next lemma generalizes Lemma 7 by taking the boundary effect into account. Its proof is very complicated and lengthy. We postpone its proof to Section 2.4 of this chapter.

**Lemma 8** Assume \( \varepsilon < 0.001 \). Suppose that \( e_1 \) and \( e_2 \) are two compatible segments in \( \mathbb{D} \) satisfying that \( \|e_1\|, \|e_2\| \in \left[ \frac{1}{1+\varepsilon} R, R \right] \) for some \( R \leq \frac{1}{200\sqrt{\pi}} \). Let \( z_1 \) and \( z_2 \) be the midpoints of \( e_1 \) and \( e_2 \), respectively. If \( \|z_1z_2\| \leq \sqrt{3}R \) and \( z_1 \) is farther away from the center of \( \mathbb{D} \) than \( z_2 \), then

\[
|\left(L_r(e_2) \setminus L_r(e_1)\right) \cap \mathbb{D}| \geq 0.0026R \|z_1z_2\|.
\]

Suppose that the four points \( a_1, b_1, a_2 \) and \( b_2 \) are on the plane \( \mathbb{R}^2 \). For each \( i = 1 \) and \( 2 \), let \( E_i \) denote the set of ordered pairs of \( \{(a_i, b_i), (b_i, a_i)\} \). Given two directed arcs \( e_1 \in E_1 \) and \( e_2 \in E_2 \). Let \( 0 < r \leq 2 \min\{\|e_1\|, \|e_2\|\} \). The two directed arcs \( e_1 \) and \( e_2 \) (or the corresponding two quasi-lunes) are said to be \( r \)-compatible if none of the four points \( a_1, b_1, a_2 \) and \( b_2 \) is contained in the union of two quasi-lunes \( L_r(e_1) \cup L_r(e_2) \).

Similar to Lemma 8 for lunes, the next lemma generates Lemma 8 to quasi-lunes and gives a lower bound for the area of the part of the difference \( L_r(e_2) \setminus L_r(e_1) \) contained in the unit-area disk \( \mathbb{D} \). The proof of Lemma 9 is also very complicated and lengthy. We postpone its proof to Section 2.5 of this chapter.

**Lemma 9** Given \( \varepsilon < 0.001 \) and \( 0 < R \leq \frac{1}{200\sqrt{\pi}} \). Let \( r = \frac{1}{1+\varepsilon} R \). Suppose that \( e_1 \in E_1 \) and \( e_2 \in E_2 \) are two \( r \)-compatible directed arcs in \( \mathbb{D} \) satisfying that \( \|e_1\|, \|e_2\| \in [r, R] \). Let \( z_1 \) and \( z_2 \) be the midpoints of \( e_1 \) and \( e_2 \) respectfully. If \( \|z_1z_2\| \leq \sqrt{3}R \) and \( z_1 \) is farther away from the center of \( \mathbb{D} \) than \( z_2 \), then

\[
|\left(L_r(e_2) \setminus L_r(e_1)\right) \cap \mathbb{D}| \geq 0.0026R \|z_1z_2\| - 16\varepsilon R^2.
\]
Next we consider the general case in which multiple lunes (or quasi-lunes) are involved. For any line segment \( e \), we define
\[
\nu(e) = |L(e) \cap \mathbb{D}|.
\]
For any geometric graph \( H \), define
\[
\nu(H) = |(\bigcup_{e \in H} L(e)) \cap \mathbb{D}|,
\]
and \( \chi(H) \) to be indicator for all edges of \( H \) are pairwise compatible. An edge \( e \in E \) is called an outermost edge of \( H \) if its midpoint is the nearest to \( \partial \mathbb{D} \). For any finite planar set \( V \) and any positive number \( r \), the \( r \)-disk graph of \( V \) is a geometric graph over \( V \) in which there is an edge between two nodes if and only if their distance is at most \( r \).

The next lemma generalizes the result of Lemma 8 when more than two lunes are involved.

**Lemma 10** Assume \( \varepsilon < 0.001 \). Suppose that \( H \) is a geometric graph over a finite subset of \( \mathbb{D} \) with at least two edges satisfying that
1. \( \chi(H) = 1 \),
2. all the edges have lengths between \( \frac{1 + \varepsilon}{2} R \) and \( R \) for some \( R \leq \frac{1}{200 \sqrt{\pi}} \), and
3. the midpoints of its edges induce a connected \( \sqrt{3}R \)-disk graph. Let \( e \) be an outermost edge of \( H \), and \( \ell \) be the largest distance between the midpoint of \( e \) and the midpoints of other edges of \( H \). Then,
\[
\nu(H) \geq \nu(e) + 0.0026 R \ell.
\]

**Proof** Let \( e' \) be the edge of \( H \) whose midpoint \( z_k \) is the farthest from the midpoint \( z_1 \) of \( e \). Let \( P = z_1 z_2 \cdots z_k \) be the min-hop path between \( z_1 \) and \( z_k \) in the \( \sqrt{3}R \)-disk graph over the midpoints of the edges in \( H \). For each \( 1 \leq i \leq k \), let \( e_i \) denote the edge of \( H \) whose midpoint is \( z_i \). Then, \( e_1 = e \) and \( e_k = e' \). For each \( 2 \leq j \leq k \), let \( H_j \) denote the subgraph
of $H$ consisting of the edges $e_i$ for $1 \leq i \leq j$. We will prove by induction on $j$ with $2 \leq j \leq k$ that

$$\nu(H_j) \geq \nu(e_1) + 0.0026R \sum_{i=1}^{j-1} \|z_i z_{i+1}\|. \quad (2.1)$$

By Lemma 8, the inequality (2.1) holds when $j = 2$. Since $P$ is the min-hop path, $\|z_1 z_3\| > \sqrt{3}R$ and $L(e_3)$ is disjoint from $L(e_1)$. Thus,

$$\nu(H_3) \geq \nu(e_1) + \nu(e_3)$$

$$\geq \nu(e_1) + \frac{1}{2} |L(e_3)| \geq \nu(e_1) + \frac{\pi \|e_3\|^2}{2\beta_0^2}$$

$$\geq \nu(e_1) + \frac{\pi R^2}{8\beta_0^2} \geq \nu(e_1) + \frac{\pi}{16\sqrt{3}\beta_0^2} R \cdot 2\sqrt{3}R$$

$$\geq \nu(e_1) + 0.044R \sum_{i=1}^{2} \|z_i z_{i+1}\|.$$

Hence, the inequality (2.1) holds when $j = 3$. Next, we assume that $j > 3$. Since $L(e_j)$ is disjoint from each $L(e_i)$ with $1 \leq i \leq j - 2$, we have

$$\nu(H_j) \geq \nu(H_{j-2}) + \nu(e_j).$$

Apply the induction hypothesis on $\nu(H_{j-2})$, we have

$$\nu(H_j) \geq \nu(e_1) + 0.0026R \sum_{i=1}^{j-3} \|z_i z_{i+1}\| + \frac{1}{2} |L(e_j)|$$

$$\geq \nu(e_1) + 0.0026R \sum_{i=1}^{j-3} \|z_i z_{i+1}\| + \frac{\pi R^2}{8\beta_0^2}$$

$$\geq \nu(e_1) + 0.0026R \sum_{i=1}^{j-3} \|z_i z_{i+1}\| + 0.044R \cdot 2\sqrt{3}R$$

$$> \nu(e_1) + 0.0026R \sum_{i=1}^{j-1} \|z_i z_{i+1}\|.$$

Thus, the inequality (2.1) holds. By the principle of induction, the inequality (2.1) holds
for every $2 \leq j \leq k$.

Since $\nu(H) \geq \nu(H_k)$ and

$$\sum_{i=1}^{k-1} \|z_i z_{i+1}\| \geq \|z_1 z_k\| = \ell,$$

the lemma holds.

Let $0 < r \leq \|e\|$, we define

$$\nu_r(e) = |L_r(e) \cap \mathbb{D}|.$$

For any directed geometric graph $H$ and let $0 < r \leq \min\{\|e\| : e \in H\}$, define

$$\nu_r(H) = |(\cup_{e \in H} L_r(e)) \cap \mathbb{D}|,$$

and $\chi_r(H)$ to be indicator for all edges of $H$ are pairwise $r$-compatible. An edge $e \in E(H)$ is called an outermost edge of $H$ if its midpoint is the nearest to $\partial \mathbb{D}$.

The next lemma generalizes the result of Lemma 9 when when more than two quasi-lunes are involved.

**Lemma 11** Suppose that $0 < R \leq \frac{1}{200\sqrt{\pi}}$ and $r \in \left[\frac{1}{1+\varepsilon} R, R\right]$. Consider a geometric graph $H$ over a finite subset of $\mathbb{D}$ with at least two edges satisfying that all the edges have length between $r$ and $R$ and the midpoints of its edges induce a connected $\sqrt{3}R$-disk graph. Let $e_1$ be an outermost edge of $H$, and $\ell$ be the largest distance between the midpoint of $e_1$ and the midpoints of other edges of $H$. Then,

$$\sum_{e \in E(H)} \nu_r(e) \geq \nu_r(e_1) + 0.0026R\ell - 16\varepsilon R^2.$$
In addition, if $\chi_r (H) = 1$ then

$$
\nu_r (H) \geq \nu_r (e_1) + 0.0026 R \ell - 16 \varepsilon R^2.
$$

**Proof** Let $e'$ be the edge of $H$ whose midpoint $z_k$ is the farthest from the midpoint $z_1$ of $e_1$. Let $P = z_1 z_2 \cdots z_k$ be the minimum-hop path between $z_1$ and $z_k$ in the $\sqrt{3}R$-disk graph over the midpoints of the edges in $H$. For each $1 \leq i \leq k$, let $e_i$ denote the edge of $H$ whose midpoint is $z_i$. Then, $e_k = e'$. For each $2 \leq j \leq k$, let $H_j$ denote the subgraph of $H$ consisting of the edges $e_i$ for $1 \leq i \leq j$. We will prove by induction on $j$ with $2 \leq j \leq k$ that

$$
\nu_r (H_j) \geq \nu_r (e_1) + 0.0026 R \sum_{i=1}^{j-1} \| z_i z_{i+1} \| - 16 \varepsilon R^2.
$$

(2.2)

Note that for any line segment $ab$ with the point $b'$ on the segment satisfying that $\| ab' \| = r$, we have

$$
L (a, b') \subseteq L_r (a, b) \subseteq L (a, b).
$$

Then

$$
| L_r (a, b) | \geq | L (a, b') | = \frac{\pi r^2}{\beta_0^2}.
$$

(2.3)

By Lemma 6, the inequality (2.2) holds when $j = 2$. Since $P$ is the minimum-hop path, $\| z_1 z_3 \| > \sqrt{3}R$ and $L (e_3)$ is disjoint from $L (e_1)$. Thus by Lemma 3 and EQ(2.3), we have

$$
\nu_r (H_3) \geq \nu_r (e_1) + \nu_r (e_3)
\geq \nu_r (e_1) + \frac{1}{2} | L_r (e_3) |
\geq \nu_r (e_1) + \frac{\pi r^2}{2 \beta_0^2}
\geq \nu_r (e_1) + \frac{\pi R^2}{2 \beta_0^2} \left( \frac{1}{1 + \varepsilon} \right)^2.
$$
\[
\begin{align*}
\geq \nu_r (e_1) + \frac{\pi R^2}{8\beta_0} \\
\geq \nu_r (e_1) + \frac{\pi}{16\sqrt{3}\beta_0^2} R \cdot 2\sqrt{3}R \\
\geq \nu_r (e_1) + 0.044R \sum_{i=1}^{2} \|z_i z_{i+1}\|.
\end{align*}
\]

Hence, the inequality (2.2) holds when \( j = 3 \). Next, we assume that \( j > 3 \). Since \( P \) is the minimum-hop path, \( \|z_i z_j\| > \sqrt{3}R \) and \( L(e_j) \) is disjoint from \( L(e_i) \) for each \( i \) between 1 and \( j - 2 \), we have

\[
\nu_r (H_j) \geq \nu_r (H_{j-2}) + \nu_r (e_j).
\]

Apply the induction hypothesis on \( \nu_r (H_{j-2}) \), we have

\[
\nu_r (H_j) \geq \nu_r (e_1) + 0.0026R \sum_{i=1}^{j-3} \|z_i z_{i+1}\| - 16\varepsilon R^2 + \nu_r (e_j)
\]

\[
\geq \nu_r (e_1) + 0.0026R \sum_{i=1}^{j-3} \|z_i z_{i+1}\| - 16\varepsilon R^2 + \frac{\pi R^2}{8\beta_0^2}
\]

\[
\geq \nu_r (e_1) + 0.0026R \sum_{i=1}^{j-3} \|z_i z_{i+1}\| - 16\varepsilon R^2 + 2 \left(0.044R \cdot \sqrt{3}R\right)
\]

\[
> \nu_r (e_1) + 0.0026R \sum_{i=1}^{j-1} \|z_i z_{i+1}\| - 16\varepsilon R^2.
\]

Thus, the inequality (2.2) holds. By the principle of induction, the inequality (2.2) holds for every \( 2 \leq j \leq k \).

Note that

\[
\sum_{e \in E(H)} \nu_r (e) \geq \nu_r (H_k)
\]

and if \( \chi_r (H) = 1 \), then we also have \( \nu_r (H) \geq \nu_r (H_k) \). Since

\[
\sum_{i=1}^{k-1} \|z_i z_{i+1}\| \geq \|z_1 z_k\| = \ell,
\]
the lemma holds.

2.4 Proof of Lemma 8

This section is dedicated to the proof of Lemma 8. Let \( R_0 = \frac{1}{\sqrt{\pi}} \) denote the radius of \( \mathbb{D} \), \( c = 0.0026 \) and \( r = \frac{1}{3+\varepsilon} R \). For each \( i = 1, 2 \), assume \( e_i = a_i b_i \), \( r_i = \|a_i b_i\| \). Then \( r_1, r_2 \in [r, R] \) and \( r > \frac{1}{1.001} R \) as \( \varepsilon < 0.001 \). By Lemma 7, Lemma 8 holds if \( L_{a_2b_2} \subset \mathbb{D} \).

So we assume \( L_{a_2b_2} \cap \mathbb{D} \neq \emptyset \). Let

\[
A = |\mathbb{D} \cap L_{a_2b_2} \setminus L_{a_1b_1}|.
\]

For any \( u, v \in \partial \mathbb{D} \), let \( C_{uv} \) denote the circular cap surrounded by the segment \( uv \) and the arc \( uv \) such that \( |C_{uv}| \leq \frac{1}{2} |\mathbb{D}| \). Since \( e_1 \) and \( e_2 \) are two compatible segments, we have \( a_2, b_2 \notin L_{a_1b_1} \) and \( a_1, b_1 \notin L_{a_2b_2} \).

We will prove stronger results than that of Lemma 8 in this section. For the results stated in Lemma 20, we relax the constraint \( a_1, b_1 \notin L_{a_2b_2} \) and allow exactly one of \( a_1 \) and \( b_1 \) to be inside of \( L_{a_2b_2} \) with the distance from the node inside to \( \partial L_{a_2b_2} \) at most \( 2.07r\varepsilon \). The results of Lemma 20 will be used in the proof of Lemma 9 in Section 2.5. But throughout this section, we follow the constraint \( a_2, b_2 \notin L_{a_1b_1} \).

We start with the following lemma which shows that segment \( a_2b_2 \) does not intersect segment \( a_1b_1 \) if the constraint \( a_1, b_1 \notin L_{a_2b_2} \) is followed.

**Lemma 12** If \( a_1, b_1 \notin L_{a_2b_2} \), then segment \( a_2b_2 \) does not intersect segment \( a_1b_1 \). In addition, \( a_2b_2 \) does not intersect with line \( a_1b_1 \) if \( a_2b_2 \) intersects with \( \partial L_{a_1b_1} \).

**Proof** We prove by contradiction and assume the contrary that \( a_2b_2 \) intersects with segment
Figure 2.10. Segment $a_2b_2$ intersects with segment $a_1b_1$.

Figure 2.11. Segment $a_2b_2$ intersects with both segment $a_1b_1$ and $\partial L_{a_1b_1}$. 
Figure 2.12. Segment $a_2b_2$ intersects with segment $a_1b_1$, but not with $\partial L_{a_1b_1}$.

$a_1b_1$ (see Fig. 2.10). Since

$$\max\{\angle b_2a_1a_2, \angle a_1a_2b_1, \angle a_2b_1b_2, \angle b_1b_2a_1\} \geq 90^\circ,$$

we assume $\angle b_1b_2a_1 \geq 90^\circ$ w.l.o.g. Then $b_2 \in L_{a_1b_1}$, which leads to a contradiction. Next assume $b_1$ is closer to line $a_2b_2$ than $a_1$ w.l.o.g., let $l_1$ denote the line tangent to $\partial L_{a_1b_1}$ at $b_1$ (see Fig. 2.11, 2.12). We claim both $a_2$ and $b_2$ are on the different side of $l_1$ from $a_1$ if $a_2b_2$ intersects with line $a_1b_1$. The claim is verified as follows: both $a_2$ and $b_2$ cannot be on the same side of $l_1$ as $a_1$ since $a_2b_2$ does not intersect segment $a_1b_1$. If one of $a_2$ and $b_2$ are on the same side of $l_1$ as $a_1$, then $\angle a_2b_1b_2 > 90^\circ$ and $b_1 \in L_{a_2b_2}$, which leads to a contradiction. Thus the claim is true. If $a_2b_2$ intersects with $\partial L_{a_1b_1}$, then $a_2$ and $b_2$ are on different sides of $l_1$, which also leads to a contradiction. Therefore, the claim holds and the lemma follows.

The following simple lemma will used frequently in the geometric proofs in this section and Section 2.5.
Lemma 13 \ Let \( c_1 \) denote the vertex of \( L_{a_1b_1} \) above \( a_1b_1 \). Assume \( m \in \text{arc } a_1c_1 \). For any point \( p \) above both line \( a_1b_1 \) and line \( b_1c_1 \), we have \(|c_1p| < |mp| < |a_1p|\) (see Fig. 2.13).

Proof \ Apply the law of cosine to \( \triangle c_1b_1p \) and \( \triangle mb_1p \), we have \(|c_1p| < |mp|\). Similarly, we have \(|mp| < |a_1p|\). Thus, the lemma is proved.

The following lemma shows that if segment \( a_2b_2 \) intersects with \( \partial L_{a_1b_1} \) at most one point and the constraint \( a_1, b_1 \notin L_{a_2b_2} \) is followed, then Lemma 8 can be proved easily by using only the half lune of \( L_{a_2b_2} \) which intersects with the lune \( L_{a_1b_1} \).

Lemma 14 \ Assume \( a_1, b_1 \notin L_{a_2b_2} \), segment \( a_2b_2 \) intersects with \( \partial L_{a_1b_1} \) at most one point. Let \( S \) denote the half lune of \( L_{a_2b_2} \) divided by \( a_2b_2 \) such that \( S \cap L_{a_1b_1} \neq \emptyset \), then \(|S \setminus L_{a_1b_1}| \geq 0.25R^2| \).

Proof \ Let \( A_1 = |S \setminus L_{a_1b_1}| \). It is easy to see that \( r_2 = r \) when \( A_1 \) is minimal. Otherwise, if \( r_2 > r \), we can shrink \( L_{a_2b_2} \) a little bit so that \( A_1 \) is strictly smaller after shrinking,
Figure 2.14. Segment $a_2b_2$ intersects with the boundary $\partial L_{a_1b_1}$ at most one point.

Figure 2.15. Segment $a_2b_2$ intersects with the boundary $\partial L_{a_1b_1}$ only at $c_1$. 
contradiction. So we assume $r_2 = r$. Let $c_1$ denote the vertex of $L_{a_1b_1}$ which is on the same side of line $a_1b_1$ as $a_2$, and $d_2$ denote the vertex of $L_{a_2b_2}$ which is on the same side of line $a_2b_2$ as $b_1$. First we consider three special cases:

(1) $a_2b_2$ is tangent to $\partial L_{a_1b_1}$. Assume $a_2b_2$ is tangent to arc $c_1b_1$ at $h$ w.l.o.g. (see Fig. 2.14). Let $||a_2h|| = \rho r_2$, then

$$\tan \angle a_2a_1h = \frac{\rho r_2}{r_1},$$
$$\tan \angle b_2a_1h = \frac{r_2(1-\rho)}{r_1}.$$

Let $a_1a_2$, $a_1b_2$ intersect $\partial B(a_1, r_1)$ at $m, n$, respectively. Then $A_2 = |\triangle a_2a_1b_2 \setminus \angle ma_1n|$ is minimal as $r_1 = R$ when $\rho$ is fixed. Thus,

$$A_2 \geq \frac{1}{2} rR - \frac{1}{2} R^2(\arctan \frac{\rho}{2} + \arctan \frac{1-\rho}{2}),$$

which is minimal as $\rho = \frac{1}{2}$. Hence,

$$A_1 \geq A_2 \geq 0.25 R^2.$$

(2) $c_1 \in a_2b_2$ (see Fig. 2.15). Fix $c_1$ and rotate $a_2b_2$ clockwise (respectively, counterclockwise) until it is tangent to arc $c_1b_1$ (respectively, arc $c_1a_1$), let $a'_2, b'_2$ (respectively, $a''_2, b''_2$) denote the new positions of $a_2, b_2$ after rotating, respectively. Let $a_1a'_2$ and $a_1b'_2$ intersect with $\partial B(a_1, r_1)$ at $m$ and $n$, respectively, $b_1a''_2$ intersect with $\partial B(b_1, r_1)$ at $m'$. It is easy to see that $b'_2, a''_2 \in S$. By symmetry,

$$|\triangle a'_2a_1c_1 \setminus \angle ma_1c_1| = |\triangle a''_2b_1c_1 \setminus \angle m'b_1c_1|. $$
Then

$$A_1 \geq |\triangle a'_2 a_1 b'_2 \setminus m a_1 n| \geq 0.25R^2.$$ 

(3) $b_1, d_2$ are on the same side of line $c_1 z_1$, $b_1 \in \partial L_{a_2 b_2}$ and $d_2 \notin L_{a_1 b_1}$ (see Fig. 2.16). Since

$$|C_{b_2 d_2}| \geq |\triangle b_2 a_2 d_2 \cap L_{a_1 b_1}|,$$

we have

$$A_1 \geq |\triangle a_2 b_2 d_2| \geq \frac{1}{2} r^2 \sin \frac{\pi}{3} \geq 0.4R^2.$$

![Diagram](image)

**Figure 2.16.** $d_2 \notin L_{a_1 b_1}$ and $b_1 \in \text{arc } a_2 d_2$.

Next we assume none of the three cases above occurs, we prove that $A_1$ is not minimal. If $d_2 \in L_{a_1 b_1}$ (see Fig. 2.17, 2.18), fix $a_2$ and rotate $L_{a_2 b_2}$ along $a_2$ clockwise by a sufficiently small angle, then $|S \cap L_{a_1 b_1}|$ becomes strictly larger and $A_1$ becomes strictly smaller after rotating. Thus, $A_1$ is not minimal. Now assume $d_2 \notin L_{a_1 b_1}$. If $c_1 \in S$, assume $b_2, d_2$ are on the different side of line $c_1 z_1$ w.l.o.g. (see Fig. 2.19). Fix $b_2$ and rotate $L_{a_2 b_2}$ along $b_2$ counterclockwise by a sufficiently small angle, then $|S \cap L_{a_1 b_1}|$ is strictly larger
Figure 2.17. $d_2 \in L_{a_1b_1}$ and $a_2 \in \partial L_{a_1b_1}$.

Figure 2.18. $d_2 \in L_{a_1b_1}$ and $a_2, b_2 \notin \partial L_{a_1b_1}$. 
and $A_1$ is strictly smaller after rotating (If $a_2, b_2$ are on the same side of line $c_1z_1$, assume $b_2$ is closer to line $a_1b_1$ than $a_2$). Thus, $A_1$ is not minimal. Next we assume $c_1 \notin S$. We claim $\partial L_{a_2b_2}$ doesn’t cross over both sides of $\partial L_{a_1b_1}$. If the claim is true, since $d_2 \notin L_{a_1b_1}$, by (3) we have $b_1 \notin \partial L_{a_2b_2}$ (This case is similar to Fig. 2.16, but $b_1 \notin \partial L_{a_2b_2}$). Fix $a_2$ and rotate $L_{a_2b_2}$ along $a_2$ clockwise by a sufficiently small angle, then $|S \cap L_{a_1b_1}|$ is strictly larger and $A_1$ is strictly smaller after rotating. Thus, $A_1$ is not minimal and the lemma holds.

So, we only need to prove the claim holds. Now we prove the claim by contradiction. Assume the contrary that $\partial L_{a_2b_2}$ crosses over both sides of $L_{a_1b_1}$. Then both arc $a_2d_2$ and arc $b_2d_2$ cross over both sides of $L_{a_1b_1}$. Assume $d_2$ is on the same side of line $c_1z_1$ as $a_1$ w.l.o.g., then both $a_2$ and $b_2$ are on the same side of line $c_1z_1$ as $b_1$. Next we show that segment $a_2b_2$ doesn’t intersect with line $a_1b_1$ by contradiction. Assume the contrary that the segment $a_2b_2$ intersects with line $a_1b_1$. Let $l_1$ denote the line tangent to $\partial L_{a_1b_1}$ at $b_1$, then $a_2, b_2$ are on the different side of $l_1$ from $a_1$ by the proof of Lemma 12. Let $m$ denote the intersection point of arc $a_1c_1$ and $\partial L_{a_2b_2}$ which is closer to $a_1$, then $\angle a_2bm > 90^\circ$. Then, $|a_2m| > r_1 \geq r_2$, contradiction. Thus, segment $a_2b_2$ doesn’t intersect with line $a_1b_1$. Since both $a_2$ and $b_2$ are on the same side of line $c_1z_1$ as $b_1$, by Lemma 13 we have $|c_1b_2| < |mb_2| \leq r_2$. Similarly, we can prove that $|c_1a_2| < r_2$. Hence, $c_1 \in S$ which leads to a contradiction. Therefore, the claim holds.

This completes the proof of the lemma.

By Lemma 12, we assume segment $a_2b_2$ does not intersect segment $a_1b_1$. If segment $a_2b_2$ intersects with line $a_1b_1$, then it does not intersect with $\partial L_{a_1b_1}$. Since $|z_1z_2| \leq \sqrt{3}R$, by Lemma 14 we have
\[
A \geq 0.25R^2 \geq cR|z_1z_2|.
\]

Thus, Lemma 8 follows.
From now on we assume that segment $a_2b_2$ does not intersect with line $a_1b_1$, but intersects with $\partial L_{a_1b_1}$ at two points. The following notations will be used in the remaining discussion of this section. Denote by $l$ the line perpendicular to $z_1z_2$ through $z$, by $l'$ the line $z_1z_2$, by $H$ the half lune of $L_{a_2b_2}$ divided by $a_2b_2$ which is on the different side of line $a_2b_2$ from $z_1$, and by $H'$ the set $L_{a_2b_2} \setminus H$. By symmetry we assume $a_1$ and $a_2$ are on the same side of $l'$, and $b_2$ is closer to line $a_1b_1$ than $a_2$. Let $c_1$ denote the vertex of $L_{a_1b_1}$ which is on the same side of line $a_1b_1$ as $a_2$, and $c_2$ and $d_2$ denote the two vertices of $L_{a_2b_2}$ such that $c_2 \in H$. Let $\alpha = \angle b_2a_1b_1$ and $\alpha' = \angle a_2b_1a_1$. Since $b_2 \notin L_{a_1b_1}$, we have $b_2$ is above line $z_2b_1$ and $l$ intersects with line $c_2d_2$.

The following lemma shows that the distance between $z$ and the intersection point of the two lines $l$ and $c_2d_2$ is bounded above by $13R$.

**Lemma 15** Assume $a_1,b_1 \notin L_{a_2b_2}$, $l$ intersects with line $c_2d_2$ at $h$, then (1) $\angle z_1z_2h \leq 86^\circ$; (2) $\|zh\| \leq 13R$. 
Figure 2.20. The line $l$ intersects $c_2d_2$ at $h$ and $\angle z_1z_2b_2 > 90^\circ$.

Figure 2.21. The point $z_2$ is not below both line $a_1n$ and line $z_1m$. 
\textbf{Proof} (1) The first part of the lemma is proved in the following two cases:

Case 1. $\angle z_1 z_2 b_2 > 90^\circ$ (see Fig. 2.20). It is easy to see that $z_2$ is on the same side of line $c_1 z_1$ as $b_1$. If $a_2 b_2$ intersects with only one side of $L_{a_1 b_1}$, say $b_1 c_1$, extend $a_1 b_1$ to $b'_1$ such that $b'_1 c_1$ is tangent to arc $b_1 c_1$ (see Fig. 2.21). Then

$$\angle z_1 z_2 a_2 \geq \angle c_1 b'_1 a_1 = 30^\circ.$$ 

Now assume $a_2 b_2$ intersects with both sides of $L_{a_1 b_1}$. Let $\varphi = \angle z_2 z_1 b_1$, then $\angle z_1 z_2 a_2 \geq \varphi$. We will prove $\varphi \geq 30^\circ$. Let $m$ denote the middle point of $b_1 c_1$, extend $a_1 m$ to $n \in \partial B(z_1, \sqrt{3}R)$ (see Fig. 2.21). We show that $z_2$ is not below both line $a_1 n$ and line $z_1 m$. By contradiction, assume $z_2$ is below both line $a_1 n$ and line $z_1 m$. If $z_2$ is above line $b_1 c_1$, then $||b_1 z_2|| \leq ||c_1 z_2|| < \frac{r_2}{2}$ by Lemma 13. Thus, $b_1 \in L_{a_2 b_2}$, contradiction. If $z_2$ is below line $b_1 c_1$, let $a_2 b_1$ intersect with $\partial B(b_1, \frac{r_2}{2})$ at $k$, then

$$||b_1 z_2|| \leq \frac{r_1}{2} \leq ||a_2 k|| < ||a_2 z_2||$$

since $z_2 \in disk(b_1, \frac{r_2}{2})$. Thus, $b_1 \in L_{a_2 b_2}$, contradiction. Hence, $z_2$ is not below both line $a_1 n$ and line $z_1 m$. Thus,

$$\varphi \geq \angle n z_1 b_1 \geq \angle na_1 b_1 = 30^\circ.$$ 

Case 2. $\angle z_1 z_2 b_2 \leq 90^\circ$ (see Fig. 2.22). Extend $z_1 c_1$ to $c'_1$ such that $||z_1 c'_1|| = \sqrt{3}R$ (see Fig. 2.23). Let $m$ denote the middle point of $a_1 c_1$. Extend $b_1 m$ to $n \in \partial B(z_1, \sqrt{3}R)$. Pick $k \in \partial B(z_1, \sqrt{3}R)$ such that $k b_1$ is tangent to arc $b_1 c_1$ and $k$ is on the same side of $a_1 b_1$ as $c_1$. Let $\varphi = \angle z_1 z_2 b_1$, we will show that the minimum of $\varphi$ occurs as $z_2 \in \text{arc} nk$ if $z_2$ is to the left of line $k z_1$. Assuming this is true, for any $p \in \text{arc} nk$, let $||b_1 p|| = x$, 

\[ ||b_1k|| = x_0, \text{ then} \]

\[
\cos \angle b_1pz_1 = \frac{||b_1p||^2 + ||z_1p||^2 - ||b_1z_1||^2}{2||z_1p|| \cdot ||b_1p||} = \frac{x^2 + (\sqrt{3}R)^2 - (r_1/2)^2}{2\sqrt{3}xR} = \frac{x^2 + x_0^2}{2\sqrt{3}xR}.
\]

which is increasing as \( x > x_0 \). Thus, \( \angle b_1kz_1 \geq \angle b_1pz_1 \geq \theta \), where \( \theta = \angle b_1nz_1 \). Hence,

\[
\sin \theta = \frac{r_1/2}{\sqrt{3}R} \sin 30^\circ \geq \frac{R}{4\sqrt{3}R} \sin 30^\circ = \frac{1}{8\sqrt{3}}.
\]

Therefore, \( \theta \geq 4^\circ \). Thus \( \angle z_1z_2b_2 \geq \varphi \geq 4^\circ \) when \( z_2 \) is to the left of line \( kz_1 \). If \( z_2 \) is to the right of line \( kz_1 \), say, \( z_2 \) is at the position of \( z'_2 \), since \( \angle z_1b_2z'_2 < 90^\circ \), \( \angle b_2z_1z'_2 \leq 90^\circ - \angle b_1kz_1 \), we have \( \angle z_1z'_2b_2 \geq \angle b_1kz_1 \geq 4^\circ \) and the claim is true. Next we assume \( z_2 \) is to the left of line \( kz_1 \) and show that the minimum of \( \varphi \) occurs as \( z_2 \in \text{arc } nk \). If \( \angle z_2z_1b_1 \leq 90^\circ \), then the minimum of \( \varphi \) occurs as \( z_2 \in \text{arc } c'k \). Now assume \( \angle z_2z_1b_1 > 90^\circ \), we prove \( z_2 \) is not below both line \( mz_1 \) and line \( nb_1 \). By contradiction, assume \( z_2 \) is below both line \( mz_1 \) and line \( nb_1 \). If \( z_2 \) is above line \( a_1c_1 \), since \( z_2b_2 \) intersects both sides of \( L_{a_1b_1} \), we have \( ||a_1z_2|| \leq ||c_1z_2|| < \frac{e^2}{2} \) by Lemma 13. Hence, \( a_1 \in L_{a_2b_2} \), contradiction. If \( z_2 \) is below line \( a_1c_1 \), say, \( z_2 \) is at the position of \( z''_2 \), let \( l'' \) denote the line parallel to \( a_1b_1 \) through \( z''_2 \), then \( a_2 \) is above \( l'' \) and below line \( b_1z''_2 \). Assume \( l'' \) intersects with arc \( b_1c_1 \) at \( e \), then \( \angle ea_1b_1 < 30^\circ \). Extend \( b_1z''_2 \) to \( d \) such that \( da_1||z_1m \), then \( a_2 \) is below line \( da_1 \) since \( z''_2 \) is below \( mz_1 \). Thus, \( \angle a_2a_1b_2 \geq \angle da_1e > 90^\circ \). Hence \( a_1 \in L_{a_2b_2} \), contradiction. Therefore, \( z_2 \) is not below both line \( mz_1 \) and line \( nb_1 \). If \( z_2 \) is above line \( nb_1 \), then \( \varphi \) is minimal as \( z_2 \in \text{arc } nc_1' \); if \( z_2 \) is below line \( nb_1 \), then \( z_2 \) is above line \( mz_1 \) and \( \varphi \) is minimal as \( z_2 \) is coincident with \( m \). Since \( \angle z_1mb_1 > \angle z_1nb_1 \), the minimum of \( \varphi \) occurs as \( z_2 \in \text{arc } nc_1' \). Thus, the first part of the lemma holds.
(2) According to (1) we have $\angle h_2z_2 \leq 86^\circ$, then

\[
||zh|| = ||z_2z|| \tan \angle h_2z_2 \\
\leq \frac{1}{2}||z_2z_1|| \tan 86^\circ \\
\leq \frac{1}{2}\sqrt{3}R \tan 86^\circ \\
\leq 13R.
\]

Thus, the second part of the lemma holds.

Figure 2.22. The line $l$ intersects $c_2d_2$ at $h$ and $\angle z_1z_2b_2 \leq 90^\circ$.

If $o$ is above or on the line $a_2b_2$, we claim $H \subset \mathbb{D}$. This claim is verified as follows: assume that $o$ is on the same side of line $c_2z_2$ as $a_2$ w.l.o.g. For any $p \in \text{arc } b_2c_2$, by Lemma 13, we have $||op|| \leq ||ob_2||$. For any point $m \in \text{arc } a_2c_2$, since $o$ is far away from $L_{a_2b_2}$, we have $\angle mob_2 < \angle b_2mo$. Note that $\angle mb_2o \leq 60^\circ$. Then $\angle b_2mo > \angle mb_2o$. Thus, $||om|| \leq ||ob_2||$. Hence, $H \subset \mathbb{D}$ and the claim holds.
Figure 2.23. The value of $\angle z_1 z_2 b_1$ is minimal as $z_2 \in arc nk$ if $z_2$ is to the left of line $kb_1$.

Therefore, Lemma 8 holds since

$$|H \setminus L_{a_1b_1}| \geq 0.039R||z_1z_2||$$

by Lemma 7. From now on, we assume $o$ is strictly below the line $a_2b_2$. Then $H' \subset D$ and $H \setminus \emptyset \neq \emptyset$. Since $||oz_1|| \geq ||oz_2||$, we have $o$ is above $l$ or $o \in l$.

Figure 2.24. Shift $D$ along the ray $zo$ away from the two lunes by a sufficiently small distance.
Figure 2.25. Rotate $\mathbb{D}$ counterclockwise along $p \in \partial \mathbb{D}$ by a sufficiently small angle.

The following lemma proves that when $a_1, b_1 \notin L_{a_2 b_2}$ and $A$ is minimal, then the center $o$ must be on the line $l$. Also, either $b_2 \in \partial \mathbb{D}$ or $a_2 \in \partial \mathbb{D}$.

**Lemma 16** Assume $a_1, b_1 \notin L_{a_2 b_2}$. When $A$ is minimal, we have

1. $o \in l$;
2. If $o$ is on the same side of $l'$ as $a_2$ (respectively, $b_2$), then $b_2 \in \partial \mathbb{D}$ (respectively, $a_2 \in \partial \mathbb{D}$);
3. If $r_2 > r_1$ and $a_2 b_2$ intersects with both sides of $L_{a_1 b_1}$, then $b_2 \in \partial L_{a_1 b_1}$.

**Proof** We assume $h$ is on the same side of $l'$ as $a_2$ without loss of generality.

1. By contradiction, assume $o \notin l$, then $o$ is above $l$. If $o$ is on the same side of $l'$ as $a_2$ (see Fig. 2.25), since $\partial \mathbb{D}$ intersects with $\partial H$,

\[
||zo|| \geq R_0 - 2(r_1 + r_2) \geq 96R > ||zh||
\]

by Lemma 15. Thus, $o$ is above line $c_2 d_2$. Hence $||oa_2|| < ||ob_2|| \leq R_0$. Let $\partial \mathbb{D}$ intersect
$\partial H$ at $p$ and $q$ such that $p$ is closer to $b_2$ than $q$. Fix $p \in \partial \mathbb{D}$ and rotate $\mathbb{D}$ counterclockwise along $p$ by a sufficiently small angle $\varepsilon$. Let $o'$ denote the new position of $o$, $\mathbb{D}'$ denote the disk centered at $o'$ with radius $R_0$. We choose $\varepsilon$ sufficiently small so that $o'$ is still above the line $l$ and both $a_2$ and $b_2$ are inside $\mathbb{D}'$. We claim that $A$ becomes strictly smaller after rotating, which leads to a contradiction. The claim is verified as follows. For any $x \in \mathbb{D}' \cap (L_{a_2b_2} \setminus L_{a_1b_1})$ which is above line $op$, since $||op|| = ||o'p||$, apply the law of cosine to $\triangle opx$ and $\triangle o'px$, we have $||ox|| < ||o'x|| \leq R_0$. Then $x \in \mathbb{D} \cap (L_{a_2b_2} \setminus L_{a_1b_1})$. For any $y \in L_{a_2b_2} \setminus L_{a_1b_1}$ which is below line $op$, since $y \in \mathbb{D}$, we have $||o'y|| < ||oy|| \leq R_0$. Then $y \in \mathbb{D} \cap \mathbb{D}'$. Also, we have $R_0 = ||oq|| < ||o'q||$ and $q$ is outside $\mathbb{D}'$. Thus,

$$\|\mathbb{D} \cap (L_{a_2b_2} \setminus L_{a_1b_1})\| > \|\mathbb{D}' \cap (L_{a_2b_2} \setminus L_{a_1b_1})\|$$

and the claim is true. If $o$ is on the same side of $l'$ as $b_2$, since $o$ is below both $l'$ and line $a_2b_2$, then $o$ is below line $c_2d_2$ and $||ob_2|| < ||oa_2|| \leq R_0$. Thus we can apply similar rotation argument as above which leads to a contradiction. Hence, this part of the lemma holds.

(2) Since $o \notin \text{line } a_2b_2$, then $o$ is below line $a_2b_2$. First we assume $o$ is on the same side of line $l'$ as $a_2$ (see Fig. 2.24). By contradiction, we assume $b_2 \notin \partial \mathbb{D}$. Similar as we did in (1), we have $||oa_2|| < ||ob_2|| < R_0$. Since $o \in l$, we shift $\mathbb{D}$ along $l$ away from the two lunes by a distance $\varepsilon$. Let $o'$ denote the new position of $o$, $\mathbb{D}'$ denote the disk centered at $o'$ with radius $R_0$. We choose $\varepsilon$ sufficiently small so that $o'$ is still below line $a_2b_2$, and $a_2, b_2$ are inside $\mathbb{D}'$. We claim $A$ is strictly smaller after shifting, contradiction. The claim is verified as follows: For any $x \in \mathbb{D}' \cap (L_{a_2b_2} \setminus L_{a_1b_1})$, pick $k \in l$ such that $xk$ is perpendicular to $l$. Since $o$ is far away from the two lunes, then $o$ is between $o'$ and $k$. Thus, $||ox|| < ||o'x|| \leq R_0$ and $x \in \mathbb{D} \cap (L_{a_2b_2} \setminus L_{a_1b_1})$. Let $\partial \mathbb{D}$ intersect with $\partial L_{a_2b_2}$ at $p$ and $q$, similarly, we have $R_0 = ||op|| < ||o'p||$, $R_0 = ||oq|| < ||o'q||$, thus both $p$ and $q$ are
outside $\mathbb{D}'$. Hence,

$$|\mathbb{D} \cap (L_{a_2b_2} \setminus L_{a_1b_1})| > |\mathbb{D}' \cap (L_{a_2b_2} \setminus L_{a_1b_1})|. $$

If $o$ is on the same side of $l'$ as $b_2$, but $a_2 \notin \partial \mathbb{D}$, since $o$ is below line $a_2b_2$, we have that $o$ is below line $c_2d_2$ and $||ob_2|| < ||oa_2|| < R_0$. Thus we can apply a similar shift argument as above which leads to a contradiction. Hence, this part of the lemma holds.

(3) We prove by contradiction and assume the contrary that $b_2 \notin \partial L_{a_1b_1}$. First we prove $a_2 \notin \partial L_{a_1b_1}$. Otherwise, if $a_2 \in \partial L_{a_2b_2}$, then $||a_2b_1|| < r_2$. Thus $\angle b_1a_2b_2 \leq \alpha' \leq 60^\circ$ and $b_1 \in L_{a_2b_2}$, contradiction. Therefore, $a_2 \notin \partial L_{a_1b_1}$. We shrink $L_{a_2b_2}$ a little bit so that $a_2, b_2 \notin L_{a_1b_1}$ after shrinking and $A$ becomes strictly smaller, which leads to a contradiction. Hence the lemma is proved.

Figure 2.26. $a_2$ is on arc $a_1c_1$ and $z_1z_2$ is perpendicular to $a_2b_2$.

The following lemma gives upper bounds for the value of $\angle z_2z_1c_1$ when $a_1, b_1 \notin L_{a_2b_2}$ and segment $a_2b_2$ intersects both sides of the lune $L_{a_1b_1}$. 
Figure 2.27. $a'_1$ on the same side of $a_1b_1$ as $a_2$ such that $a'_1b_2||a_1b_1$ and $||a'_1b_2|| = ||a_1b_1||$. 

**Lemma 17** Assume $a_1, b_1 \in L_{a_2b_2}$, segment $a_2b_2$ intersects both sides of $L_{a_1b_1}$, then $c_1 \in L_{a_2b_2} \cup \partial L_{a_2b_2}$. Let $\delta = \angle z_2z_1c_1$. (1) Assume $\angle z_2z_1b_1 < 90^\circ$, then $\delta \leq \frac{3}{4}\alpha'$; (2) Assume $\angle z_2z_1b_1 > 90^\circ$, then $\delta \leq \frac{\alpha}{2}$ if one of the following two conditions holds: (i) $r_1 \geq r_2$, or (ii) $r_2 > r_1$ and $b_2 \in \partial L_{a_1b_1}$, $\alpha < 30^\circ$.

**Proof** Let $a_2b_2$ intersect arc $a_1c_1$ at $m$. By Lemma 13, we have $||c_1b_2|| < ||mb_2|| \leq r_2$. Similarly, $||c_1a_2|| \leq r_2$. Thus, $c_1 \in L_{a_2b_2} \cup \partial L_{a_2b_2}$.

(1) First we assume $r_1 \geq r_2$. If $r_1, r_2, \alpha', \angle b_2a_2b_1$ are fixed, $\delta$ is maximal as $a_2 \in arc a_1c_1$, we relax the constraint $b_1 \notin L_{a_2b_2}$ and allow $b_1 \in L_{a_2b_2}$ when $\delta$ is maximal. If $r_1, \alpha', \angle b_2a_2b_1$ are fixed, $a_2 \in arc a_1c_1$, the maximum of $\delta$ occurs as $r_2 = r_1$ since $r_2 \leq r_1$. If $\alpha'$ is fixed, $\delta$ is maximal as $z_1z_2$ is perpendicular to $a_2b_2$. Thus in the case of $\delta$ maximum, we have $r_2 = r_1, a_2 \in arc a_1c_1$ and $z_1z_2$ is perpendicular to $a_2b_2$ (see Fig. 2.26). Pick $b'_2 \in arc b_2c_2$ such that $a_2b'_2||a_1b_1$, let $z'_2, h$ denote respectively the middle
points of segments $a_2 b_2, a_2 b_1$. Since $a_1 b_1 b_2 a_2$ is a parallelogram, $\angle z'_{2}z_{1}c_{1} = \frac{\alpha'}{2}$. Since

$$\angle z_{2}z_{1}z'_{2} \leq \frac{1}{2} \angle z'_{2}hz_{2} = \frac{1}{4} \angle z'_{2}a_{2}z_{2} \leq \frac{\alpha'}{4},$$

we have $\delta \leq \frac{3}{4}\alpha'$. Next we assume $r_1 \leq r_2$. If $r_1, r_2, \alpha', \angle b_2 a_2 b_1$ are fixed, $\delta$ is maximal as $b_1 \in \partial L_{a_2 b_2}$. So we assume $b_1 \in \partial L_{a_2 b_2}$. Since $r_1 \leq r_2$, it is easy to see that $\delta$ is maximal as $r_1 = r_2$. By the discussion above we have $\delta \leq \frac{3}{4}\alpha'$.

(2) Pick $a'_1$ on the same side of $a_1 b_1$ as $a_2$ such that $a'_1 b_2 || a_1 b_1$ and $||a'_1 b_2|| = ||a_1 b_1||$ (see Fig. 2.27).

(i) We claim the maximum of $\delta$ occurs if $b_2 \in \partial L_{a_1 b_1}, a_2 b_2 || a_1 b_1$ and $r_1 = r_2$. This claim is verified as follows: if $r_1, r_2$ and $\angle a_2 b_2 a'_1$ are fixed, $\delta$ is maximal as $b_2 \in \partial L_{a_1 b_1}$.

When $r_1, r_2$ are fixed and $b_2 \in \partial L_{a_1 b_1}$, the maximum of $\delta$ occurs as $a_2 b_2 || a_1 b_1$. If $b_2 \in \partial L_{a_1 b_1}$ and $a_2 b_2 || a_1 b_1$, $\delta$ is maximal as $r_1 = r_2$ since $r_1 \geq r_2$. Thus, the claim is true. In the case of $\delta$ maximum, $a_2$ is coincide with $a'_1$ and $\delta = \frac{\alpha}{2}$.

(ii) Since $r_2 > r_1, b_2 \in \partial L_{a_1 b_1}$, if $r_1, r_2$ are fixed, $\delta$ is maximal as $a_1 \in \partial L_{a_2 b_2}$, so we assume $a_1 \in \partial L_{a_2 b_2}$. Extend $a_1 a'_1$ to $a'_2$ on the line $a_2 b_2$, then

$$\angle a'_2 a_1 b_2 = 90^\circ - \frac{\alpha}{2} > 75^\circ$$

and

$$\angle a_2 b_2 a_1 \leq \angle c_1 b_2 a_1 = 60^\circ + \frac{\alpha}{2} < 75^\circ,$$

thus $||a'_2 a_1|| < ||a'_2 b_2||$. Since $r_2 > r_1$, then $\angle b_2 a_2 a_1 < 60^\circ$. We claim $a_2$ is between $a'_2$ and $b_2$, otherwise, if $a'_2$ is between $a_2$ and $b_2$, then

$$||a_1 a_2|| < ||a'_2 a_1|| + ||a'_2 a_2|| < ||a'_2 b_2|| + ||a'_2 a_2|| = ||a_2 b_2||$$
and $a_1 \in L_{a_2 b_2}$, contradiction. Thus, $a_2$ is between $a'_2$ and $b_2$. Let $z'_2$ denote the middle point of $a'_2 b_2$, then $z_2$ is between $z'_2$ and $b_2$. Since $\angle z'_2 z_1 b_1 = 90^\circ + \frac{\alpha}{2}$, we have $\delta \leq \frac{\alpha}{2}$. This proves Lemma 17.

![Figure 2.28](image)

Figure 2.28. If $o$ is on the same side of line $z_1 z_2$ as $a_1$, then $\angle b_2 oz \leq \arcsin \frac{\sqrt{3}}{200}$.

We will prove stronger results than that of Lemma 8. To this end, from now on we relax the constraint $a_1, b_1 \notin L_{a_2 b_2}$ and allow at most one of $a_1$ and $b_1$ is inside $L_{a_2 b_2}$ with the distance from the node inside to $\partial L_{a_2 b_2}$ at most $2.07r\varepsilon$. Under this assumption, the following lemma gives upper bounds of the angle $\angle b_2 oz$ or $\angle a_2 oz$.

**Lemma 18** Assume at most one of $a_1$ and $b_1$ is in $L_{a_2 b_2}$ with the distance from the node inside to $\partial L_{a_2 b_2}$ at most $2.07r\varepsilon$, then the following are true when $A$ is minimal: (1) $\angle b_2 oz$ (respectively, $\angle a_2 oz$) $\leq \arcsin \frac{\sqrt{3}}{200}$ if $o$ is on the same side of $l'$ as $a_1$ (respectively, $b_1$); (2) If $b_2$ is above $l$, $o$ is on the same side of $l'$ as $a_1$ and $\angle z_2 z_1 b_1 > 90^\circ$, then $\angle b_2 oz \leq \frac{1}{50}\alpha$.

**Proof** (1) First we assume $o$ is on the same side of $l'$ as $a_1$ (see Fig. 2.28). Pick $h \in l$ such
that \( b_2h \) is perpendicular to \( l \). If \( b_2 \) is above \( l \), then

\[
\|\|hb_2\|\| \leq \frac{|z_1z_2|}{2} \leq \frac{\sqrt{3}R}{2}.
\]

If \( b_2 \) is below \( l \), then

\[
\|\|hb_2\|\| \leq \frac{r_2}{2} \leq \frac{R}{2}.
\]

Thus,

\[
\sin \angle b_2oz = \frac{\|hb_2\|}{\|ob_2\|} \leq \frac{\sqrt{3}R}{2R_0} \leq \frac{\sqrt{3}}{200}.
\]

Similarly, \( \sin \angle a_2oz \leq \frac{\sqrt{3}}{200} \) if \( o \) is on the same side of \( l' \) as \( b_1 \).

(2) Let \( l'' \) denote the line parallel to \( a_1b_1 \) through \( z \), then \( o \) is below \( l'' \) since \( \angle z_2z_1b_1 > 90^\circ \). Assume \( l \) intersects with \( \partial \mathbb{D}, a_1b_2 \) at \( m, n \), respectively, then \( \angle mn b_2 \leq \alpha \). Since \( \|nb_2\| \leq r_1 + r_2 \), then

\[
\|\|mb_2\|\| \leq \|nb_2\|\angle mn b_2 \leq (r_1 + r_2)\alpha.
\]

Thus,

\[
\angle b_2oz = \frac{\|\|mb_2\|\|}{R_0} \leq \frac{(r_1 + r_2)\alpha}{R_0} \leq \frac{\alpha}{50}
\]

and the lemma is proved.

The following lemma gives upper bounds of the angle \( \angle z_2z_1c_1 \) in terms of either \( \alpha \) or \( \alpha' \) when exactly one of \( a_1 \) and \( b_1 \) is allowed to be inside of \( L_{a_2b_2} \) and the distance from the node inside to \( \partial L_{a_2b_2} \) is at most \( 2.07r\varepsilon \).

Lemma 19 Assume exactly one of \( a_1 \) and \( b_1 \) is in \( L_{a_2b_2} \) and the distance from the node inside to \( \partial L_{a_2b_2} \) is at most \( 2.07r\varepsilon \). Suppose segment \( a_2b_2 \) intersects both sides of \( L_{a_1b_1} \).
Figure 2.29. $a_1 \notin L_{a_2 b_2}$ and $b_1 \in L_{a_2 b_2}$.

Figure 2.30. $a_1 \in L_{a_2 b_2}$ and $b_1 \notin L_{a_2 b_2}$. 
Since one of $a_1$ and $b_1$ is in $L_{a_2b_2}$, we have $r_1 < r_2$. (1) First we assume $a_1 \notin L_{a_2b_2}$, then $b_1 \in L_{a_2b_2}$ and $\text{dist}(b_1, \partial L_{a_2b_2}) \leq 2.07r\varepsilon$. Since $a_1, a_2$ are on the same side of $l'$ and $b_2$ is closer to line $a_1b_1$ than $a_2$, we have $b_1 \notin L_{r-1.07r\varepsilon}(a_2, b_2)$ (see Fig. 2.29). Let $a_2b_2$ intersect with $\partial B(a_2, ||a_2b_1||)$ at $b''_2$, let $z''_2 = \frac{1}{2}(a_2 + b''_2)$, then $||z_2z''_2|| \leq 1.035\varepsilon R$. By Lemma 17(1), $\angle z''_2z_1c_1 \leq \frac{3}{4}\alpha'$. Since

$$\sin \angle z''_2z_1z_2 \leq \frac{||z_2z''_2||}{||z_1z_2||} \leq \frac{1.035\varepsilon R}{200\varepsilon R} \leq \frac{1.035}{200},$$

(2.4)

we have $\angle z''_2z_1z_2 \leq 0.0052$ and $\delta \leq \frac{3}{4}\alpha' + 0.0052$. Next we assume $a_1 \in L_{a_2b_2}$, then $b_1 \notin L_{a_2b_2}$ (see Fig. 2.30). Let $a_2b_2$ intersect with arc $c_1a_1$ at $m$, then $||mb_2|| < ||a_1b_2|| < r_2$ by Lemma 13. Pick $u \in a_2b_2$ such that $||ub_2|| = ||a_1b_2||$, then $u \notin L_{a_1b_1}$ and

$$||ua_2|| = r_2 - ||a_1b_2|| \leq r_2 - r_1 \leq 2.07r\varepsilon.$$

Let $u' = \frac{1}{2}(b_2 + u)$, then

$$\delta \leq \angle u'z_1c_1 \leq \frac{3}{4}(\alpha' + \angle ub_1a_2)$$

by applying Lemma 17(1). Since

$$\sin \angle ub_1a_2 \leq \frac{||ua_2||}{||b_1a_2||} \leq \frac{||ua_2||}{r_1} \leq \frac{2.07r\varepsilon}{r - 1.07r\varepsilon} < 0.003,$$

we have $\angle ub_1a_2 < 0.004$ and

$$\delta \leq \angle u'z_1c_1 < \frac{3}{4}\alpha' + 0.0052.$$
(2) First we assume \( a_1 \notin L_{a_2b_2} \), then \( b_1 \in L_{a_2b_2} \) and \( \text{dist}(b_1, \partial L_{a_2b_2}) \leq 2.07r\varepsilon \) (see Fig. 2.31). Let \( a_2b_2 \) intersect with \( c_1z_1 \) at \( k \), pick \( h \) on the line \( a_2b_2 \) such that \( c_1z_1 || hb_1 \).

Since \( b_1 \in L_{a_2b_2} \), \( \angle a_2b_2b_1 < \frac{\pi}{2} \). Then \( h \) is between \( k \) and \( b_2 \). Thus,

\[
\frac{r_1}{2} = ||z_1b_1|| \leq ||kh|| \leq ||kb_2||.
\]

Hence,

\[
||kz_2|| \leq \frac{r_2 - r_1}{2} \leq 1.035\varepsilon_n R.
\]

Similar to EQ(2.4), we have \( \delta \leq 0.0052 \). Next we assume \( b_1 \notin L_{a_2b_2} \), then \( a_1 \in L_{a_2b_2} \) and \( \text{dist}(a_1, \partial L_{a_2b_2}) \leq 2.07r\varepsilon \). Similar to Lemma 17(2), the maximum value of \( \delta \) is achieved as \( b_2 \in \partial L_{a_1b_1} \) and \( a_2b_2||a_1b_1 \) (see Fig. 2.32). In the case of \( \delta \) maximum, pick \( a''_2 \in a_2b_2 \) such that \( ||a''_2b_2|| = ||a_1b_1|| \). Let \( z''_2 = \frac{1}{2}(a_2 + b''_2) \), then \( \angle z''_2z_1c_1 = \frac{\alpha}{2} \) and \( ||z_2z''_2|| \leq 1.035\varepsilon R \). Similar to EQ(2.4), we have \( \angle z''_2z_1z_2 \leq 0.0052 \). Hence, \( \delta \leq \frac{\alpha}{2} + 0.0052 \). Thus, Lemma 19 is proved.

The following lemma proved stronger results than that of Lemma 8. The result of

![Figure 2.31](image-url)
Figure 2.32. The maximum value of $\angle z_2z_1c_1$ is achieved when $b_2 \in \partial L_{a_1b_1}$ and \(a_2b_2 \parallel a_1b_1\).

Figure 2.33. $m \in \text{arc } a_1c_1$ and $n \in \text{arc } b_1c_1$ such that $mz_2$ and $nz_2$ are tangent to \(\text{arc } a_1c_1\) and \(\text{arc } b_1c_1\), respectively.
Figure 2.34. If \( o \) is above line \( ub_2 \), then \( \angle ob_2p \leq \beta + \psi + \xi \).

Figure 2.35. If \( o \) is below line \( ub_2 \), then \( z \) is below line \( ub_2 \) and \( \angle z_2z_1b_1 < 90^\circ \).
Lemma 8 is included in part (1) of Lemma 20. Part (2) of Lemma 20 will be used for the proof of Lemma 9 in next section.

**Lemma 20** Suppose segment $a_2b_2$ doesn’t intersect with line $a_1b_1$, but intersects with $\partial L_{a_2b_1}$ at two points. Assume $o \in l$, either $a_2 \in \partial D$ or $b_2 \in \partial D$. Let $r_1, r_2 \in [r - 1.07r \varepsilon, R]$ and $\|z_1z_2\| \leq \sqrt{3}R$.

1. The inequality $A \geq cR\|z_1z_2\|$ holds if either segment $a_2b_2$ intersects only one side of $L_{a_1b_1}$ or $a_1, b_1 \notin L_{a_2b_2}$;

2. Assume segment $a_2b_2$ intersects both sides of $L_{a_1b_1}$, exactly one of $a_1$ and $b_1$ is in $L_{a_2b_2}$ with the distance from the node inside to $\partial L_{a_2b_2}$ at most $2.07r \varepsilon$. If $\|z_1z_2\| > 200 \varepsilon R$, then

$$|D \cap H \setminus L_{a_1b_1}| \geq cR\|z_1z_2\| - \varepsilon R^2.$$  

**Proof** Since $r_1, r_2 \in [r - 1.07r \varepsilon, R]$, we have

$$0.997 \leq \frac{r_1}{r_2}, \frac{r_2}{r_1} \leq 1.003$$

and $\min\{r_1, r_2\} \geq 0.997R$. Note that $\|z_1z_2\| \leq \sqrt{3}R$, then

$$cR\|z_1z_2\| \leq 0.0047r_2^2.$$  

So under the conditions of (1) it is sufficient to prove

$$A \geq \min\{0.0047r_2^2, cR\|z_1z_2\|\};$$  

(2.5)

Under the conditions of (2) it is sufficient to prove

$$|D \cap H \setminus L_{a_1b_1}| \geq \min\{0.0047r_2^2, cR\|z_1z_2\|\} - \varepsilon R^2.$$  

(2.6)
By symmetry we assume \( a_1, a_2 \) are on the same side of \( l' \) and \( b_2 \) is closer to line \( a_1b_1 \) than \( a_2 \). If \( o \) is on the same side of \( l' \) as \( a_1 \) (respectively, \( b_1 \)), then \( b_2 \in \partial \mathbb{D} \) (respectively, \( a_2 \in \partial \mathbb{D} \)); let \( p \notin \{a_2, b_2\} \) denote the other intersection point of \( \partial \mathbb{D} \) and \( \partial L_{a_2b_2} \), let \( \beta = \angle a_2b_2p, \gamma = \angle b_2op, \eta = \angle b_2oz \) (respectively, \( \beta = \angle b_2a_2p, \gamma = \angle a_2op, \eta = \angle a_2oz \)). Let \( \delta = \angle z_2z_1c_1 \).

Pick \( u \) on arc \( a_1c_1 \) such that \( ub_2 || a_1b_1 \) (see Fig. 2.34, 2.35). Since \( r_1 \leq 1.003r_2 \), we have

\[
|C_{c_1b_1}| = \frac{1}{2} r_1^2 \left( \frac{\pi}{3} - \sin \frac{\pi}{3} \right) < 0.1 r_2^2.
\]

Case 1. Suppose segment \( a_2b_2 \) intersects only one side of \( L_{a_1b_1} \). Assume \( a_2b_2 \) intersects with \( \text{arc } c_1b_1 \) at two points w.l.o.g. We will prove \( A \geq 0.08r_2^2 \), which implies EQ(2.5). If \( c_2 \in \mathbb{D} \), then

\[
A \geq \frac{1}{2} r_2^2 \left( \frac{\pi}{3} \right) - |C_{c_1b_1}| \geq 0.42r_2^2.
\] (2.7)

So we assume \( c_2 \notin \mathbb{D} \). Let \( H' = L_{a_2b_2} \setminus H \). If \( \angle b_1z_1z_2 \geq 90^\circ \), we prove \( |H' \setminus L_{a_1b_1}| \geq 0.13R^2 \). Pick \( m \in \text{arc } a_1c_1 \) and \( n \in \text{arc } b_1c_1 \) such that \( mz_2 \) and \( nz_2 \) are respectively tangent to \( \text{arc } a_1c_1 \) and \( \text{arc } b_1c_1 \). Let \( a_1n \) and \( b_1m \) intersect at \( h \) (see Fig. 2.33). Then \( \angle nz_2m + \angle nhm = 180^\circ \). Thus,

\[
\angle nz_2m = \angle na_1b_1 + \angle mb_1a_1 \leq 120^\circ
\]

and \( \angle mz_2a_2 \geq 60^\circ \). Pick \( v \) on the ray \( z_2m \) such that \( ||z_2v|| = ||a_2z_2|| \), then

\[
|H' \setminus L_{a_1b_1}| \geq |\angle vz_2a_2| \geq \frac{1}{2} \left( \frac{r_2}{2} \right)^2 \frac{\pi}{3} \geq 0.13R^2.
\]

Next we assume \( \angle z_2z_1b_1 < 90^\circ \). Since \( \sin \frac{\pi}{2} \leq \frac{r_2}{2} \), then \( \frac{\pi}{2} \leq \arcsin \frac{1}{200} \). By Lemma 18, \( \eta \leq \arcsin \frac{\sqrt{3}}{200} \). Thus,

\[
\frac{\gamma}{2} + \eta < 0.014.
\] (2.8)
Subcase 1.1. Assume $o$ is on the same side of $l'$ as $a_1$. Let $\xi = \angle a_2b_2c_1$, $\psi = \angle c_1b_2u$, then

$$\psi \leq \angle c_1ub_2 \leq \frac{\pi}{3} - \frac{\alpha}{2}.$$ 

We claim that

$$\angle ob_2p \leq \beta + \psi + \xi + \eta,$$

which is true if $o$ is above line $ub_2$ since $\angle ob_2p \leq \beta + \psi + \xi$ (see Fig. 2.34). If $o$ is below line $ub_2$, then $z$ is below line $ub_2$ since $\angle z_2z_1b_1 < 90^\circ$ (see Fig. 2.35). Thus $o$ is between line $ub_2$ and line $l''$ which is the line parallel to $ub_2$ through $z$. Hence the claim is true as $\angle ob_2u \leq \eta$. Since $2\angle ob_2p + \gamma = \pi$, we have

$$\beta + \xi \geq \frac{\pi}{6} + \frac{\alpha}{2} - \frac{\gamma}{2} - \frac{\eta}{2} \geq \frac{\pi}{6} + \frac{\alpha}{2} - 0.014.$$

If $c_1 \in L_{a_2b_2}$, extend $b_2c_1$ to $d \in \partial L_{a_2b_2}$, then

$$A \geq |\angle pb_2d| - |C_{c_1q}| \geq \frac{1}{2} r_2^2 (\beta + \xi) - |C_{c_1q}| \geq \frac{1}{2} r_2^2 (\frac{\pi}{6} + \frac{\alpha}{2} - \frac{\gamma}{2} - \frac{\eta}{2}) - \frac{1}{2} r_2^2 \left[ (\frac{\pi}{3} - \alpha) - \sin (\frac{\pi}{3} - \alpha) \right] \geq \frac{1}{2} r_2^2 (\frac{\pi}{6} + \frac{\alpha}{2} - 0.014) - \frac{1}{2} (1.003r_2)^2 \left[ (\frac{\pi}{3} - \alpha) - \sin (\frac{\pi}{3} - \alpha) \right],$$

which increases as $\alpha \geq 0$, thus $A \geq 0.16r_2^2$. Next we assume $c_1 \notin L_{a_2b_2}$ (see Fig. 2.34). Let $arc \ c_1b_1$ intersect $a_1b_2$ at $q$, $arc \ c_1v$ intersect $\partial L_{a_2b_2}$ at $h$, since $\angle a_1hb_2 \geq \angle a_1hq > 60^\circ$ and $\angle b_2hc_2 > \angle b_2d_2c_2 = 30^\circ$, we have $\angle a_1hc_2 > 90^\circ$. Pick $k \in arc \ b_2c_2$ such that $hk$ is perpendicular to $a_1h$. Since

$$\angle hb_2c_1 \leq \anglehqc_1 = \frac{1}{2} \angle ha_1c_1 \leq \frac{1}{2} (\frac{\pi}{3} - \angle ha_1k)$$

and

$$\angle pb_2h = \beta + \xi - \angle hb_2c_1,$$
we have

\[
\angle ha_1 k = \arctan \frac{|hk|}{r_1} \geq \arctan \frac{r_2}{r_1}
\geq \arctan 0.997 \approx 0.7838.
\]

Thus,

\[
A \geq |\angle pb_2 h| - |C_{c_1 b_1}|
\geq \frac{1}{2} r_2^2 \left( \frac{1}{2} \angle ha_1 k - 0.014 \right) - 0.1 r_2^2
> 0.08 r_2^2.
\]

Figure 2.36. If \( \angle z_1 z_2 b_2 \leq 90^\circ \), then \( \angle oa_2 c_2 < 60.5^\circ \) and \( c_2 \in \mathbb{D} \).

Subcase 1.2. Assume \( o \) is on the same side of \( l' \) as \( b_1 \). Extend \( a_2 b_2 \) and \( a_1 b_1 \), assume they intersect at \( h \), pick \( k \in \text{line} \ a_1 h \) such that \( a_2 k \parallel l \). If \( \angle z_1 z_2 b_2 \leq 90^\circ \) (see Fig. 2.36), we claim \( \angle oa_2 c_2 < 60.5^\circ \), which implies \( c_2 \in \mathbb{D} \) since \( \angle oa_2 p > 89^\circ \) and the lemma holds by EQ(2.7). This claim is verified as follows. If \( b_2 \) is below line \( a_2 o \), clearly \( \angle oa_2 c_2 \leq \angle b_2 a_2 c_2 = 60^\circ \). When \( b_2 \) is above line \( a_2 o \), we have \( \angle oa_2 c_2 < 60.5^\circ \) since \( \angle oa_2 b_2 \leq \angle
Figure 2.37. If $\angle z_1z_2b_2 > 90^\circ$, then $k$ is between $b_1$ and $h$, and $\angle oa_2b_2 < 30.5^\circ$.

$\angle oa_2k < 0.5^\circ$ by Lemma 18. Therefore, the claim is true.

Next we assume $\angle z_1z_2b_2 > 90^\circ$, then $k$ is between $b_1$ and $h$. We claim $\angle oa_2b_2 < 30.5^\circ$. Let $m$ and $n$ be the intersection points of segment $a_2b_2$ and arc $b_1c_1$ such that $m$ is closer to $a_2$ than $n$ (see Fig. 2.37). Then $\angle b_1a_1m \leq 60^\circ$. Since segment $a_2b_2$ intersects with arc $b_1c_1$ at two points, we have

$$\angle b_1mb_2 \leq \frac{1}{2} \angle b_1a_1m \leq 30^\circ.$$ 

Thus,

$$\angle ka_2h \leq \angle b_1a_2h \leq \angle b_1mb_2 \leq 30^\circ.$$ 

Since $\angle ao_2k = \angle a_2oz < 0.5^\circ$, we have

$$\angle oa_2b_2 = \angle ao_2k + \angle ka_2b_2 < 30.5^\circ$$

and the claim holds.
Since
\[ \beta + \angle oab_2 + \frac{\gamma}{2} = \frac{\pi}{2}, \]
then \( \beta \geq 59.5^\circ - \frac{\gamma}{2} \). Thus,
\[ A \geq |\angle b_2a_2p| - |C_{mn}| \]
\[ \geq \frac{1}{2}r_2^2\beta - \frac{1}{2}r_1^2(\angle ma_1n - \sin \angle ma_1n) \]
\[ \geq \frac{1}{2}r_2^2(59.5^\circ - \arcsin \frac{1}{200}) - \frac{1}{2}(1.003r_2)^2(60^\circ - \sin 60^\circ) \]
\[ \geq 0.42r_2^2. \]

Case 2. Suppose segment \( a_2b_2 \) intersects with both sides of \( L_{a_1b_1} \). First we assume \( a_1, b_1 \notin L_{a_2b_2} \) and \( r_2 \geq r_1 \), then \( b_2 \in \partial L_{a_1b_1} \) by Lemma 16(3). If \( \alpha \geq 30^\circ \), we claim \( |H' \setminus L_{a_1b_1}| \geq 0.06r_2^2 \), which implies EQ(2.5). This claim is verified as follows: Fix \( b_2 \in \partial L_{a_1b_1} \) and rotate \( L_{a_2b_2} \) along \( b_2 \) counterclockwise until \( a_2b_2||a_1b_1, |H' \cap L_{a_1b_1}| \) is increasing during rotating (see Fig. 2.38). Thus \( |H' \setminus L_{a_1b_1}| \) achieves minimum as \( a_2b_2||a_1b_1 \). So we assume \( a_2b_2||a_1b_1 \). It is easy to see that \( |H' \setminus L_{a_1b_1}| \) achieves minimum as \( r_2 = r_1 \) since \( r_2 \geq r_1 \). Thus we further assume \( r_2 = r_1 \). Let \( \text{arc } a_1c_1 \) intersect with \( a_2b_2 \) at \( u \), since \( |C_{a_1u}| = |C_{a_1a_2}| \), we have \( |H' \setminus L_{a_1b_1}| \geq |\triangle a_1a_2u| \). Since \( \angle a_2a_1u = \alpha \) and \( \angle b_2ua_1 = \frac{\pi}{2} + \frac{\alpha}{2} \), apply the law of sine to \( \triangle b_2ua_1 \), we have

\[ ||ua_1|| = r_1 \sin \alpha / \sin \left( \frac{\pi}{2} + \frac{\alpha}{2} \right) = r_1 \sin \alpha / \cos \frac{\alpha}{2}. \]
\[ |\triangle a_1a_2u| = \frac{1}{2}||ua_1||^2 \sin \alpha = r_1^2 (\sin \alpha - \frac{1}{2} \sin 2\alpha), \]

which increases as \( \alpha \geq 30^\circ \), thus
\[ |H' \setminus L_{a_1b_1}| \geq 0.06R^2 \geq 0.06r_2^2 \]
Figure 2.38. $|H \setminus L_{a_1b_1}|$ achieves minimum as $a_2b_2|a_1b_1$, $r_2 = r_1$ and $b_2 \in \partial L_{a_1b_1}$.

and the claim is true. Hence when $a_1, b_1 \notin L_{a_2b_2}$ and $r_2 > r_1$, we can assume $\alpha < 30^\circ$.

Therefore, by Lemma 17(2) and Lemma 19(2), when $\angle b_1z_1z_2 > 90^\circ$ we have

$$\delta \leq \frac{\alpha}{2} + \delta_0. \quad (2.9)$$

By Lemma 17, $c_1 \in L_{a_2b_2}$, extend $b_2c_1$ to $d \in \text{arc } a_2c_2$ (see Fig. 2.42). Let $\theta = \angle a_2b_2d$, $\theta' = \angle a_2b_2u$. Since $\angle b_2a_1c_1 = \frac{\pi}{3} - \alpha$ and $\angle a_1qz_1 \geq \angle a_1b_2c_1$, then $\angle a_1b_2c_1 \leq \frac{\pi}{3} + \frac{\alpha}{2}$. Since $\angle a_1b_2c_1 = \alpha + \theta + \theta'$, we have

$$\angle c_2b_2d = \frac{\pi}{3} - \theta \geq \frac{\alpha}{2} + \theta'. \quad (2.10)$$

First we prove the following two inequalities:

$$\frac{1}{2}r_2^2\left(\frac{\alpha}{2} + \theta'\right) \geq 0.026R||z_1z_2|| \text{ if } a_1, b_1 \notin L_{a_2b_2}; \quad (2.11)$$

$$\frac{1}{2}r_2^2\left(\frac{\alpha}{2} + \theta'\right) \geq 0.026R||z_1z_2|| - \varepsilon R^2 \text{ if } a_1 \text{ or } b_1 \text{ is in } L_{a_2b_2}. \quad (2.12)$$
If \( b_2 \in \partial L_{a_1b_1} \), EQ(2.11) holds by Lemma 7. When \( b_2 \notin \partial L_{a_1b_1} \), let \( a_2b_2 \) intersect with arc \( b_1c_1 \) at \( b''_2 \), extend \( b''_2a_2 \) to \( a'_2 \) such that \( ||a''_2b''_2|| = r_2 \), let \( z''_2 \) denote the middle point of \( a''_2b''_2 \), let \( t, t' \) denote respectively the distances from \( b_2, b''_2 \) to line \( a_1b_1 \), then \( t' \leq t + r_2\theta' \) (see Fig. 2.39). From the discussion in the appendix of [34], if we shift \( L_{a_2b_2} \) along its waist but do not let both \( a_2 \) and \( b_2 \) cross the boundary \( \partial L_{a_1b_1} \), \( ||z_1z_2|| \) achieves its maximum as \( b_2 \notin \partial L_{a_1b_1} \) during shifting. Further, from the appendix of [34] we have

\[
||z_1z_2|| \leq ||z_1z'_2|| \leq \frac{1 + \sqrt{3}}{\sqrt{3}}t' + r_2\theta'
\leq \frac{1 + \sqrt{3}}{\sqrt{3}}t + \frac{1 + 2\sqrt{3}}{\sqrt{3}}r_2\theta'.
\]

Thus,

\[
\frac{1}{2}r_2^2\left(\frac{\alpha}{2} + \theta'\right) \geq \frac{1}{4}r_2^2 \sin \alpha + \frac{1}{2}r_2^2\theta'
\geq \frac{1}{4}r_2^2\left(\frac{t}{r_1 + r_2}\right) + \frac{1}{2}r_2^2\theta'
\geq \frac{1}{4}r_2t\left(\frac{r_2}{r_1 + r_2}\right) + \frac{1}{4}R(r_2\theta')
\geq \frac{1}{24}R(t + 6r_2\theta')
= \frac{\sqrt{3}}{24(1 + \sqrt{3})}R\left(\frac{1 + \sqrt{3}}{\sqrt{3}}t + \frac{6 + 6\sqrt{3}}{\sqrt{3}}r_2\theta'\right)
\geq \frac{\sqrt{3}}{24(1 + \sqrt{3})}R||z_1z_2||
\approx 0.026R||z_1z_2||.
\]

Thus EQ(2.11) holds.

Now we prove EQ(2.12). If \( b_1 \notin L_{a_2b_2} \), then \( a_1 \in L_{a_2b_2} \) and \( \text{dis}(a_1, \partial L_{a_2b_2}) \leq 2.07r\varepsilon \) (see Fig. 2.40). Let \( a_2b_2 \) intersect with \( \partial B(b_2, ||a_1b_2||) \) at \( u \), and \( u' = \frac{1}{2}(u + b_2) \), then \( ||z_2u'|| \leq 1.035r\varepsilon \) and \( ||z_1u'|| \geq ||z_1z_2|| - 1.035r\varepsilon \). By Lemma 12(2), \( u \notin L_{a_1b_1} \).
Figure 2.39. Shift the lune $L_{a_2b_2}$ along its waist $a_2b_2$.

According to EQ(2.11), we have

$$\frac{1}{2}||ub_2||^2(\frac{\alpha}{2} + \theta') \geq 0.026R||z_1u'||.$$

Thus, EQ(2.12) holds. If $a_1 \notin L_{a_2b_2}$, then $b_1 \in L_{a_2b_2}$ and $\text{dis}(b_1, \partial L_{a_2b_2}) \leq 2.07r\varepsilon$ (see Fig. 2.41). Let $a_2b_2$ intersect with $\partial B(a_2, ||b_1a_2||)$ at $u$, and $u' = \frac{1}{2}(u + a_2)$. then $||z_2u'|| \leq 1.035r\varepsilon$ and $||z_1u'|| \geq ||z_1z_2|| - 1.035r\varepsilon$. Let $t$ denote the distance from $b_2$ to line $ua_1$, then $t \leq ||ub_2|| \leq 2.07r\varepsilon$. Since $\varepsilon < 0.001$, we have

$$\sin \angle ua_1b_2 = \frac{t}{||a_1b_2||} \leq \frac{2.07r\varepsilon}{r - 1.07r\varepsilon} < 2.08\varepsilon,$$

and

$$\angle ua_1b_2 < \arcsin 2.08\varepsilon < 4\varepsilon.$$

By EQ(2.11) we have

$$\frac{1}{2}||ua_2||^2(\frac{1}{2}\angle ua_1b_1 + \theta') \geq 0.026R||z_1u'||.$$
Therefore,

\[
\frac{1}{2} r_2^2 \left( \frac{\alpha}{2} + \theta' \right) \geq \frac{1}{2} \| u a_2 \|^2 \left( \frac{1}{2} \angle u a_1 b_1 + \theta' - \frac{1}{2} \angle u a_1 b_2 \right) \\
\geq 0.026 R \| z_1 u' \| - \frac{1}{4} \| u a_2 \|^2 \angle u a_1 b_2 \\
\geq 0.026 R \| z_1 u' \| - \varepsilon R^2.
\]

Thus, EQ(2.12) holds.

Let \( \delta_0 = 0.0052 \). We further consider two subcases:

Subcase 2.1. Assume \( o \) is on the same side of \( l' \) as \( a_1 \). First we prove

\[
\angle o b_2 p \leq \beta + \theta' + \frac{26}{50} \alpha + \delta_0. \tag{2.13}
\]

This is true if \( o \) is above or on line \( u b_2 \) since \( \angle o b_2 p \leq \beta + \theta' \) (see Fig. 2.42). Next we assume \( o \) is below line \( u b_2 \). Since \( u b_2 \parallel a_1 b_1 \), let \( \mu = \angle u b_2 o \), then \( \angle o b_2 p = \beta + \theta' + \mu \). We
Figure 2.41. $a_1 \notin L_{a_2 b_2}$ and $b_1 \in L_{a_2 b_2}$.

Figure 2.42. The center $o$ is above or on line $ub_2$. 
will prove
\[ \mu \leq \frac{26}{50} \alpha + \delta_0. \]

Let \( l'' \) denote the line parallel to \( a_1b_1 \) through \( z \). If \( \angle z_2z_1b_1 \leq 90^\circ \), then \( o \) is between \( l'' \) and line \( ub_2 \) and
\[ \mu = \angle ub_2o \leq \angle b_2oz = \eta \]
(see Fig. 2.43). Let \( a_1b_2 \) intersect line \( l \) at \( e \), then
\[
||eb_2|| \leq ||a_1b_2|| \leq r_1 + r_2
\]
(see Fig. 2.44). Apply the law of sine to \( \triangle oeb_2 \), we have
\[
sin\eta = \frac{||eb_2||}{||ob_2||} \sin(\eta + \angle ob_2e) \leq \frac{(r_1 + r_2)\sin(\eta + \alpha)}{R_0} \leq \frac{\sin\eta + \sin\alpha}{50}.
\]
Thus, $\sin \eta \leq \frac{\alpha}{49}$ and

$$\mu \leq \eta \leq 2\sin \eta \leq \frac{2\alpha}{49}$$

Now we assume $\angle z_2 z_1 b_1 > 90^\circ$. If $z$ is above $ub_2$, let line $ub_2$ intersect with $oz$ at $k$, then

$$\mu \leq \angle zkb_2 = \delta \leq \frac{\alpha}{2} + \delta_0$$

by EQ(2.9) (see Fig. 2.45). So we assume $z$ is below $ub_2$. If $b_2$ is above $l$, then $\eta \leq \frac{\alpha}{50}$ by Lemma 18. Extend $ub_2$ to $k$ such that $k \in l$, we have $\angle oku = \delta$ and

$$\mu = \delta + \eta \leq \frac{26}{50} \alpha + \delta_0$$

(see Fig. 2.46). If $b_2$ is below $l$, then

$$\mu \leq \delta \leq \frac{\alpha}{2} + \delta_0.$$
Therefore, EQ(2.13) holds.

Let segment $c_1b_2$ intersect arc $c_1b_1$ at $v$. Since $\alpha \leq \frac{\pi}{3}$, we have

$$\delta_0 + \frac{\gamma}{2} + \frac{\alpha}{50} < 0.032.$$ 

**Subcase 2.1.1.** $c_2 \notin \mathcal{D}$ (see Fig. 2.43). Since $2\angle ob_2p + \gamma = \pi$, we have

$$\beta - \theta \geq \frac{\pi}{6} - (\delta_0 + \frac{\gamma}{2} + \frac{\alpha}{50}) > \frac{\pi}{6} - 0.032$$

by EQ(2.10) and EQ(2.13). Then

$$A \geq |\angle pb_2d| - |C_{c_1v}| \geq \frac{1}{2}r_2^2(\beta - \theta) - \frac{1}{2}r_1^2\left[\frac{\pi}{3} - \alpha - \sin\left(\frac{\pi}{3} - \alpha\right)\right]$$

$$\geq \frac{1}{2}r_2^2\left(\frac{\pi}{6} - 0.032\right) - \frac{1}{2}(1.003r_2)^2\left[\frac{\pi}{3} - \alpha - \sin\left(\frac{\pi}{3} - \alpha\right)\right],$$

which increases as $\alpha \geq 0$, thus $A \geq 0.15r_2^2$.

**Subcase 2.1.2.** $c_2 \in \mathcal{D}$ (see Fig. 2.44). Let $\zeta = \angle pb_2c_2$, then $\beta = \frac{\pi}{3} + \zeta$. Since
2∠ob_2p + γ = π, by EQ(2.13) we have

$$\frac{\pi}{6} - \zeta \leq \left( \frac{\alpha}{2} + \theta' \right) + \frac{\alpha}{50} + \frac{\gamma}{2} + \delta_0,$$

then

$$\sin\frac{\gamma}{2} = \frac{||pb_2||}{2R_0} = \frac{r_2 \sin\left(\frac{\pi}{6} - \zeta\right)}{R_0} \leq \frac{\sin\left(\frac{26}{50}\alpha + \theta' + \delta_0\right) + \sin\frac{\gamma}{2}}{100},$$

(2.15)

$$\gamma \leq 3\sin\frac{\gamma}{2} \leq \frac{1}{33}\sin\left(\frac{26}{50}\alpha + \theta' + \delta_0\right) \leq \frac{26}{825}\left(\frac{\alpha}{2} + \theta' + \delta_0\right).$$

Let

$$\Phi = 2\left(\frac{\alpha}{2} + \theta'\right) + \frac{\alpha}{25} + \gamma + 2\delta_0.$$

Then $\angle p_0a_2b_2 = \frac{\pi}{3} - 2\zeta \leq \Phi$. If $\Phi > \frac{\pi}{3}$, we have

$$\frac{\alpha}{2} + \theta' > \frac{\pi}{6} - (\delta_0 + \frac{\gamma}{2} + \frac{\alpha}{50}).$$
Thus $A \geq 0.15r_2^2$ by EQ(2.10) and EQ(2.14). Next we assume $\Phi \leq \frac{\pi}{3}$. It is easy to see that

$$|C_{c_1v}| \leq |C_{c_2b_2}|.$$  (2.16)

This is true if $r_1 \geq r_2$ since $||c_1v|| \leq ||c_2b_2||$; if $r_1 < r_2$, then

$$|C_{c_1v}| \leq |C_{c_2b_1}| \leq |C_{c_2b_2}|.$$

Let $S$ denote the region surrounded by segments $c_2b_2$, $pb_2$ and $arc$ $c_2p$, then

$$A \geq \angle c_2b_2d + |S| - |C_{c_1v}|$$
$$= \angle c_2b_2d + (|S| + |C_{pb_2}|) - |C_{c_1v}| - |C_{pb_2}|$$
$$\geq \angle c_2b_2d - |C_{pb_2}| \geq \frac{1}{2}r_2^2(\frac{\alpha}{2} + \theta') - \frac{1}{2}r_2^2(\Phi - \sin \Phi)$$
$$\geq \frac{1}{2}r_2^2(\frac{\alpha}{2} + \theta') - \frac{1}{2}r_2^2 \left[ 2(\frac{\alpha}{2} + \theta') + \frac{\alpha}{25} + \gamma + 2\delta_0 - \sin \left( 2(\frac{\alpha}{2} + \theta') + \frac{\alpha}{25} + \gamma + 2\delta_0 \right) \right]$$
$$\geq \frac{1}{2}r_2^2 \left[ \sin 2(\frac{\alpha}{2} + \theta' + \delta_0) - \left( \frac{\alpha}{2} + \theta' + \delta_0 \right) - \frac{\alpha}{25} - \gamma - \delta_0 \right]$$
$$\geq \frac{1}{2}r_2^2 \left[ \frac{1}{4}(\frac{\alpha}{2} + \theta' + \delta_0) - \frac{2}{25}(\frac{\alpha}{2} + \theta' + \delta_0) - \frac{26}{825}(\frac{\alpha}{2} + \theta' + \delta_0) - \delta_0 \right]$$
$$\geq \frac{1}{2}r_2^2 \left[ \frac{\alpha}{2} + \theta' + \delta_0 \right] \cdot 0.14 - \delta_0 \right].$$

The second last inequality holds since $\sin 2x - x \geq \frac{x}{4}$ as $0 \leq x \leq \frac{\pi}{4}$ and EQ(2.15). If $\frac{\alpha}{2} + \theta' \geq 0.1$, then $A \geq 0.0047r_2^2$ and EQ(2.5) holds. Next we assume $\frac{\alpha}{2} + \theta' < 0.1$, then $\alpha < 0.2$. Let $arc$ $b_1c_1$ intersect with $a_2p$ at $p''$, $a_2b_2$ and $a_2c_2$ intersect with $\partial B(a_2, ||a_2p''||)$ at $b_2''$ and $c_2''$, respectively (see Fig. 2.47). Let $b_2b_2''p''p$ denote the arc quadrangle surrounded by $arc$ $b_2p$, $arc$ $b_2''p''$, segment $b_2b_2''$ and segment $p''p$. Then

$$\frac{\angle p_a b_2}{\angle c_2 a_2 b_2} \leq \frac{\Phi}{\pi/3} \leq \frac{2(\theta' + \frac{\alpha}{2} + \frac{\alpha}{25} + \frac{\gamma}{2} + \delta_0)}{\pi/3} < \frac{1}{4},$$
Figure 2.47. \(c_2 \in \mathbb{D}, \ \frac{\alpha}{2} + \theta' < 0.1\) and \(A > \frac{3}{4} |\angle c_2b_2d|\).

\[
\frac{|C_{pb_2}|}{A} \leq \frac{|b_2b'_2p'p|}{|c_2c'_2p'p|} = \frac{\angle pa_2b_2}{\angle pa_2c_2} < \frac{1}{3}.
\]

Thus,

\[
A \geq |\angle c_2b_2d| - |C_{pb_2}| > |\angle c_2b_2d| - \frac{1}{3}A.
\]

Hence, by EQ(2.10) we have

\[
A > \frac{3}{4} |\angle c_2b_2d| \geq \frac{3}{4} \cdot \frac{1}{2} r_2^2 (\frac{\alpha}{2} + \theta').
\]

Thus EQ(2.5) and EQ(2.6) hold by EQ(2.11) and EQ(2.12), respectively.

Subcase 2.2. Assume \(o\) is on the same side of \(l'\) as \(b_1\). If \(c_2 \in \mathbb{D}\), let \(w = \frac{1}{2}(a_2 + c_2)\), then \(|b_2w| = \frac{\sqrt{3}}{2} r_2\) (see Fig. 2.48). Let \(b_2d\) and \(b_2c_2\) intersect respectively \(\partial B(b_2, |b_2w|)\) at \(h\) and \(k\), then \(A \geq |\angle h_b_2k|\) since \(|C_{c_1v}| \leq |C_{c_2b_2}|\) by EQ(2.16). Then

\[
A \geq |\angle h_b_2k| = \frac{1}{2} |b_2w|^2 (\frac{\pi}{3} - \theta) \geq \frac{1}{2} |b_2w|^2 (\frac{\alpha}{2} + \theta') \geq \frac{3}{4} \cdot \frac{1}{2} r_2^2 (\frac{\alpha}{2} + \theta').
\]
Thus EQ(2.5) and EQ(2.6) hold by EQ(2.11) and EQ(2.12), respectively.

Next we assume $c_2 \notin \mathbb{D}$ (see Fig. 2.49). Extend $a_2c_1$ to $d' \in \text{arc } c_2$, we claim

$$\angle d'a_2o \leq \frac{\pi}{3} + \frac{1}{4}\alpha' + \eta + \delta_0.$$ 

Let $\text{arc } c_1a_1$ intersect $a_2b_1$ at $q$. Pick $q', a'_2 \in \text{arc } c_1$ such that $qq'||a_2a'_2||a_1b_1$, pick $z' \in z_1z_2$ such that $a_2z'||l$. Then

$$\angle c_1a_2a'_2 \leq \angle c_1qq' = \frac{\pi}{3} - \frac{1}{2}\alpha'$$

and $\angle z'a_2o = \eta$. By Lemma 19, $\angle a'_2a_2z' = \delta \leq \frac{\pi}{4}\alpha' + \delta_0$. Thus

$$\angle d'a_2o \leq \angle c_1a_2a'_2 + \angle a'_2a_2z' + \angle z'a_2o$$

$$\leq \frac{\pi}{3} + \frac{1}{4}\alpha' + \eta + \delta_0.$$
Hence the claim is true. Since $2\angle o a_2 p + \gamma = \pi$, we have by EQ(2.8)

$$
\angle d' a_2 p \geq \frac{\pi}{6} - \frac{1}{4}\alpha' - \left(\frac{\gamma}{2} + \eta + \delta_0\right) \geq \frac{\pi}{6} - \frac{1}{4}\alpha' - 0.0192.
$$

$$
A \geq |\angle p o a_2 d'| - |C_{c_1 q}|
$$

$$
\geq \frac{1}{2}r_2^2\left(\frac{\pi}{6} - \frac{1}{4}\alpha' - 0.0192\right) - \frac{1}{2}r_1^2\left[\frac{\pi}{3} - \alpha' - \sin\left(\frac{\pi}{3} - \alpha'\right)\right]
$$

$$
\geq \frac{1}{2}r_2^2\left(\frac{\pi}{6} - \frac{1}{4}\alpha' - 0.0192\right) - \frac{1}{2}(1.003r_2)^2\left[\frac{\pi}{3} - \alpha' - \sin\left(\frac{\pi}{3} - \alpha'\right)\right],
$$

which achieves minimum as $\alpha' = \frac{\pi}{3}$ since its second derivative w.r.t. $\alpha'$ is negative. Thus

$$
A \geq 0.12r_2^2.
$$

This completes the proof of Lemma 20.

Thus, the proof for Lemma 8 is complete.

### 2.5 Proof of Lemma 9

This section is dedicated to the proof of Lemma 9. Let $R_0 = \frac{1}{\sqrt{\pi}}$ denote the radius
of \( \mathbb{D} \) and \( c = 0.0026 \). Let \( r_i = \| e_i \|, i = 1, 2 \), then \( r_1, r_2 \in [r, R] \). Since \( \varepsilon < 0.001 \), we have \( \frac{1}{1.001} < \frac{r_1}{r_2}, \frac{r_2}{r_1} < 1.001 \). For any disk \( B \) and any \( u, v \in \partial B \), denote by \( C_{uv} \) the circular cap surrounded by segment \( uv \) and arc \( uv \) satisfying that \( |C_{uv}| \) is at most a half of \( |B| \).

Without loss of generality, we assume \( e_1 = (a_1, b_1) \) and \( e_2 = (a_2, b_2) \). Then none of the four points \( a_1, b_1, a_2 \) and \( b_2 \) is contained in the union of \( L_r(e_1) \cup L_r(e_2) \). Let

\[
A_1 = |\mathbb{D} \cap L_r(e_2) \setminus L_r(e_1)|.
\]

We will prove the following inequality of Lemma 9

\[
A_1 \geq cR \| z_1z_2 \| - 16\varepsilon R^2.
\]

(2.17)

This is trivially true if \( \| z_1z_2 \| \leq 200\varepsilon R \), so from now on we assume \( \| z_1z_2 \| > 200\varepsilon R \). Also, we assume the segment \( a_1b_1 \) is always on a horizontal line. By symmetry, we assume \( a_2 \) is above the line \( a_1b_1 \).

The following notations will be used throughout this section for the proof of Lemma 9. \( H \) denotes the half lune of \( L(a_2, b_2) \) divided by \( a_2b_2 \) which is on the different side of \( a_2b_2 \) from \( z_1 \). Let \( c_2 \) and \( d_2 \) denote the two vertices of \( L(a_2, b_2) \) such that \( c_2 \in H \), i.e., \( c_2 \) is on the different side of \( a_2b_2 \) from \( z_1 \). Let \( c_1 \) and \( d_1 \) the two vertices of \( L(a_1, b_1) \) such that \( c_1 \) is above \( a_1b_1 \). Let \( z = \frac{1}{2} (z_1 + z_2) \) be the middle point of the segment \( z_1z_2 \). Denote by \( l \) the line perpendicular to \( z_1z_2 \) through \( z \), by \( l' \) the line \( z_1z_2 \). For each \( i = 1, 2 \), let segment \( a_ib_i \) intersect respectively with \( \partial B(a_i, r) \) and \( \partial B(b_i, r) \) at \( b'_i \) and \( a'_i \), then \( \|a_i a'_i\| = ||b'_i b'_i|| \leq \varepsilon r \).

We start with a simple lemma which will be used frequently in this section.

**Lemma 21** For each \( i = 1, 2 \), pick \( z'_i \in a_i z_i \) and \( z''_i \in b_i z_i \) such that \( \|z'_i z_i\| = \|z''_i z_i\| = \varepsilon r \).
Figure 2.50. For each $i = 1, 2$, \( \|z'_i z_i\| = \|z''_i z_i\| = \varepsilon r \), and $x_i$ is a point between $z'_i$ and $z''_i$.

then for any $x_i$ between $z'_i$ and $z''_i$, we have

\[
\min\{\|x_1 z_2\|, \|x_2 z_1\|\} \geq \|z_1 z_2\| - \varepsilon R
\]

(see Fig. 2.50).

**Proof** By symmetry, we only need to prove \( \|z_1 x_2\| \geq \|z_1 z_2\| - \varepsilon R \), which follows from the following inequalities

\[
\|x_1 z_2\| \geq \|z_1 z_2\| - \|z_2 x_2\| - \|z_1 x_1\| \geq \|z_1 z_2\| - \varepsilon R.
\]

Thus, the lemma is proved.

Now we are ready to prove EQ(2.17). Here is the overview of the proof for EQ(2.17).

1. We first prove EQ(2.17) holds when $a_2 b_2$ intersects with $\partial L(a'_1, b'_1)$ at most one point in Lemma 22;

2. Then we can assume $a_2 b_2$ intersects with $\partial L(a'_1, b'_1)$ at two points. When $A_1$ is minimal, we can determine the position of the center $o$ and prove in Lemma 25 that $o \in l$,
and that either \( a_2 \in \partial \mathbb{D} \) or \( b_2 \in \partial \mathbb{D} \) if \( o \) and \( d_2 \) are on the same side of line \( a_2b_2 \). To prove Lemma 25, we need the fact that the intersection point of the two lines \( l \) and \( c_2d_2 \) is not far away from \( z \). This fact is proved in Lemma 24;

3. When the segment \( a_2b_2 \) does not intersect with line \( a_1b_1 \), EQ(2.17) is proved in Lemma 26;

4. When the segment \( a_2b_2 \) intersects with line \( a_1b_1 \), EQ(2.17) is proved in Lemma 27.

Let

\[
A = |\mathbb{D} \cap L(e_2) \setminus L(e_1)|.
\]

By Lemma 4, we have \( A_1 \geq A - 2.2\varepsilon R^2 \). Thus, to prove EQ(2.17), it is sufficient to prove the following inequality

\[
A \geq cR \|z_1z_2\| - 13.8\varepsilon R^2.
\]

(2.18)

Figure 2.51. The segment \( a_2b_2 \) intersects with \( \partial L(a'_1, b'_1) \) at most one point.

We now prove EQ(2.17) holds when \( a_2b_2 \) intersects with \( \partial L(a'_1, b'_1) \) at most one point.
Lemma 22 If segment $a_2b_2$ intersects with $\partial L(a'_1, b'_1)$ at most one point, then EQ(2.17) holds.

Proof Clearly, $a_2, b_2 \notin L(a'_1, b'_1)$. Let $S$ denote the half lune of $L(a_2, b_2)$ divided by $a_2b_2$ such that $S \cap L(a'_1, b'_1) \neq \emptyset$, we will prove

$$|S \setminus L(a'_1, b'_1)| \geq 0.25R^2.$$

Then

$$|S \setminus L(a_1, b_1)| \geq 0.25R^2 - 8\varepsilon R^2$$

by part (2) of Lemma 4. Thus EQ(2.18) holds as $|z_1z_2| \leq \sqrt{3}R$ and EQ(2.17) follows. Next we prove EQ(2.19). If $a'_1, b'_1 \notin L(a_2, b_2)$, then EQ(2.19) holds by Lemma 14. If exactly one of $a'_1$ and $b'_1$ is in $L(a_2, b_2) \setminus L_r(a_2, b_2)$, say $b'_1 \in L(a_2, b_2) \setminus L_r(a_2, b_2)$, then $a'_1 \notin L(a_2, b_2)$ (see Fig. 2.51). Pick $u \in a_2b_2$ such that $||a_2u|| = ||a_2b'_1||$. Then $a'_1, b'_1 \notin L(a_2, u)$ and $a_2, u \notin L(a'_1, b'_1)$. Thus EQ(2.19) holds by applying Lemma 14 to $L(a_2, u)$ and $L(a'_1, b'_1)$. If $a'_1, b'_1 \in L(a_2, b_2)$, pick $u \in a_2b_2$ such that $||a_2u|| = \min\{||a_2b'_1||, ||a_2a'_1||\}$. Then $a'_1, b'_1 \notin L(a_2, u)$ and $a_2, u \notin L(a'_1, b'_1)$. Thus EQ(2.19) holds by applying Lemma 14 to $L(a_2, u)$ and $L(a'_1, b'_1)$. Hence Lemma 22 is proved.

The following lemma shows that the two segments $a_2b_2$ and $a_1b_1$ cannot intersect with each other if $||a_1a_2|| < r$.

Lemma 23 If segment $a_2b_2$ intersects with line $a_1b_1$ at $h$ and $||a_1a_2|| < r$, then $h \notin L(a_1, b_1)$ and $h$ is on the ray $a_1b_1$.

Proof By contradiction, we assume $h \in L(a_1, b_1)$. We claim $||b_2a_1|| < r_2$, which is obviously true if both $a_1$ and $b_2$ are on the same side of line $c_1z_1$ since $\angle b_2a_1a_2 > 90^\circ$ (see Fig.
Figure 2.52. Both $a_1$ and $b_2$ are on the same side of the line $c_1 z_1$.

Figure 2.53. Both $b_1$ and $b_2$ are on the same side of the line $c_1 z_1$. 
2.52). If both $b_1$ and $b_2$ are on the same side of line $c_1z_1$, then $||b_2a_1|| < r_2$ since

$$\angle b_2a_2a_1 < \angle b_1a_2a_1 \leq \angle b_1a_1a_2 < \angle b_2a_1a_2$$

(see Fig. 2.53). Thus, $||b_2a_1|| < r_2$ which implies $a_1 \in L_r(a_2, b_2)$, contradiction. Hence, $h \not\in L(a_1, b_1)$. Now we prove $h$ is on the ray $a_1b_1$. By contradiction, assume $h$ is on the ray $b_1a_1$. Then $\angle b_2a_1a_2 > 90^\circ$ and $a_1 \in L(a_2, b_2)$. Since $||a_2a_1|| < r$, then $a_1 \in L_r(a_2, b_2)$ which leads to a contradiction. Thus $h$ is on the ray $a_1b_1$. Thus Lemma 23 is proved.

![Figure 2.54. $a_2, b_2 \not\in L(a_1, b_1)$](image)

According to Lemma 22, from now on we assume segment $a_2b_2$ intersects with $\partial L(a'_1, b'_1)$ at two points. The following lemma proves that the distance between $z$ and the intersection point of the two lines $l$ and $c_2d_2$ is bounded above by $17R$.

**Lemma 24** Assume segment $a_2b_2$ intersects with $\partial L(a'_1, b'_1)$ at two points. Let $l$ intersect with line $c_2d_2$ at $k$, then $||zk|| \leq 17R$. 
Proof. We claim $\angle z_1z_2k \leq 87^\circ$. Assuming the claim is true, then

$$||zk|| = ||z_2z|| \tan \angle z_1z_2k$$
$$\leq \frac{\sqrt{3}}{2} R \cdot \tan 87^\circ$$
$$\leq 17R.$$ 

Thus the lemma holds. Next we prove the claim.

Case 1. Suppose $a_2, b_2 \notin L(a_1, b_1)$. Clearly, segment $a_2b_2$ doesn’t intersect with segment $a_1b_1$. Otherwise if segment $a_2b_2$ intersects with segment $a_1b_1$, then at least one of $a_1$ and $b_1$ is in $L_r(a_2, b_2)$ which leads to a contradiction. First we assume $a_1, a_2$ are on the same side of $l'$. It is easy to see that the claim can be proved by following similar arguments if $a_1, b_2$ are on the same side of $l'$. If $a_1 \in L(a_2, b_2) \setminus L_r(a_2, b_2)$, $b_1 \notin L(a_2, b_2)$ (see Fig. 2.54), let $a_2b_2$ intersect with $\partial B(b_1, r_1)$ at $m$, then $||mb_2|| < ||a_1b_2|| < ||a_2b_2||$ by Lemma 13. Pick $u \in a_2m$ such that $||a_1b_2|| = ||ub_2||$. Since $||a_1b_2|| \geq r_1$, we have

$$||a_2u|| \leq r_2 - r_1 \leq \varepsilon r.$$ 

Let $u' = \frac{1}{2}(b_2 + u)$, then $||z_2u'|| \leq \frac{1}{2} \varepsilon r$ and

$$\sin \angle z_2z_1u' \leq \frac{||z_2u'||}{||z_2z_1||} < \frac{1}{400}. $$

Thus $\angle z_2z_1u' < 1^\circ$. Pick $k' \in l$ such that $k'u||kz_2$. By Lemma 15, we have $\angle z_1u'k' \leq 86^\circ$. Then

$$\angle z_1z_2k = \angle z_2z_1u' + \angle z_1u'k' \leq 87^\circ.$$ 

If $b_1 \in L(a_2, b_2) \setminus L_r(a_2, b_2)$ and $a_1 \notin L(a_2, b_2)$ (see Fig. 2.55), pick $u \in a_2b_2$ such that $||a_1a_2|| = ||ua_2||$. Then $||b_2u|| \leq \varepsilon r$. Let $u' = \frac{1}{2}(a_2 + u)$, pick $k' \in l$ such that $k'u||kz_2$. Then $\angle z_1z_2k \leq \angle z_1u'k' \leq 86^\circ$ by Lemma 15. If $a_1, b_1 \in L(a_2, b_2) \setminus L_r(a_2, b_2)$, then we
shrink $L(a_2, b_2)$ until $a_1 \in \partial L(a_2, b_2)$ or $b_1 \in \partial L(a_2, b_2)$. It is easy to verify that $a_2$ and $b_2$
don’t cross $\partial L(a_1, b_1)$ during shrinking. Thus $\angle z_1 z_2 k \leq 87^\circ$ by the discussion above.

![Figure 2.55](image)

Figure 2.55. $b_1 \in L(a_2, b_2) \setminus L_r(a_2, b_2)$ and $a_1 \notin L(a_2, b_2)$.

Case 2. Suppose segment $a_2 b_2$ doesn’t intersect with segment $a_1 b_1$. By Case 1, we assume at least one of $a_2, b_2$ is in $L(a_1, b_1) \setminus L_r(a_1, b_1)$. If $a_2, b_2 \in L(a_1, b_1) \setminus L_r(a_1, b_1)$ (see Fig. 2.56), pick $v \in a_1 b_1$ such that $||a_1 v|| = \min\{||a_1 b_2||, ||a_1 a_2||\}$. Pick $u \in a_1 v$ such that $||a_1 u|| = ||v b_1||$, then $a_2, b_2 \notin L(u, v)$. Thus the claim is true by applying Case 1 to $L(u, v)$ and $L(a_2, b_2)$. If $a_2 \in L(a_1, b_1) \setminus L_r(a_1, b_1), b_2 \notin L(a_1, b_1)$, pick $v \in a_1 b_1$ such that $||a_1 v|| = ||a_1 a_2||$. Pick $u \in a_1 v$ such that $||a_1 u|| = ||v b_1||$, then $a_2, b_2 \notin L(u, v)$ and the claim is true by Case 1. Similarly, we can prove $\angle z_1 z_2 k \leq 87^\circ$ if $b_2 \in L(a_1, b_1) \setminus L_r(a_1, b_1), a_2 \notin L(a_1, b_1)$.

Case 3. Suppose segment $a_2 b_2$ intersects with segment $a_1 b_1$ at $h$ and at least one of $a_2$ and $b_2$ is in $L(a_1, b_1) \setminus L_r(a_1, b_1)$. By Lemma 23, we have $||a_2 a_1|| \geq r$. It is sufficient to prove $\min\{\angle z_1 z_2 a_2, \angle z_1 z_2 b_2\} \geq 3^\circ$. By contradiction, we assume $\min\{\angle z_1 z_2 a_2, \angle z_1 z_2 b_2\} < 3^\circ$.

Subcase 3.1. $a_2, b_2 \in B(a_1, r) \cup B(b_1, r)$. Then $||a_2 b_1|| < r$ and $||b_2 b_1|| \geq r_2$
Figure 2.56. $a_2, b_2 \in L(a_1, b_1) \setminus L_r(a_1, b_1)$.

Figure 2.57. The point $z_1$ is above $b_2a_2$ and $c_2$ is below $b_2a_2$. 
as \( b_1 \notin L_r(a_2, b_2) \). Hence \( ||a_1b_2|| < r \), \( a_2 \in L(a_1, b_1) \) and \( b_2 \notin L(a_1, b_1) \). First we assume \( z_1 \) is above \( b_2a_2 \), then \( c_2 \) is below \( b_2a_2 \) and \( ||b_2z_1|| \geq ||a_2z_1|| \) (see Fig. 2.57). Thus \( \angle z_1z_2a_2 \leq 90^\circ \). Hence \( \angle z_1z_2a_2 < 3^\circ \). Pick \( z''_1 \in z_2a_2 \) such that \( z_2a_2 \) is perpendicular to \( z_1z''_1 \). Let \( a_1b_1 \) intersect with \( \partial B(1, ||a_1b_2||) \) at \( a''_1 \), then \( ||a''_2b_1|| \leq \varepsilon r \) as \( a_2 \notin L_r(a_1, b_1) \). Thus

\[
\frac{r_1}{2} - \varepsilon r \leq ||z_1b_1|| - ||a''_2b_1|| \leq ||a_2z''_1|| \leq \frac{r_2}{2} - ||z_1z_2|| \cos 3^\circ .
\]

Hence \( ||z_1z_2|| < 2\varepsilon R \) which contradicts with the fact that \( ||z_1z_2|| > 200\varepsilon R \). Next we assume \( z_1 \) is below \( b_2a_2 \). Then \( c_2 \) is above \( b_2a_2 \). If \( \angle z_1z_2a_2 \leq 90^\circ \), then \( \angle z_1z_2a_2 < 3^\circ \) (see Fig. 2.58). Pick \( z''_1 \in z_2a_2 \) such that \( z_2a_2 \) is perpendicular to \( z_1z''_1 \). Let \( b_2a_2 \) intersect with \( \partial B(2, ||a_1a_2||) \) at \( a''_1 \), then \( ||a''_2b_2|| \leq \varepsilon r \) as \( a_1 \notin L_r(a_2, b_2) \). Thus

\[
\frac{r_1}{2} = ||a_1z_1|| \geq ||a''_1z''_1||
\]

\[
= ||z''_1z_2|| + \frac{r_2}{2} - ||a''_2b_2||
\]

\[
\geq ||z_1z_2|| \cos 3^\circ + \frac{r_2}{2} - \varepsilon r.
\]

Hence \( ||z_1z_2|| < 2\varepsilon R \) which contradicts with the fact that \( ||z_1z_2|| > 200\varepsilon R \). So we assume

![Figure 2.58. The point \( z_1 \) is below \( b_2a_2 \) and \( \angle z_1z_2a_2 \leq 90^\circ \).](image-url)
\[ \angle z_1z_2a_2 > 90^\circ, \text{ then } \angle z_1z_2b_2 < 3^\circ \] (see Fig. 2.59). Extend \( b_2z_1 \) to \( a'_2 \) such that \( ||z_1a'_2|| = ||z_1a_2'|| \), then \( ||b_2a'_2|| \geq ||b_2b_1|| \geq r_2 \) and \( \angle z_2z_1b_2 \leq \angle z_1z_2b_2 < 3^\circ \). Note that

\[ \angle z_1b_2a_2 \leq \angle b_1b_2a_2 \leq \angle b_1a_1a_2 < 60^\circ, \]

which leads to a contradiction.

![Figure 2.59. The point \( z_1 \) is below \( b_2a_2 \) and \( \angle z_1z_2a_2 > 90^\circ \).](image)

Subcase 3.2. \( a_2 \notin B(a_1, r) \cup B(b_1, r) \). By Lemma 5, we have

\[ \angle z_1z_2a_2 > \angle z_1ha_2 > \angle z_1b_1a_2 > 59.9^\circ. \]

We will prove \( \angle z_1z_2b_2 \geq 30^\circ \), which leads to a contradiction as \( \min\{\angle z_1z_2a_2, \angle z_1z_2b_2\} < 3^\circ \).

Subcase 3.2.1. \( a_2 \in L(a_1, b_1) \setminus L_r(a_1, b_1) \), then \( b_2 \notin L(a_1, b_1) \) (see Fig. 2.60). By symmetry we assume \( b_2 \) is on the same side of line \( c_1z_1 \) as \( b_1 \). When \( a_2 \) and \( L(a_1, b_1) \) are fixed, \( \angle z_1z_2b_2 \) is minimal as \( b_2 \in \partial L(a_1, b_1) \). So we assume \( b_2 \in \partial L(a_1, b_1) \). Extend \( b_2z_1 \)
Figure 2.60. $a_2 \in L(a_1, b_1) \setminus L_r(a_1, b_1)$ and $b_2 \notin L(a_1, b_1)$.

to $b_2'' \in \partial L(a_1, b_1)$, then

$$\angle z_1 z_2 b_2 = \angle b_2'' a_2 b_2 \geq \angle b_2'' c_1 b_2 \geq \angle b_2'' c_1 z_1 \geq 30^\circ.$$  

Subcase 3.2.2. $b_2 \in L(a_1, b_1) \setminus L_r(a_1, b_1)$, then $a_2 \notin L(a_1, b_1)$ (see Fig. 2.61).
When $a_2$ and $L(a_1, b_1)$ are fixed, $\angle z_1 z_2 b_2$ is minimal as $b_2 \in \partial L_r(a_1, b_1)$. So we assume $b_2 \in \partial L_r(a_1, b_1)$. Extend $b_2 z_1$ to $b_2'' \in \partial L_r(b_1, a_1)$. Let $\theta_1 = \angle b_2'' a_2 b_2$, $\theta_2 = \angle a_2 b_2'' b_2$, then $\theta_1 + \theta_2 \geq 90^\circ$ as $\angle a_2 b_2 b_2'' \leq \angle a_2 b_2 a_1 \leq 90^\circ$. Since

$$\sin \theta_1 = \frac{||b_2 b_2''||}{||b_2 a_2||} \sin \theta_2 \geq \frac{r_1 - 2\varepsilon r}{r_2} \sin \theta_2 > 0.99 \sin \theta_2,$$

we have $\angle z_1 z_2 b_2 = \theta_1 \geq 30^\circ$.

Subcase 3.3. $b_2 \notin B(a_1, r) \cup B(b_1, r)$ (see Fig. 2.62). Since $||a_2 b_1|| < r$, we have $||b_2 b_1|| \geq ||b_2 a_2||$ as $b_1 \notin L_r(a_2, b_2)$. Then $a_2 \in L(a_1, b_1) \setminus L_r(a_1, b_1)$. By Lemma 5, we have $\angle z_1 z_2 b_2 > \angle z_1 h b_2 > \angle z_1 b_1 b_2 > 59.9^\circ$. Next we prove $\angle z_1 z_2 a_2 \geq 30^\circ$.

When $b_2$ and $L(a_1, b_1)$ are fixed, $\angle z_1 z_2 a_2$ is minimal as $a_2 \in \partial L_r(a_1, b_1)$. So we assume
Figure 2.61. \( b_2 \in L(a_1, b_1) \setminus L_r(a_1, b_1) \) and \( a_2 \notin L(a_1, b_1) \).

Figure 2.62. \( b_2 \notin B(a_1, r) \cup B(b_1, r) \).
$a_2 \in \partial L_r(a_1, b_1)$. Similar to subcase 3.2.2, we have $\angle z_1 z_2 a_2 \geq 30^\circ$, which leads to a contradiction as $\min\{\angle z_1 z_2 a_2, \angle z_1 z_2 b_2\} < 3^\circ$.

Therefore, the claim is true and Lemma 24 is proved.

Based on the upper bound of the distance between $z$ and the intersection point of the two lines $l$ and $c_2 d_2$ obtained in Lemma 24, the following lemma proves that $o \in l$, and either $a_2 \in \partial D$ or $b_2 \in \partial D$ when $A$ is minimal.

**Lemma 25** Assume segment $a_2 b_2$ intersects with $\partial L(a_1', b_1')$ at two points and $o$ is on the same side of line $a_2 b_2$ as $d_2$. When $A$ is minimal, we have (1) $o \in l$; (2) either $a_2 \in \partial D$ or $b_2 \in \partial D$.

**Proof** Let $l$ intersect with line $c_2 d_2$ at $k$, by Lemma 24 we have $||z k|| \leq 17R$. Without loss of generality we assume $a_1, a_2$ and $k$ are on the same side of $l'$.

Figure 2.63. Fix $p \in \partial D$ and rotate $D$ counterclockwise along $p$ by a sufficiently small angle.

(1) We prove by contradiction and assume the contrary that $o \notin l$. Then $o$ is above $l$. First we assume $o$ is on the same side of $l'$ as $a_1$ (see Fig. 2.63). Since $H \setminus \mathbb{D} \neq \emptyset$, we
have

\[ |z_0| \geq R_0 - 2(r_1 + r_2) \geq 96R > |zk|. \]

Then \( o \) is above line \( c_2d_2 \). Hence \( |oa_2| < |ob_2| \leq R_0 \). Let \( \partial\mathbb{D} \) intersect \( \partial H \) at \( p \) and \( q \) such that \( p \) is closer to \( b_2 \) than \( q \). Fix \( p \in \partial\mathbb{D} \) and rotate \( \mathbb{D} \) counterclockwise along \( p \) by a sufficiently small angle \( \varepsilon \). Let \( o' \) denote the new position of \( o \), \( \mathbb{D}' \) denote the disk centered at \( o' \) with radius \( R_0 \). We choose \( \varepsilon \) sufficiently small so that \( o' \) is still above the line \( l \) and both \( a_2 \) and \( b_2 \) are inside \( \mathbb{D}' \). We claim that \( A \) becomes strictly smaller after rotating, which leads to a contradiction. This claim can be verified by using the exact same argument as in the proof for part (1) of Lemma 16 in Section 2.4. If \( o \) is on the same side of \( l' \) as \( b_1 \), we can apply a similar rotation argument as above which also leads to a contradiction. Thus, this part of the lemma holds.

![Figure 2.64](image-url)  

Figure 2.64. Shift \( \mathbb{D} \) along \( l \) away from the two lunes by a small distance \( \varepsilon \).

(2) First we assume \( o \) is on the same side of line \( l' \) as \( a_1 \) (see Fig. 2.64). By contradiction, assume \( a_2, b_2 \not\in \partial\mathbb{D} \). Similar to (1), we have \( |oa_2| < |ob_2| < R_0 \). Since \( o \in l \), we shift \( \mathbb{D} \) along \( l \) away from the two lunes by a sufficiently small distance \( \varepsilon \). Let \( o' \) denote the new position of \( o \), \( \mathbb{D}' \) denote the disk centered at \( o' \) with radius \( R_0 \). We choose \( \varepsilon \) sufficiently small so that \( o' \) is still below line \( a_2b_2 \) and both \( a_2 \) and \( b_2 \) are inside \( \mathbb{D}' \). We
claim $A$ becomes strictly smaller after shifting, which leads to a contradiction. The claim can be verified by using the same argument as in the proof for part (2) of Lemma 16 in Section 2.4. If $o$ is on the same side of $l'$ as $b_1$, we can apply a similar shift argument as above which also leads to a contradiction. Hence the lemma is proved.

The following lemma proves that EQ(2.17) holds when segment $a_2b_2$ doesn’t intersect with line $a_1b_1$, but intersects with $\partial L(a'_1, b'_1)$ at two points.

**Lemma 26** Assume segment $a_2b_2$ doesn’t intersect with line $a_1b_1$, but intersects with $\partial L(a'_1, b'_1)$ at two points, then EQ(2.17) holds.

**Proof** The lemma is proved in the following two cases:

**Case 1.** $H \subset \mathbb{D}$. Since

$$A = |\mathbb{D} \cap \mathcal{L}(a_2, b_2) \setminus \mathcal{L}(a_1, b_1)| \geq |H \setminus \mathcal{L}(a_1, b_1)|,$$
we will prove the following inequality which implies EQ(2.17):

\[ |H \setminus L(a_1, b_1)| \geq cR \|z_1z_2\| - 13.88R^2. \]  

(2.20)

First we can assume \(a_1, a_2\) are on the same side of line \(l'\). It is easy to see that EQ(2.20) can be proved by following similar arguments if \(a_1, b_2\) are on the same side of line \(l'\) since the unit-area disk \(D\) is not considered in this case.

**Subcase 1.1.** \(a_2, b_2 \notin L(a_1, b_1)\). EQ(2.20) holds by Lemma 7 if \(a_1, b_1 \notin L(a_2, b_2)\).

So we assume at least one of \(a_1\) and \(b_1\) is in \(L(a_2, b_2)\). First we assume \(b_1 \in L(a_2, b_2) \setminus L_r(a_2, b_2)\), pick \(u \in a_2b_2\) such that \(\|a_2u\| = \|a_2b_1\|\), then \(u \notin L(a_1, b_1)\) and \(b_1 \notin L(a_2, u)\).

If \(a_1 \notin L(a_2, u)\), let \(u' = \frac{1}{2}(a_2 + u)\), then \(|H \setminus L(a_1, b_1)| \geq cR \|u'z_1\|\) by Lemma 7 (see Fig. 2.65). Thus EQ(2.20) holds by Lemma 21.

If \(a_1 \in L(a_2, u)\), pick \(v \in a_2u\) such that \(\|uv\| = \|a_1u\|\), then \(v \notin L(a_1, b_1)\) by Lemma 13 (see Fig. 2.66). Also, \(\|a_2v\| = \|a_2u\| - \|uv\| \leq r_2 - r_1 \leq \epsilon r\). Let \(w = \frac{1}{2}(u + v)\), then \(|H \setminus L(a_1, b_1)| \geq cR \|wz_1\|\) by Lemma 7. Thus EQ(2.20) holds by
Lemma 21.

Next we assume \( b_1 \notin L(a_2, b_2) \), then \( a_1 \in L(a_2, b_2) \setminus L_r(a_2, b_2) \). Pick \( u \in a_2b_2 \) such that \( ||b_2a_1|| = ||ub_2|| \), then \( ||ua_2|| \leq r_2 - r_1 \leq \epsilon r \) (see Fig. 2.67). Also, \( u \notin L(a_1, b_1) \) by Lemma 13 since \( ||a_1b_2|| = ||ub_2|| \). Let \( u' = \frac{1}{2}(a_2 + u) \), then \( |H \setminus L(a_1, b_1)| \geq cR \|u'z_1\| \) by Lemma 7. Thus EQ(2.20) holds by Lemma 21.

Subcase 1.2. At least one of \( a_2 \) and \( b_2 \) is in \( L(a_1, b_1) \). If \( a_2, b_2 \in L(a_1, b_1) \setminus L_r(a_1, b_1) \),
pick $u \in a_1b_1$ such that $||a_1u|| = \min\{||a_1b_2||, ||a_1a_2||\}$, then $a_2, b_2 \notin L(a_1, u)$ (see Fig. 2.68). Thus EQ(2.20) holds by applying Subcase 1.1 above to $L(a_1, u)$ and $L(a_2, b_2)$. If $b_2 \in L(a_1, b_1) \setminus L_r(a_1, b_1)$ but $a_2 \notin L(a_1, b_1)$, pick $u \in a_1b_1$ such that $||a_1u|| = ||a_1b_2||$. Then $a_2, b_2 \notin L(a_1, u)$. Thus EQ(2.20) holds by applying Subcase 1.1 above to $L(a_1, u)$ and $L(a_2, b_2)$. Similarly, EQ(2.20) can be proved if $a_2 \in L(a_1, b_1) \setminus L_r(a_1, b_1)$ but $b_2 \notin L(a_1, b_1)$.

Case 2. Assume $H \setminus \mathbb{D} \neq \emptyset$, then $o$ is below line $a_2b_2$. By Lemma 25, when $A$ is minimal we have $(i) o \in l$; $(ii) a_2 \in \partial \mathbb{D}$ or $b_2 \in \partial \mathbb{D}$.

Subcase 2.1. $a_2, b_2 \notin L(a_1, b_1)$. If $a_1, b_1 \notin L(a_2, b_2)$, then EQ(2.18) holds by part (1) of Lemma 20 in Section 2.4. If exactly one of $a_1, b_1$ is in $L(a_2, b_2) \setminus L_r(a_2, b_2)$, then EQ(2.18) holds by part (2) of Lemma 20. So we assume both $a_1$ and $b_1$ are inside $L(a_2, b_2) \setminus L_r(a_2, b_2)$, then $b_2$ is closer to line $a_1b_1$ than $a_2$. Since $a_2, b_2 \notin L(a_1, b_1)$, by symmetry we assume $b_2$ is on the same side of $l'$ as $b_1$. Let $a_2b_2$ intersect with $\partial B(b_1, r_1)$ and $\partial B(a_2, ||a_2b_1||)$ at $u$ and $v$, respectively.
First we assume \(|a_2u| \geq |vb_2|\) (see Fig. 2.69). Pick \(v' \in a_2u\) such that \(|a_2v'| = |vb_2|\), then \(b_1 \notin L(v, v')\) and \(v, v' \notin L(a_1, b_1)\). If \(a_1 \notin L(v, v')\), then EQ(2.18) holds by applying part (1) of Lemma 20 to the two lunes \(L(a_1, b_1)\) and \(L(v, v')\). If \(a_1 \in L(v, v')\), then EQ(2.18) holds by applying part (2) of Lemma 20 to the two lunes \(L(a_1, b_1)\) and \(L(v, v')\).

Next we assume \(|a_2u| < |vb_2|\) (see Fig. 2.70). Pick \(u' \in vb_2\) such that \(|a_1u'| = |ub_1|\), then it is easy to verify that \(|a_1u'| > |uu'|\) by following a similar argument as in the proof for Lemma 13. Thus \(a_1 \notin L(u, u')\) and \(b_1 \in L(u, u')\). Hence EQ(2.18) holds by applying part (2) of Lemma 20 to the two lunes \(L(a_1, b_1)\) and \(L(u, u')\).

Subcase 2.2. At least one of \(a_2\) and \(b_2\) is in \(L(a_1, b_1) \setminus L_r(a_1, b_1)\). First we assume \(a_2, b_2 \in L(a_1, b_1) \setminus L_r(a_1, b_1)\) (see Fig. 2.71). Pick \(u \in z_1b_1\) such that \(|a_1u| = \min\{||a_1b_2||, ||a_1a_2||\}\), then pick \(v \in a_1z_1\) such that \(|a_1v| = |ab_1|\). Then \(a_2, b_2 \notin L(u, v)\) and EQ(2.18) holds by applying Subcase 2.1 to the two lunes \(L(u, v)\) and \(L(a_2, b_2)\). Next we assume \(a_2 \notin L(a_1, b_1)\), then \(b_2 \in L(a_1, b_1) \setminus L_r(a_1, b_1)\) (see Fig. 2.72). Pick \(u \in z_1b_1\) such that \(|a_1u| = ||a_1b_2||\), pick \(v \in a_1z_1\) such that \(|a_1v| = |ab_1|\). Then \(a_2, b_2 \notin L(u, v)\)
and EQ(2.18) holds by applying Subcase 2.1 to the two lunes $L(u, v)$ and $L(a_2, b_2)$. Similarly, EQ(2.18) can be proved if $b_2 \notin L(a_1, b_1)$ and $a_2 \in L(a_1, b_1) \setminus L_r(a_1, b_1)$.

This completes the proof of Lemma 26.

The following lemma proves that EQ(2.17) holds when segment $a_2b_2$ intersects with line $a_1b_1$.

**Lemma 27** Assume segment $a_2b_2$ intersects with line $a_1b_1$ at $h$ and intersects with $\partial B(a_1', b_1')$ at two points, then EQ(2.17) holds.

**Proof** Since $a_2$ is above $a_1b_1$, we have that $b_2$ is below $a_1b_1$. Let segment $a_2b_2$ intersect with $\partial B(a_1, r)$ at $x$ and $y$ such that $x$ is closer to $a_2$ than $y$ (see Fig. 2.73). Clearly, $|C_{xy}|$ is maximal as $a_2, b_2 \in \partial B(a_1, r)$. In the case of $|C_{xy}|$ maximum, we have $\angle a_2a_1b_2 < 61^\circ$ since

$$\sin \frac{1}{2} \angle a_2a_1b_2 \leq \frac{r_2}{2r} < \frac{1.001}{2}.$$
Thus,

\[
|C_{xy}| \leq \frac{1}{2} r^2 (\angle a_2 a_1 b_2 - \sin \angle a_2 a_1 b_2) \tag{2.21}
\]

\[
\leq \frac{1}{2} r^2 \left( \frac{61}{180} \pi - \sin 61^\circ \right)
\]

\[
\leq 0.096 r^2.
\]

First we assume \( ||a_2 a_1|| < r \). By Lemma 23, \( h \notin L(a_1, b_1) \) and \( h \) is on the ray \( a_1 b_1 \). Hence, \( \angle b_2 b_1 a_2 > 90^\circ \), \( b_1 \in L(a_2, b_2) \) and \( ||a_2 b_1|| \geq r_1 \) (see Fig. 2.74). Pick \( u \in a_2 b_2 \) such that \( ||a_2 b_1|| = ||a_2 u|| \). If \( H \subset \mathbb{D} \), then by applying Lemma 7 to the two lunes \( L(a_1, b_1) \) and \( L(a_2, u) \), we have \( A \geq cR ||z_1 z_2'|| \), where \( z_2' \) is the middle point of \( a_2 u \). Thus, EQ(2.18) holds by Lemma 21. So we assume \( H \setminus \mathbb{D} \neq \emptyset \), then \( o \) is below line \( a_2 b_2 \). By Lemma 25, we have \( o \in l \); either \( a_2 \in \partial \mathbb{D} \) or \( b_2 \in \partial \mathbb{D} \). Pick \( a_1'', b_1'' \in a_1 b_1 \) such that \( ||a_1'' a_1|| = ||b_1'' b_1|| = \varepsilon r \). Pick \( v \in a_2 z_2 \) such that \( ||va_2|| = ||ub_2|| \). Since \( ||a_2 b_1|| = ||a_2 u|| \geq r_1 \), we have \( ||va_2|| = ||ub_2|| \leq \varepsilon r \). Thus \( u, v \notin L(a_1'', b_1'') \). Applying Lemma 20
Figure 2.73. Segment $a_2b_2$ intersects with $\partial B(a_1, r)$ at $x$ and $y$ such that $x$ is closer to $a_2$ than $y$.

to the two lunes $L(u, v)$ and $L(a''_1, b''_1)$, we have

$$|H \cap \mathbb{D} \setminus L(a''_1, b''_1)| \geq cR \|z_1z_2\| - \varepsilon R^2.$$

By part (2) of Lemma 4, $|L(a_1, b_1) \setminus L(a''_1, b''_1)| \leq 8\varepsilon R^2$. Thus,

$$|H \cap \mathbb{D} \setminus L(a_1, b_1)| \geq cR \|z_1z_2\| - 9\varepsilon R^2.$$

Hence EQ(2.18) holds and EQ(2.17) follows.

Next we assume $||a_2a_1|| \geq r$ and the lemma is proved in the following three cases:

Case 1. $a_2, b_2 \in B(a_1, r) \cup B(b_1, r)$. Then $||a_2b_1|| < r$ and $||b_2b_1|| \geq r_2$ as $b_1 \notin L_r(a_2, b_2)$. Hence $||a_1b_2|| < r, a_2 \in L(a_1, b_1) \setminus L_r(a_1, b_1)$ and $b_2 \notin L(a_1, b_1)$ (see Fig. 2.75). If $z_1 \in a_2b_2$, then $||z_1z_2|| \leq \varepsilon R$ and EQ(2.18) holds. So we assume $z_1 \notin a_2b_2$. We further consider two subcases:

Subcase 1.1. $z_1$ is below $b_2a_2$ (see Fig. 2.75). Let $a_2b_2$ intersect with $\partial L(a_1, b_1)$ at
Figure 2.74. $|a_2a_1| < r$, $h \notin L(a_1, b_1)$ and $h$ is on the ray $a_1b_1$.

$b'_2$, pick $a'_2 \in a_2b'_2$ such that $|a'_2a_2| = |b'_2b_2|$. Extend $b_2a_2$ to $u \in \partial L(a_1, b_1)$, extend $b'_2z_1$ to $e \in \partial L(a'_2, b'_2)$ and to $v \in \partial L(a_1, b_1)$. Since $a_1 \notin L_r(a_2, b_2)$, we have $|b_2b'_2| \leq \varepsilon r$. Pick $w \in z_1v$ such that $|a_1w| = |a_1a_2|$, then $|b'_2a_2| > |b'_2w|$. Let arc $c_1a_1$ intersect with arc $c'_2a'_2$ at $f$, $H' \subset H$ denote the half lune of $L(a'_2, b'_2)$ divided by $a'_2b'_2$. Let $c'_2$ and $c_3 \in H$ the vertices of $L(a'_2, b'_2)$ and $L(b'_2, e)$, respectively. Let $r'_2 = |a'_2b'_2|$, then $r'_2 \leq |b'_2b_1| = r_1$ since $b_1 \notin L(a_2, b_2)$. First we assume $H' \subset \mathbb{D}$.

EQ(2.18) follows immediately from the following two inequalities

\[ |H' \setminus L(a_1, b_1)| \geq |H' \setminus L(b'_2, e)| - 6.3\varepsilon R^2. \tag{2.22} \]

\[ |H' \setminus L(b'_2, e)| \geq 0.99R |z_1z_2| - 1.1\varepsilon R^2. \tag{2.23} \]

So it is sufficient to prove EQ(2.22) and EQ(2.23). First we prove EQ(2.22). It holds if $c_3 \notin L(a_1, b_1)$ since $r'_2 \leq r_1$ and arc $b'_2f$ is contained in $L(b'_2, e)$. If $c_3 \in L(a_1, b_1)$, then

\[ |ec_3| \geq r_2 - 2\varepsilon r \geq r_1 - 3\varepsilon r \geq |ef| - 3\varepsilon r. \]
Let $ef$ intersect with arc $b'_2c_3$ at $g$, then

$$\|gc_3\| \leq \|gf\| \leq 3\varepsilon r < 0.003r.$$ 

Thus, $\angle egc_3 > 89^\circ$ by Lemma 5(3). Hence,

$$\|fc_3\| \leq 2\|fg\| \leq 6\varepsilon r.$$ 

Let $S$ denote the region surrounded by arc $b'_2c_3$, arc $b'_2f$ and arc $fc_3$, $M = \max \left\{ \|b'_2c_3\|, \|b'_2f\| \right\}$, then $M \leq \frac{\pi}{3}R$ and

$$|S| \leq M\|fc_3\| < 6.3\varepsilon R^2.$$ 

Note that

$$|H'\setminus L(a_1, b_1)| = |H'\setminus L_{eb_2}| - |S|.$$ 

Thus, EQ(2.22) holds.

To prove EQ(2.23), we claim $\|wv\| \leq 1.07\varepsilon r$. Assuming the claim is true, note that $\|ew\| \leq \|a'_2a_2\|$, we have

$$\|ev\| \leq \|ew\| + \|wv\| \leq 2.07\varepsilon r.$$ 

Since $\angle eb'_2a'_2 = \angle c_3b'_2c'_2$, we have

$$r_2 \|z_1z_2\| = \frac{r_2}{2}\|va'_2\| \leq \frac{r_2}{2}(\|ve\| + \|ea'_2\|)$$

$$\leq 1.035\varepsilon r_2 + \frac{1}{2}r_2^2\angle eb'_2a'_2$$

$$\leq 1.1\varepsilon R^2 + |H'\setminus L(b_2, e)|.$$
Hence EQ(2.23) holds. Next we prove the claim. Let \( \varphi = \angle a_1 w b_1, \psi = \angle a_1 b_1 w \). Since

\[
||wb_1|| \leq ||a_2 b_1|| < r \leq ||wa_1|| \leq ||a_1 b_1||,
\]

we have \( \varphi > 60^\circ \). By the law of sine, we have

\[
\frac{\sin \varphi}{\sin \psi} = \frac{||a_1 b_1||}{||a_1 w||} \leq \frac{R}{r} = 1 + \varepsilon < 1.001.
\]

If \( \psi \leq 59.9^\circ \), then

\[
\sin \varphi < 1.001 \sin \psi < \sin 60^\circ,
\]

which leads to a contradiction. Thus \( \psi > 59.9^\circ \). Extend \( a_1 w \) to \( v' \in \partial L(a_1, b_1) \) (see Fig. 2.76). Let \( \alpha = \angle v w v', \beta = \angle v b'_2 b_1 \) and \( \gamma = \angle v a_1 v' \). If \( \beta \geq 20.1^\circ \), then \( \angle a_1 b_1 b'_2 > \beta \geq 20.1^\circ \). Thus

\[
\angle b_1 w b'_2 < 80^\circ < \angle wb_1 b'_2.
\]
Figure 2.76. $|wv'| < |vv'| < \varepsilon r$ and $|wv| < 1.07\varepsilon r$.

Hence,

$$||b_2b_1|| < ||b_2w|| < ||b_2a_2|| \leq ||b_2a_2||.$$

Therefore, $b_1 \in L_r(a_2, b_2)$, which leads to a contradiction. Then $\beta < 20.1^\circ$ and

$$\alpha = \beta + \gamma < 2\beta < 40.2^\circ.$$

Thus $\angle wvv' > 49.8^\circ$ and $|vv'| < |ww'| \leq \varepsilon r$. Since

$$\sin \gamma < |vv'| / r_1 \leq \varepsilon < 0.001,$$

we have $\alpha = \beta + \gamma < 20.16^\circ$. Then

$$|wv| \leq \frac{|wv'|}{\cos \alpha} < 1.07\varepsilon r$$

and the claim is true.

Next we assume $H \setminus D \neq \emptyset$, then $o$ is on the same side of line $a_2b_2$ as $d_2$. Then by
Lemma 25, we have \( o \in l \); if \( o \) is on the same side of \( l' \) as \( a_2 \) (respectively, \( b_2 \)), then \( b_2 \in \partial \mathbb{D} \) (respectively, \( a_2 \in \partial \mathbb{D} \)). Pick \( e' \in b_2e \) such that \( \|b_2'e'\| = \|ev\| \) (see Fig. 2.75). Then \( \|b_2'e'\| \leq 2.07 \varepsilon r \) and

\[
e' \in L(a_2', b_2') \setminus L(a_2', b_2')(r - 1.07 \varepsilon r).
\]

By Lemma 20,

\[ |D \cap L(a_2', b_2') \setminus L(e', e)| \geq cR \|z_1z_2\| - \varepsilon R^2. \]

Since \( r'_2 \leq r_1 < \|b_2'v\| \), we have \( L(a_2', b_2') \setminus H' \subset L(v, b_2') \). Thus by Lemma 4,

\[
A_1 \geq |D \cap L(a_2', b_2') \setminus L(a_1, b_1)| - 2.2 \varepsilon R^2 \\
\geq |D \cap L(a_2', b_2') \setminus L(v, b_2')| - 2.2 \varepsilon R^2 \\
\geq |D \cap L(a_2', b_2') \setminus L(e', e)\| - 15 \varepsilon R^2 \\
\geq cR \|z_1z_2\| - 16 \varepsilon R^2.
\]

Subcase 1.2. \( z_1 \) is above \( b_2a_2 \) (see Fig. 2.77). Pick \( b_2'' \in z_1b_1 \) such that \( \|a_1a_2\| = \)

![Figure 2.77. The point \( z_1 \) is above \( b_2a_2 \) and \( a_2, b_2 \in B(a_1, r) \cup B(b_1, r) \).](image-url)
\[ \|a_1b_1''\|, \text{ then } \|b_1b_1''\| \leq \varepsilon r. \] Pick \( a_1'' \in a_1z_1 \) such that \( \|a_1a_1''\| = \|b_1b_1''\| \), extend \( a_2z_1 \) to \( b_2'' \) such that \( \|z_1a_2\| = \|z_1b_2''\| \), then \( a_2 \notin L(a_1'',b_1'') \) and \( b_1'' \in L(a_2,b_2). \) Let \( r_1' = ||a_1''b_1''||, \) \( r_2' = ||a_2b_2''|| \), then

\[ r_2 - 3\varepsilon r \leq r_1 - 2\varepsilon r \leq r_1' < r_2'. \]

Since \( b_2 \notin L(a_1,b_1), \) \( b_2'' \in L(a_1,b_1), \) we have \( r_2' < r_2. \) Let \( \partial B(a_2,r_2') \) intersect with segment \( b_2a_2 \) and arc \( d_2a_2 \) at \( e, f, \) respectively, then \( ||b_2e|| = r_2 - r_2' \leq 3\varepsilon r. \) By Lemma 5, \( \angle fa_2e < 61^\circ < 1.07. \) First we assume \( H \subset \mathbb{D}. \) By Lemma 4,

\[ A_1 \geq |H \setminus L(a_1,b_1)| - 2.2\varepsilon R^2 \]
\[ \geq |H \setminus L(a_1'',b_1'')| - 10.2\varepsilon R^2. \]

We will prove

\[ |H \setminus L(a_1'',b_1'')| \geq |H \setminus L(a_2,b_2)| - 3.3\varepsilon R^2. \] (2.24)

\[ |H \setminus L(a_2,b_2)| \geq 0.99R \|z_1z_2\| - 1.5\varepsilon R^2. \] (2.25)

Let \( S \) denote the region surrounded by \( b_2e \) and arcs \( b_2c_2, ef \) and \( c_2f, \) then

\[ |S| < \frac{1}{2}(r_2^2 - r_2'^2) \cdot \angle fa_2e < 3.3\varepsilon R^2. \]

Thus EQ(2.24) holds from \( |H \setminus L(a_1'',b_1'')| \geq |H \setminus L(a_2,b_2)| - |S|. \) To prove EQ(2.25), let \( c_2'' \) denote the vertex of \( L(a_2,b_2) \) below line \( a_2b_2. \) Since \( \angle b_2'a_2b_2 = \angle c_2''a_2c_2, \) we have \( \angle c_2''a_2b_2 < 60^\circ \) and \( c_2'' \in H. \) Extend \( a_2c_2'' \) to \( g \in \text{arc } b_2c_2. \) Since \( r_2 \geq r_2' \), we have \( |C_{a_2c_2}| \geq |C_{a_2c_2}|. \) Then

\[ |H \setminus L(a_2,b_2)| \geq |\angle c_2a_2g| + |C_{a_2c_2}| - |C_{a_2c_2}| \geq |\angle b_2'a_2e|. \]
Thus

\[ r_2 \|z_1 z_2\| = \frac{r_2^2 \|b_2 b_2''\|}{2} \leq \frac{r_2^2}{2} (\|b_2 e\| + \|e b_2''\|) \leq 1.5 \varepsilon r_2^2 + |\mathcal{L} b_2'' a_2 e| \leq 1.5 \varepsilon R^2 + |H \setminus L(a_2, b_2)|. \]

Hence EQ(2.25) holds and EQ(2.18) follows. Next we assume \( H \setminus \mathbb{D} \neq \emptyset \), then \( o \) is on the same side of line \( a_2 b_2 \) as \( d_2 \). Thus by Lemma 25, we have \( o \in l \); if \( o \) is on the same side of \( l' \) as \( a_2 \) (respectively, \( b_2 \)), then \( b_2 \in \partial \mathbb{D} \) (respectively, \( a_2 \in \partial \mathbb{D} \)). By Lemma 20, we have

\[ |\mathbb{D} \cap H \setminus L(a_2, b_2)| \geq c R \|z_1 z_2\| - \varepsilon R^2. \]

By Lemma 4 and EQ(2.24),

\[ A_1 \geq |\mathbb{D} \cap H \setminus L(a_1, b_1)| - 2.2 \varepsilon R^2 \geq |\mathbb{D} \cap H \setminus L(a_2, b_2)| - 15 \varepsilon R^2 \geq c R \|z_1 z_2\| - 16 \varepsilon R^2. \]

Case 2. \( a_2 \notin B(a_1, r) \cup B(b_1, r) \). Let \( a_2 b_2 \) intersect with \( \partial L_r(a_1, b_1) \) (respectively, \( \partial L_r(b_1, a_1) \)) at \( x \) and \( y \) if \( b_2 \) is on the same side of line \( c_1 z_1 \) as \( b_1 \) (respectively, \( a_1 \)). If \( H \subset \mathbb{D} \), then EQ(2.17) holds since \( |H| = \frac{2 \pi}{35} r_2^2 \) and

\[ A_1 \geq |H| - |C_{xy}| \geq 2.35 r_2^2 \]

by EQ(2.21). Next we assume \( H \setminus \mathbb{D} \neq \emptyset \), then \( o \) is below line \( a_2 b_2 \). By Lemma 25, we have \( o \in l \); either \( a_2 \in \partial \mathbb{D} \) or \( b_2 \in \partial \mathbb{D} \).

Case 2.1. Assume \( h \notin L(a_1, b_1) \), then \( b_2 \notin L(a_1, b_1) \) (see Fig. 2.78). By symmetry
we assume $b_2$ is on the same side of line $c_1z_1$ as $b_1$. Pick $a''_1, b''_1 \in a_1b_1$ such that $||a''_1a_1|| = ||b''_1b_1|| = \varepsilon r$, then $||a''_1b''_1|| = r_1 - 2\varepsilon r$. Pick $u \in a_2b_2$ such that $||a_2b_1|| = ||a_2u||$, pick $v \in a_2z_2$ such that $||va_2|| = ||ub_2||$. Then $u \notin L(a''_1, b''_1)$ and

$$||va_2|| = ||ub_2|| \leq ||b_2b_2'|| \leq \varepsilon r.$$ 

Thus

$$||a''_1v|| \geq ||a_1v|| - \varepsilon r \geq ||a_1a_2|| - 2\varepsilon r \geq r_1 - 3\varepsilon r = ||a''_1b''_1|| - \varepsilon r.$$ 

EQ(2.17) holds by applying Lemma 26 to $L(u, v)$ and $L(a''_1, b''_1)$.

Subcase 2.2. Assume $h \in L(a_1, b_1)$ (see Fig. 2.79). If we relax the constraint $b_2 \notin L_r(a_1, b_1)$ and allow

$$b_2 \in L(a_1, b_1) \setminus (L_r(b_1, a_1) \cap L_r(a_1, b_1)).$$ 

Then by symmetry we assume $b_2$ is on the same side of line $c_1z_1$ as $b_1$.

Let $\psi = \angle b_2z_2z_1$ and we will prove $\psi < 80^\circ$. It is easy to see that $\psi$ achieves
maximum as \( b_1 \in a_2 b_2 \). Assuming \( b_1 \in a_2 b_2 \). Since \( b_1 \notin L_r(a_2, b_2) \), we have \( ||b_1 z_2|| \geq r \frac{2}{2} - \epsilon r \). By Lemma 5(2), we have \( \angle a_2 b_1 a_1 > 59.9^\circ \). Let \( \varphi = \angle b_2 z_1 z_2 \), then \( \varphi + \psi < 120.1^\circ \).

Since \( \sin \psi = \frac{||b_1 z_1||}{||b_1 z_2||} \leq \frac{r_1/2}{r_2/2 - \epsilon r} \leq \frac{r_1}{r_1 - 3\epsilon r} \leq \frac{1}{1 - 3\epsilon} < 1.004 \),

If \( \psi \geq 80^\circ \), then \( \varphi > 78^\circ \) which leads to a contradiction. Let \( l \) intersect with line \( a_2 b_2 \) at \( z' \), then

\[ ||z'z|| = ||z_2z|| \tan \psi < \frac{\sqrt{3}}{2}R \tan 80^\circ < 5R. \]

Thus \( o \) is on the same side of \( l' \) as \( a_1 \) and \( b_2 \in \partial \mathbb{D} \). Let arc \( a_2 d_2 \) intersect with \( \partial B(a_1, r) \) at \( a''_2 \) and \( z''_2 = \frac{1}{2}(a''_2 + b_2) \), extend \( b_2 z_1 \) to \( b''_2 \) such that \( ||b''_2 z_1|| = ||b_2 z_1|| \). Then by Lemma 5(1), we have

\[ \angle a''_2 b_2 b''_2 \leq \angle a''_2 b_2 a_1 \leq \angle a''_2 b''_2 a_1 < 60.1^\circ. \]

Next let \( \phi = \angle b_2 a''_2 b''_2 \) and prove \( \phi > 45^\circ \). Let \( \phi' = \angle b_2 b''_2 a''_2 \), then \( \phi' + \phi > 119.9^\circ \).

By the law of sine, we have

\[ \frac{\sin \phi}{\sin \phi'} = \frac{||b_2 b''_2||}{||a''_2 b_2||} \geq \frac{||a''_2 b''_2||}{||a_2 b_2||} \geq \frac{r_1 - 2\epsilon r}{r_2} \geq \frac{r_2 - 3\epsilon r}{r_2} > 0.997. \]
If \( \phi \leq 45^\circ \), then \( \phi' < 46^\circ \) which leads to a contradiction. Thus, \( \phi > 45^\circ \). Let \( z_1z_2 \) intersect with \( \partial \mathbb{D} \) at \( k \), then \( \angle b_2kz_1 > \phi > 45^\circ \). Thus, \( \angle a_2''z'o < 45^\circ \). Let \( t \) denote the distance from \( b_2 \) to \( l \), then
\[
t \leq \frac{1}{2}||z_1z_2|| \leq \frac{\sqrt{3}}{2}R.
\]
Since
\[
\sin \angle zob_2 = \frac{t}{R_0} < \frac{\sqrt{3}}{400},
\]
we have \( \angle zob_2 < 1^\circ \).

Hence,
\[
\angle a_2''b_2o \leq \angle zob_2 + \angle a_2''z'o < 46^\circ.
\]

Let \( p \) denote the intersect point of \( \partial \mathbb{D} \) and \( \partial L(a_2, b_2) \) other than \( b_2 \), then \( \angle ob_2p > 89^\circ \) by Lemma 5(3). Thus \( \angle a_2''b_2p > 43^\circ \) and we have by EQ(2.21)
\[
A_1 \geq |\angle a_2''b_2p| - |C_{xy}|
\]
\[
\geq \frac{1}{2}t_2(43^\circ) - 0.096r_2^2
\]
\[
\geq 0.27r_2^2 \geq 0.15R||z_1z_2||.
\]

Case 3. \( b_2 \notin B(a_1, r) \cup B(b_1, r) \) (see Fig. 2.80). Then \( ||a_2b_1|| < r \) and \( ||b_2b_1|| \geq r_2 \) as \( b_1 \notin L_r(a_2, b_2) \). Hence \( a_2 \in L(a_1, b_1) \setminus L_r(a_1, b_1) \) and \( h \in L(a_1, b_1) \). Therefore, this case is symmetry to Subcase 2.2. Let \( a_2b_2 \) intersect with \( \partial L_r(a_1, b_1) \) at \( x \) and \( y \). If \( H \subset \mathbb{D} \), then EQ(2.17) holds as
\[
A_1 \geq |H| - |C_{xy}| \geq 2.35r_2^2.
\]

Next we assume \( H \setminus \mathbb{D} \neq \emptyset \), then \( o \) is above line \( a_2b_2 \). By Lemma 25, we have \( o \in l \); either \( a_2 \in \partial \mathbb{D} \) or \( b_2 \in \partial \mathbb{D} \). Similar to subcase 2.2, we have \( o \) is on the same side of \( l' \) as \( a_1 \) and \( a_2 \in \partial \mathbb{D} \). Let arc \( b_2d_2 \) intersect with \( \partial B(a_1, r) \) at \( b_2'' \), \( p \) denote the intersect point of \( \partial \mathbb{D} \) and
\( \partial L(a_2, b_2) \) other than \( a_2 \), similar to Subcase 2.2 we have

\[
A_1 \geq |\angle b_2'a_2p| - |C_{xy}| \geq 0.15R \| z_1 z_2 \|.
\]

Therefore, Lemma 27 is proved.

![Figure 2.80](image)

**Figure 2.80.** \( h \in \text{L}(a_1, b_1) \) and \( b_2 \notin B(a_1, r) \cup B(b_1, r) \).

Thus, the proof for Lemma 9 is complete.
CHAPTER 3
ASYMPTOTIC PROBABILITY DISTRIBUTION OF THE MAXIMUM EDGE LENGTH OF RNG

Relative neighborhood graph (abbreviated by RNG) has been widely used in localized topology control and geographic routing in wireless ad hoc and sensor networks. If all the nodes have the same transmission radii, the maximum edge length of the RNG is the smallest transmission radius for constructing the RNG by using only 1-hop neighbor information and it is referred to as the critical transmission radius of the RNG. The maximum edge length of the RNG is the minimum requirement on the maximum transmission radius by those applications of RNG. In this chapter, we derive the precise asymptotic probability distribution of the maximum edge length of the RNG on a Poisson point process over a unit-area disk. Since the maximum RNG edge length is a lower bound on the critical transmission radius for greedy forward routing, our result in this chapter also leads to an improved asymptotic almost sure lower bound on the critical transmission radius for greedy forward routing in wireless ad hoc and sensor networks.

3.1 Introduction

The RNG of a finite planar set was originally introduced first by [33] with applications in pattern recognition. It is a bounded-degree planar graph containing the Euclidean minimum spanning tree as a subgraph. Due to its simple construction and maintenance, RNG has found many important applications in localized topology control (e.g., [13], [21], [23]) and geographic routing (e.g., [3], [14], [30]) in wireless ad hoc and sensor networks. All these applications require the maximum transmission radius of the networking nodes be no shorter than the maximum edge length in the RNG. While the maximum edge length in the RNG can be computed in polynomial time, little is known about its random behavior when the underlying vertex is a random point set.
In this chapter, we assume a wireless ad hoc network is represented by a Poisson point process over the unit-area disk $\mathbb{D}$ with density $n$, which is denoted by $\mathcal{P}_n$, and all nodes have the same maximal transmission radius which is a function of $n$. We derive the precise asymptotic probability distribution of the maximum edge length in the RNG of $\mathcal{P}_n$. Denote the maximum edge length of a geometric graph $G$ by $\lambda(G)$, and the RNG of a finite planar set $V$ by $\text{RNG}(V)$. Let

$$\beta_0 = \frac{1}{\sqrt{\frac{2}{3} - \frac{\sqrt{3}}{2\pi}}} \approx 1.6.$$ 

The main result of this chapter is stated in the following theorem.

**Theorem 28** For any constant $\xi$, we have

$$\lim_{n \to \infty} \Pr \left[ \lambda(\text{RNG}(\mathcal{P}_n)) \leq \beta_0 \sqrt{\frac{\ln n + \xi}{\pi n}} \right] = e^{-\frac{\beta_0^2}{16} e^{-\xi}}.$$

It is interesting to compare the maximum edge length of the RNG with the maximum edge length of the (Euclidean) minimum spanning tree (MST), which is also known the critical transmission radius for connectivity [11], and the maximum edge length of the Gabriel graph (abbreviated by GG) [9], which also has many applications in wireless ad hoc networks. Let $\text{MST}(V)$ and $\text{GG}(V)$ denote the MST and the GG of finite planar set $V$. It's well-known that for any finite planar set,

$$\text{MST}(V) \subseteq \text{RNG}(V) \subseteq \text{GG}(V).$$

Thus,

$$\lambda(\text{MST}(V)) \leq \lambda(\text{RNG}(V)) \leq \lambda(\text{GG}(V)).$$

The asymptotic distributions of $\lambda(\text{MST}(\mathcal{P}_n))$ and $\lambda(\text{GG}(\mathcal{P}_n))$ were derived in [26] (based
on an earlier result [6]) and in [35] respectfully. Specifically, for any constant $\xi$,

$$\lim_{n \to \infty} \Pr \left[ \lambda(MST(P_n)) \leq \sqrt{\frac{\ln n + \xi}{\pi n}} \right] = e^{-e^{-\xi}},$$

$$\lim_{n \to \infty} \Pr \left[ \lambda(GG(P_n)) \leq 2 \sqrt{\frac{\ln n + \xi}{\pi n}} \right] = e^{-2e^{-\xi}}.$$

So roughly speaking, the maximum edge length of the RNG (respectfully, GG) of a Poisson point process is asymptotically about 1.6 times (respectfully, twice) its critical transmission radius for connectivity.

Another parameter closely related to the maximum edge length of the RNG is the critical transmission radius for greedy forward routing [8, 32]. In greedy forward routing, each node discards a packet if none of its neighbors is closer to the destination of the packet than itself, or otherwise forwards the packet to the neighbor closest to the destination of the packet. The critical transmission radius of a planar node set $V$ for greedy forward routing, denoted by $\sigma(V)$, is the smallest transmission radius by $V$ which ensures successful delivery of any packets from any source node in $V$ to any destination node in $V$. Clearly, $\lambda(RNG(V)) \leq \sigma(V)$. It was recently proved in [34] that for any constant $\epsilon > 0$, it is asymptotically almost sure (abbreviated by a.a.s.) that

$$(1 - \epsilon) \beta_0 \sqrt{\frac{\ln n}{\pi n}} \leq \sigma(P_n) \leq (1 + \epsilon) \beta_0 \sqrt{\frac{\ln n}{\pi n}}.$$

This immediately implies that for any constant $\epsilon > 0$, it is a.a.s. that

$$\lambda(RNG(P_n)) \leq (1 + \epsilon) \beta_0 \sqrt{\frac{\ln n}{\pi n}}.$$

In other words, $(1 + \epsilon) \beta_0 \sqrt{\frac{\ln n}{\pi n}}$ is an a.a.s. upper bound on $\lambda(RNG(P_n))$. While this a.a.s. bound is weaker than Theorem 28, it had inspired us to conjecture and then prove
Theorem 28. This a.a.s. bound will also be used in the proof of Theorem 28. As the immediate consequence of Theorem 28, a tighter a.a.s. lower bound on $\sigma(P_n)$ can be obtained: Suppose that $\lim_{n \to \infty} \xi_n = \infty$ and $\lim_{n \to \infty} \xi_n / \ln n = 0$. Then it is a.a.s. that 

$$\sigma(P_n) \geq \beta_0 \sqrt{\frac{\ln n - \xi_n}{\pi n}}.$$ 

The remaining of this chapter is organized as follows. In Section 3.2, we obtain the asymptotic values of several integrals which will be used to prove the main theorem. In Section 3.3, we give the proof for the main theorem of this chapter. In Section 3.4, we summarize this chapter.

3.2 Integral Ingredients

In this section, we derive the asymptotic values of several useful integrals. We will frequently change the integral variables using a technique introduced in [35]. Consider a tree topology on $k$ planar points $x_1, x_2, \ldots, x_k$, and assume without loss of generality that $x_{k-1}x_k$ is an edge in this tree. Let $z_{k-1}$, $\rho$, and $\omega$ be the midpoint, half-length and the slope of $x_{k-1}x_k$ respectively. We root the tree at $x_k$. For $1 \leq i \leq k - 2$, let $z_i$ be the midpoint of the edge between $x_i$ and its parent in such rooted tree. Then, we replace $x_1, x_2, \ldots, x_k$ by $z_1, \ldots, z_{k-1}, \rho, \omega$. The Jacobian determinant of this change is

$$\frac{\partial (x_1, \ldots, x_{k-1}, x_k)}{\partial (z_1, \ldots, z_{k-1}, \rho, \omega)} = \frac{\partial (x_1 + p(x_1), \ldots, x_{k-1} + p(x_{k-1}), x_k)}{\partial (z_1, \ldots, z_{k-1}, \rho, \omega)}$$

$$= 4^{k-1} \frac{\partial \left(\frac{x_1 + p(x_1)}{2}, \ldots, \frac{x_{k-1} + p(x_{k-1})}{2}, x_k\right)}{\partial (z_1, \ldots, z_{k-1}, \rho, \omega)}$$

$$= 4^{k-1} \frac{\partial (z_1, \ldots, z_{k-1}, x_k - z_{k-1})}{\partial (z_1, \ldots, z_{k-1}, \rho, \omega)}$$
\[ I_2 \cdots 0 \quad 0 \\
\vdots \cdots \cdot \cdots \vdots \\
0 \cdots I_2 \quad 0 \\
n0 \cdots 0 \quad \cos \omega - \rho \sin \omega \\
0 \cdots 0 \quad \sin \omega \quad \rho \cos \omega \\
= 4^{k-1} \rho. \]

Fix a constant \( \xi \) and a sequence \((\xi_n)\) with \( \xi_n = o(\ln n) \) and \( \xi_n \to \infty \). Let

\[
\begin{align*}
  r_n &= \beta_0 \sqrt{\frac{\ln n + \xi}{\pi n}}, \\
  R_n &= \beta_0 \sqrt{\frac{\ln n + \xi_n}{\pi n}}, \\
  R'_n &= 1.1 \beta_0 \sqrt{\frac{\ln n}{\pi n}}.
\end{align*}
\]

Then, for sufficiently large \( n \), we have \( r_n < R_n < R'_n < 2r_n \). Given \( 0 < \varepsilon < 0.001 \). Since \( \frac{R_n}{r_n} \to 1 \) and \( R_n \to 0 \) as \( n \to \infty \), we can choose \( n \) sufficiently large so that

\[
\frac{1}{1 + \varepsilon} R_n < r_n < R_n < \frac{1}{200\sqrt{n}}.
\]

Define

\[
\begin{align*}
  \Omega &= \{(x_1, x_2) \in \mathbb{D}^2 : r_n < \|x_1 x_2\| \leq R_n\}, \\
  \Omega' &= \{(x_1, x_2) \in \mathbb{D}^2 : R_n < \|x_1 x_2\| \leq R'_n\}.
\end{align*}
\]
Lemma 29 The following are true:

\[
\frac{n^2}{2} \int_{\Omega} e^{-nv(x_1x_2)} dx_1 dx_2 \sim \frac{\beta_0^2}{2} e^{-\xi},
\]

\[
\frac{n^2}{2} \int_{\Omega'} e^{-nv(x_1x_2)} dx_1 dx_2 = o(1).
\]

Proof Let \( \rho = \rho(x_1, x_2) \) be the half-length of \( x_1x_2 \), and \( z = z(x_1, x_2) \) be the midpoint of \( x_1x_2 \). Let \( \Omega_1 \) be the set of \( (x_1, x_2) \in \Omega \) satisfying that \( z \in \mathbb{D}_{\sqrt{3}\rho}(0) \), and let \( \Omega_2 = \Omega \setminus \Omega_1 \).

First, we calculate the integration over \( \Omega_1 \). If \( (x_1, x_2) \in \Omega_1 \), \( L_{x_1x_2} \) is fully contained in \( \mathbb{D} \) and \( v(x_1x_2) = \frac{1}{\rho_0^2} \pi \rho^2 \). Changing the integration variable \( x_1 \) and \( x_2 \) by \( z, \rho \), and the slope of \( x_1x_2 \) yields

\[
\frac{n^2}{2} \int_{\Omega_1} e^{-nv(x_1x_2)} dx_1 dx_2
= 4\pi n^2 \int_{\frac{r_\mathbb{D}}{2}}^{\frac{r_\mathbb{D}}{2}} e^{-\frac{4}{\rho_0^2} n\pi \rho^2} \rho d\rho \int_{\mathbb{D}_{\sqrt{3}\rho}(0)} dz
\sim 4\pi n^2 \int_{\frac{r_\mathbb{D}}{2}}^{\frac{r_\mathbb{D}}{2}} e^{-\frac{4}{\rho_0^2} n\pi \rho^2} \rho d\rho
= -\frac{\beta^2}{2} ne^{-\frac{4}{\rho_0^2} n\pi \rho^2} \bigg|_{\frac{r_\mathbb{D}}{2}}^{\frac{r_\mathbb{D}}{2}} \sim \frac{\beta_0^2}{2} e^{-\xi}.
\]

Next, we calculate the integration over \( \Omega_2 \). Let \( t = t(z) \) be the distance between \( z \) and \( \partial \mathbb{D} \).

By Lemma 2, we have

\[
v(x_1x_2) \geq \frac{2}{\beta_0^2} \pi \rho^2 + \rho t
\]

and

\[
\rho \theta(z, \rho) \leq 2\pi t.
\]
Changing the integration variable as above yields

\[
\frac{n^2}{2} \int_{\Omega} e^{-nv(x_1,x_2)} \, dx_1 dx_2 \leq \frac{n^2}{2} \int_{\Omega} e^{-n\left(\frac{2}{n^2} \pi \rho^2 + \rho t\right)} \, dx_1 dx_2.
\]

\[
= 2n^2 \int_{\nu}^{R_n} \, d\rho \int_{D \cap D_{\rho}(1) \setminus D_{\rho}(2)} e^{-n\left(\frac{2}{n^2} \pi \rho^2 + \rho t\right)} \rho \theta(z,\rho) \, dz
\]

\[
= 2n^2 \int_{\nu}^{R_n} \, d\rho \int_{D \cap D_{\rho}(1) \setminus D_{\rho}(2)} e^{-n\left(\frac{2}{n^2} \pi \rho^2 + \rho t\right)} \rho \theta(z,\rho) \, dz
\]

\[
\leq 4\pi n^2 e^{-\frac{1}{n^2 \pi} n \pi \nu^2} \int_{\nu}^{R_n} \, d\rho \int \, e^{-n\rho t} \, dz
\]

\[
= O(1) n^{-1.5} \int_{\nu}^{R_n} \, d\rho \int_0^{1/\sqrt{n}} \, e^{-n\rho t} \, dt
\]

\[
\leq O(1) n^{-1.5} \int_{\nu}^{R_n} \, d\rho \int_0^{\infty} \, e^{-n\rho t} \, dt
\]

\[
= O(1) \frac{1}{\sqrt{n}} \int_{\nu}^{R_n} \, d\rho \leq O(1) \frac{1}{\sqrt{n}} r_n^{-2} R_n
\]

\[
= O(1) \frac{1}{\sqrt{n} r_n} = O(1) \frac{1}{\sqrt{\ln n}} = o(1).
\]

Therefore,

\[
\frac{n^2}{2} \int_{\Omega} e^{-nv(x_1,x_2)} \, dx_1 dx_2 \sim \frac{\beta_0^2}{2} e^{-\xi}.
\]

Note that \(\Omega \cup \Omega'\) consists of \((x_1, x_2) \in D^2\) satisfying that \(r_n < \|x_1, x_2\| \leq R_n'.\) Using the same argument as above, we can show that

\[
\frac{n^2}{2} \int_{\Omega \cup \Omega'} e^{-nv(x_1,x_2)} \, dx_1 dx_2 \sim \frac{\beta_0^2}{2} e^{-\xi}.
\]

Thus, the second asymptotic equality in the lemma holds.
A topology with numbered vertices is specified by a collection of the pairs of the indices of the numbered vertices. For any topology \( \tau \) on \( m \) numbered vertices and a planar set \( U \) of \( m \) numbered points, we denote by \( \tau (U) \) the graph on \( U \) with topology \( \tau \). Suppose that \( \tau \) is a topology with \( m \) numbered vertices and without isolated vertices. We denote by \( \Gamma (\tau) \) the set of \( x = (x_1, \cdots, x_m) \in \mathbb{D}^m \) satisfying that the length of each edge in \( \tau (x) \) is more than \( r_n \) but at most \( R_n \). Note that for each \( x \in \Gamma (\tau) \), the \( \sqrt{3}R_n \)-disk graph on the midpoints of the edges in any connected component of \( \tau (x) \) is connected. Thus, the \( \sqrt{3}R_n \)-disk graph on the midpoints of the edges in \( \tau (x) \) has no more connected components than \( \tau (x) \) itself. For any positive integer \( l \) no more than the number of connected components of \( \tau \), we denote by \( \Gamma_l (\tau) \) the set of \( x \in \Gamma (\tau) \) such that the \( \sqrt{3}R_n \)-disk graph on the midpoints of the edges in \( \tau (x) \) has \( l \) connected components. For any positive integer \( k \), we denote by \( C_k \) the forest on \( 2k \) numbered vertices \( v_1, v_2, \cdots, v_{2k} \) which consists of \( k \) edges \( v_{2i-1}v_{2i} \) for \( 1 \leq i \leq k \). Then, \( C_k \) has \( k \) tree components, each consisting of a single edge, and \( \Gamma (C_k) = \Omega^k \).

**Lemma 30** For any fixed integer \( k \geq 2 \),

\[
\left( \frac{n^2}{2} \right)^k \int_{\Gamma_1 (C_k)} \chi (C_k (x)) e^{-n \nu (C_k (x))} dx = o (1).
\]

**Proof** For each \( x = (x_1, \cdots, x_{2k}) \in \Gamma_1 (C_k) \), let \( z_i \) and \( \rho_i \) be the midpoint and half-length of \( x_{2i-1}x_{2i} \) respectively. We denote by \( S \) the set of \( x = (x_1, \cdots, x_{2k}) \in \Gamma_1 (C_k) \) satisfying that \( x_1x_2 \) is the outermost edge in \( C_k (x) \) and \( z_2 \) is the farthest from \( z_1 \). It suffices to prove

\[
n^{2k} \int_{S} \chi (C_k (x)) e^{-n \nu (C_k (x))} dx = o (1).
\]
By Lemma 10 in Chapter 2, for any \( x = (x_1, \cdots, x_{2k}) \in S \) with \( \chi(C_k(x)) = 1 \),

\[
\nu(C_k(x)) \geq \nu(x_1x_2) + cR_n \|z_1z_2\|
\]

for some constant \( c \). So, it is sufficient to show that

\[
n^{2k} \int_S e^{-n(\nu(x_1x_2) + cR_n \|z_1z_2\|)} \prod_{i=1}^{2k} dx_i = o(1).
\]

For each \( 2 \leq i \leq k \), we replace \( x_{2i-1} \) and \( x_{2i} \) by \( z_i, \rho_i \) and the slope of \( x_{2i-1}x_{2i} \). Note that for any \( 3 \leq i \leq k \), \( z_i \in \overline{D}(z_1, \|z_2z_1\|) \). Thus,

\[
n^{2k} \int_S e^{-n(\nu(x_1x_2) + cR_n \|z_1z_2\|)} \prod_{i=1}^{2k} dx_i \\
\leq O(1) n^{2k} \left( \int_{\Omega} e^{-n\nu(x_1x_2)} dx_1 dx_2 \right) \left( \int_{\mathbb{R}^2} \rho d\rho \right)^{k-1} \\
\cdot \left( \int_{\mathbb{R}^2} e^{-ncR_n \|z_2z_1\|} \|z_2z_1\|^2 \right) \left( \int_{\overline{D}(z_1, \|z_2z_1\|)} dz \right)^{k-2} \\
\sim O(1) n^{2k-2} (R_n^2 - r_n^2)^{k-1} \\
\cdot \int_{\mathbb{R}^2} e^{-ncR_n \|z_2z_1\|} \|z_2z_1\|^{2(k-2)} dz_2 \\
\leq O(1) n^{2k-2} (R_n^2 - r_n^2)^{k-1} \int_0^\infty e^{-ncR_n t} t^{2k-3} dt \\
= O(1) \frac{n^{2k-2} (R_n^2 - r_n^2)^{k-1}}{(nR_n)^{2(k-1)}} \\
= O(1) \left( \frac{nR_n^2 - nrr_n}{nR_n^2} \right)^{k-1} \\
= O(1) \left( \frac{\xi_n - \xi}{\ln n} \right)^{k-1} = o(1),
\]

where the second asymptotic equality follows from Lemma 29, and the last equality is based on \( \xi_n = o(\ln n) \).
Lemma 31 For any fixed integers 2 ≤ l < k.

\[
\left(\frac{n^2}{2}\right)^k \int_{\Gamma_l(C_k)} \chi(C_k(x)) \, e^{-n\nu(C_k(x))} \, dx = o(1) .
\]

Proof For any nontrivial l-partition \( \Pi = \{P_1, P_2, \ldots, P_l\} \) of \( \{1, 2, \ldots, k\} \), let \( S(\Pi) \) denote the set of \( x = (x_1, \ldots, x_{2k}) \in \Gamma_l(C_k) \) such that for each \( 1 \leq j \leq l \), the set \( \{z_i : i \in P_j\} \) is a connected components of the \( \sqrt{3}R_n \)-disk graph on \( z_1, z_2, \ldots, z_k \). Then \( \Gamma_l(C_k) \) is the union of \( S(\Pi) \) over all nontrivial l-partitions \( \Pi \) of \( \{1, 2, \ldots, k\} \). So, it is sufficient to show that for any l-partition \( \Pi \) of \( \{1, 2, \ldots, k\} \),

\[
n^{2k} \int_{S(\Pi)} \chi(C_k(x)) \, e^{-n\nu(C_k(x))} \, dx = o(1) .
\]

Now fix an l-partition \( \Pi = \{P_1, P_2, \ldots, P_l\} \) of \( \{1, 2, \ldots, k\} \), and let \( p_j = |P_j| \) for \( 1 \leq j \leq l \). Then,

\[
S(\Pi) \subseteq \prod_{j=1}^l \Gamma_1(C_j) .
\]

For any \( x = (x_1, x_2, \ldots, x_{2k}) \in S(\Pi) \), let \( x^{(j)} \) denote the subsequence of \( (x_{2i-1}, x_{2i}) \) with \( i \in P_j \) for \( 1 \leq j \leq l \). Then,

\[
\nu(C_k(x)) = \sum_{j=1}^l \nu(C_j(x^{(j)}))
\]

\[
\chi(C_k(x)) \leq \prod_{j=1}^l \chi(C_j(x^{(j)})) .
\]

Thus,

\[
n^{2k} \int_{S(\Pi)} \chi(C_k(x)) \, e^{-n\nu(C_k(x))} \, dx \leq n^{2k} \int_{S(\Pi)} \prod_{j=1}^l \chi(C_j(x^{(j)})) \, e^{-n\nu(C_j(x^{(j)}))} \, dx
\]
\[
\leq n^{2k} \int_{\prod_{j=1}^{l} \Gamma_1(C_j)} \prod_{j=1}^{l} \chi(C_j(x^{(j)})) e^{-n\nu(C_j(x^{(j)}))} d\mathbf{x}
\]
\[
= \prod_{j=1}^{l} \left( n^{2p_j} \int_{\Gamma_1(C_j)} \chi(C_j(x^{(j)})) e^{-n\nu(C_j(x^{(j)}))} d\mathbf{x}^{(j)} \right)
\]
\[
= o(1),
\]
where the last equality follows from Lemma 30 and the fact that at least one \( p_j \geq 2 \).

**Lemma 32** For any fixed integer \( k \geq 2 \),
\[
\left( \frac{n^2}{2} \right)^k \int_{\Gamma_k(C_k)} e^{-n\nu(C_k(x))} d\mathbf{x} \sim \left( \frac{\beta^2}{2} e^{-\xi} \right)^k.
\]

**Proof** For any \( \mathbf{x} = (x_1, \cdots, x_{2k}) \in \Gamma_k(C_k) \),
\[
\nu(C_k(\mathbf{x})) = \sum_{i=1}^{k} \nu(x_{2i-1}x_{2i}).
\]
Thus,
\[
\left( \frac{n^2}{2} \right)^k \int_{\Gamma_k(C_k)} e^{-n\sum_{i=1}^{k} \nu(x_{2i-1}x_{2i})} d\mathbf{x}
\]
\[
= \left( \frac{n^2}{2} \right)^k \int_{\Gamma_k(C_k)} e^{-n\sum_{i=1}^{k} \nu(x_{2i-1}x_{2i})} d\mathbf{x}
\]
\[
= \left( \frac{n^2}{2} \right)^k \int_{\Gamma(C_k)} e^{-n\sum_{i=1}^{k} \nu(x_{2i-1}x_{2i})} d\mathbf{x}
\]
\[
- \sum_{l=1}^{k-1} \left( \frac{n^2}{2} \right)^k \int_{\Gamma_l(C_k)} e^{-n\sum_{i=1}^{k} \nu(x_{2i-1}x_{2i})} d\mathbf{x}.
\]
We shall show that the first term is asymptotically equal to \( \left( \frac{\beta^2}{2} e^{-\xi} \right)^k \), and the second term
is vanishing. Indeed,

$$\left( \frac{n^2}{2} \right)^k \int_{\Gamma(C_k)} e^{-n \sum_{i=1}^{k} \nu(x_{2i-1}x_{2i})} d\mathbf{x} = \prod_{i=1}^{k} \left( \frac{n^2}{2} \int_{\Omega} e^{-n\nu(x_{2i-1}x_{2i})} dx_{2i-1} dx_{2i} \right) \sim \left( \frac{\beta_0^2}{2} e^{-\xi} \right)^k,$$

where the last equality follows from Lemma 29. For any $x = (x_1, \cdots, x_{2k}) \in \Gamma_1(C_k)$, if $x_1x_2$ is the outermost edge in $C_k(x)$ and $z_2$ is the farthest from $z_1$, it can be proved that

$$\sum_{i=1}^{k} \nu(x_{2i-1}x_{2i}) \geq \nu(x_1x_2) + cR_n \|z_1z_2\|.$$

Following the same argument in Lemma 30, we can show that

$$\left( \frac{n^2}{2} \right)^k \int_{\Gamma_1(C_k)} e^{-n \sum_{i=1}^{k} \nu(x_{2i-1}x_{2i})} d\mathbf{x} = o(1).$$

Then, following the same argument in Lemma 31, we can show that for any $2 \leq l \leq k-1,$

$$\left( \frac{n^2}{2} \right)^k \int_{\Gamma_l(C_k)} e^{-n \sum_{i=1}^{k} \nu(x_{2i-1}x_{2i})} d\mathbf{x} = o(1).$$

Thus, the lemma holds.

**Lemma 33** Let $F$ be a forest on $m$ numbered vertices with maximum degree at least two and minimum degree at least one. Then,

$$n^m \int_{\Gamma(F)} \chi(F(x)) e^{-n\nu(F(x))} d\mathbf{x} = o(1).$$

**Proof** Let $\kappa$ be the number of tree components of $F$. Then, $m \geq \kappa + 2$, and $F$ has exactly $m - \kappa$ edges denoted by $e_1, \cdots, e_{m-\kappa}$. For any $x = (x_1, \cdots, x_m) \in \Gamma(F)$, let $z_i$ denote
the middle point of $e_i$ in $F(x)$ for each $1 \leq i \leq m - \kappa$. We first show that

$$n^m \int_{\Gamma_1(F)} \chi(F(x)) e^{-n\nu(F(x))} d\mathbf{x} = o(1).$$

For any pair of distinct integers $p$ and $q$ between 1 and $m - \kappa$, let $S_{pq}$ denote the set of $\mathbf{x} = (x_1, \cdots, x_m) \in \Gamma_1(F)$ satisfying that $e_p$ is an outermost edge in $F(x)$ and $z_q$ is the farthest from $z_p$ among all $z_1, \cdots, z_{m-\kappa}$. Then, it suffices to prove for any such $p$ and $q$,

$$n^m \int_{S_{pq}} \chi(F(x)) e^{-n\nu(F(x))} d\mathbf{x} = o(1).$$

Fix a pair of distinct integers $p$ and $q$ between 1 and $m-\kappa$. Let $p'$ and $p''$ be the indices of the two endpoints of the edges $e_p$. Then, for any $\mathbf{x} = (x_1, \cdots, x_m) \in S_{pq}$ with $\chi(F(\mathbf{x})) = 1$,

$$\nu(F(\mathbf{x})) \geq \nu(x_{p'}x_{p''}) + cR_n \|z_pz_q\|$$

for some constant $c > 0$. Thus, we only need to show that

$$n^m \int_{S_{pq}} e^{-n(\nu(x_{p'}x_{p''})+cR_n \|z_pz_q\|)} d\mathbf{x} = o(1).$$

We change the integral variables $x_1, \cdots, x_m$ as follows. For the tree component containing $e_p$, we replace the $x_i$’s in this tree by the midpoints of the edges in this tree except $x_p$ and $x_{p'}$, $x_{p''}$ (both of which are kept). For any other tree component, we use the method introduced at the beginning of this section: pick an arbitrary edge as the rooted edge. We replace $x_i$’s in this tree by the midpoints of all the edges in this tree together with the half-length and slope of the root edge. Such change of integration variables yields

$$n^m \int_{S_{pq}} e^{-n(\nu(x_{p'}x_{p''})+cR_n \|z_pz_q\|)} d\mathbf{x}$$
≤ O(1) n^{m-2} \left( R_n^2 - r_n^2 \right)^{\kappa-1} \cdot \int_{\mathbb{R}^2} e^{-ncR_n\|zp\|\|zp\|^{2(m-\kappa-2)}} dz

\sim O(1) n^{m-2} \left( R_n^2 - r_n^2 \right)^{\kappa-1} \cdot \int_{\mathbb{R}^2} e^{-ncR_n\|zp\|\|zp\|^{2(m-\kappa-2)}} dz

≤ O(1) n^{m-2} \left( R_n^2 - r_n^2 \right)^{\kappa-1} \cdot \int_{0}^{\infty} e^{-ncR_n\mu\mu^{2(m-\kappa-3)}} d\mu

= O(1) n^{m-2} \frac{(R_n^2 - r_n^2)^{\kappa-1}}{(nR_n)^{2(m-\kappa-1)}}

= O(1) n^{m-2} \frac{(nR_n^2 - nr_n^2)^{\kappa-1}}{(nR_n^2)^{m-\kappa-1}}

= O(1) n^{m-2} \frac{(\xi_n - \xi)^{\kappa-1}}{\ln^{m-\kappa-1} n} = o(1)

where the asymptotic equality follows from Lemma 29, and the last equality follows from \( \xi_n = o(\ln n) \) and \( m - \kappa - 1 \geq 1 \).

Following the same decomposition argument as in the proof of Lemma 31, we can show that for any \( 2 \leq l \leq \kappa \),

\[ n^m \int_{\Gamma_l(F)} \chi(F(x)) e^{-n\nu(F(x))} d\textbf{x} = o(1) \]

Thus, the lemma holds.

### 3.3 Proof of The Main Theorem
We first give a brief overview on our approach to prove Theorem 28. Let $M_n$ denote the number of edges in $RNG(P_n)$ longer than $r_n$ but not shorter than $R_n$, $M'_n$ denote the number of edges in $RNG(P_n)$ longer than $R_n$ but not shorter than $R'_n$, and $M''_n$ denote the number of edges in $RNG(P_n)$ longer than $R'_n$. Then, $\lambda(RNG(P_n)) \leq r_n$ if and only if $M_n + M'_n + M''_n = 0$. According the discussion in Section ??, $M''_n = 0$ is a.a.s.. In Lemma 36, we will prove that $E[M'_n] = o(1)$, which implies that $M'_n = 0$ is a.a.s. by Markov’s inequality. In Lemma 37, we will prove that $M_n$ is asymptotically Poisson with mean $\frac{\beta^2}{2} e^{-\xi}$. Consequently,

$$\lim_{n \to \infty} \Pr[\lambda(RNG(P_n)) \leq r_n] = \lim_{n \to \infty} \Pr[M_n + M'_n + M''_n = 0] = \lim_{n \to \infty} \Pr[M_n = 0] = e^{-\frac{\beta^2}{2} e^{-\xi}}.$$

Two key techniques used in our proof are the Palm theory for Poisson processes (see, e.g., Theorem 1.6 in [27]) and the Brun’s sieve (see, e.g., Theorem 10 in [35]), which are stated below.

**Theorem 34** Suppose that $h(U,V)$ is a bounded measurable function defined on all pairs of the form $(U,V)$ with $V$ being a finite planar set and $U$ being a subset of $V$. Then any positive integer $k$,

$$E \left[ \sum_{U \subseteq P_n, |U| = k} h(U,P_n) \right] = \frac{n^k}{k!} E[h(X_k, X_k \cup P_n)].$$

**Theorem 35** Suppose that $N$ is a non-negative integer random variable, and $B_1, \ldots, B_N$ are $N$ Bernoulli random variables. If there is a constant $\mu$ such that for every fixed positive
integer $k$,  
\[ \mathbb{E} \left[ \sum_{I \subseteq \{1, \ldots, N\}, |I| = k} \prod_{i \in I} B_i \right] \sim \frac{1}{k!} \mu^k, \]

then $\sum_{i=1}^{N} B_i$ is asymptotically Poisson with mean $\mu$.

Now, we apply Palm theory to show that $\mathbb{E} [M'_n]$ is vanishing.

**Lemma 36** $\mathbb{E} [M'_n] = o(1)$.

**Proof** For any edge $e \in K (P_n)$, define $B' (e)$ to be the Bernoulli random variable which equals to one if and only if $e \in RNG (P_n)$ and $R_n < \| e \| \leq R^*_n$. Then $M'_n = \sum_{e \in K (P_n)} B' (e)$. Let $\mathcal{X}_2 = \{ X_1 X_2 \}$ and define $B'_1$ to be the Bernoulli random variable which equals to one if and only if $X_1 X_2 \in RNG (\mathcal{X}_{2k} \cup P_n)$ and $R_n < \| X_1 X_2 \| \leq R^*_n$. By treating each edge of $K (P_n)$ as a subset of two points in $P_n$ and with the application of Theorem 34, we have

\[ \mathbb{E} [M'_n] = \mathbb{E} \left[ \sum_{e \in K (P_n)} B' (e) \right] = \frac{n^2}{2} \mathbb{E} [B'_1]. \]

By Lemma 29,

\[ \frac{n^2}{2} \mathbb{E} [B'_1] = \frac{n^2}{2} \int_{\Omega'} \Pr [B'_1 = 1 \mid \mathcal{X}_2 = x] \, dx \]
\[ = \frac{n^2}{2} \int_{\Omega'} e^{-nx(x_1 x_2)} \, dx_1 dx_2 = o(1). \]

Therefore, $\mathbb{E} [M'_n] = o(1)$.

Next, we apply the Brun’s sieve together with the Palm theory to prove $M_n$ is asymptotically Poisson.

**Lemma 37** $M_n$ is asymptotically Poisson with mean $\frac{\beta_n^2}{2} e^{-\xi}$. 

Proof For any edge $e \in K (\mathcal{P}_n)$, define $B (e)$ to be the Bernoulli random variable which equals to one if and only if $e \in RNG (\mathcal{P}_n)$ and $r_\alpha < ||e|| \leq R_\alpha$. Then $M_n = \sum_{e \in K (\mathcal{P}_n)} B (e)$.

For any subgraph $H$ of $K (\mathcal{P}_n)$, define $B (H) = \prod_{e \in H} B (e)$. Denote by $\mathcal{T}_m$ the set of topologies on $m$ numbered vertices in which there are exactly $k$ edges and no vertex is isolated. Denote by $k^* = \left\lceil \frac{1 + \sqrt{1 + 4k^2}}{2} \right\rceil$. Then, $\mathcal{T}_m = \emptyset$ unless $k^* \leq m \leq 2k$. For any topology $\tau$ on $m$ numbered vertices and a planar set $U$ of $m$ numbered points, we denote by $\tau (U)$ the graph on $U$ with topology $\tau$. By Theorem 35, we only need to prove that

$$E \left\lbrack \sum_{m = k^*}^{2k} \sum_{U \subset \mathcal{P}_n, |U| = m} \sum_{\tau \in \mathcal{T}_m} B (\tau (U)) \right\rbrack \sim \frac{1}{k!} \left( \frac{\beta_1^2}{2} e^{-\xi} \right)^k.$$  \hspace{1cm} (3.1)

For each $e \in K (\mathcal{X}_m)$, define $B_m (e)$ to be the Bernoulli random variable which equals to one if and only if $e \in RNG (\mathcal{X}_{2k} \cup \mathcal{P}_n)$ and $r_\alpha < ||e|| \leq R_\alpha$. For any subgraph $H$ of $K (\mathcal{X}_m)$, define $B_m (H) = \prod_{e \in H} B_m (e)$. By Theorem 34,

$$E \left\lbrack \sum_{m = k^*}^{2k} \sum_{U \subset \mathcal{P}_n, |U| = m} \sum_{\tau \in \mathcal{T}_m} B (\tau (U)) \right\rbrack = \sum_{m = k^*}^{2k} \frac{n^m}{m!} E \left[ \sum_{\tau \in \mathcal{T}_m} B_m (\tau (\mathcal{X}_m)) \right].$$

We will prove that

$$\frac{n^{2k}}{(2k)!} E \left[ \sum_{\tau \in \mathcal{T}_{2k}} B_{2k} (\tau (\mathcal{X}_{2k})) \right] \sim \frac{1}{k!} \left( \frac{\beta_1^2}{2} e^{-\xi} \right)^k,$$  \hspace{1cm} (3.2)

and for each $\tau \in \mathcal{T}_m$ with $k^* \leq m < 2k$

$$n^m E [B_m (\tau (\mathcal{X}_m))] = o (1).$$  \hspace{1cm} (3.3)
These asymptotic equalities imply the asymptotic equality (3.1) immediately.

We first prove the asymptotic equality (3.2). Since

\[ \text{card}(\mathcal{T}_{2k}) = \frac{1}{k!} \binom{2k}{2, 2, \ldots, 2} = \frac{(2k)!}{k!2^k}, \]

and all topologies in \( \mathcal{T}_{2k} \) are isomorphic to each other, we have

\[
\frac{n^{2k}}{(2k)!} \mathbb{E} \left[ \sum_{\tau \in \mathcal{T}_{2k}} B_{2k}(\tau(\mathcal{X}_{2k})) \right] = \frac{1}{k!} \left( \frac{n^2}{2} \right)^k \mathbb{E} [B_{2k}(C_k(\mathcal{X}_{2k}))].
\]

It is sufficient to show that

\[
\left( \frac{n^2}{2} \right)^k \mathbb{E} [B_{2k}(C_k(\mathcal{X}_{2k}))] \sim \left( \frac{\beta_0^2}{2} e^{-\xi} \right)^k.
\] (3.4)

For \( k = 1 \), by Lemma 29 we have

\[
\frac{n^2}{2} \mathbb{E} [B_2(C_1(\mathcal{X}_2))] = \frac{n^2}{2} \int_{\Omega} \Pr [B_2(C_1(\mathcal{X}_2)) = 1 \mid \mathcal{X}_2 = x] \, dx = \frac{n^2}{2} \int_{\Omega} e^{-\nu(x_1x_2)} \, dx_1 \, dx_2 \sim \frac{\beta_0^2}{2} e^{-\xi}.
\]

So, the asymptotic equality (3.4) is true for \( k = 1 \). Now, suppose that \( k \geq 2 \), we have

\[
\left( \frac{n^2}{2} \right)^k \mathbb{E} [B_{2k}(C_k(\mathcal{X}_{2k}))] = \left( \frac{n^2}{2} \right)^k \int_{\Gamma(C_k)} \Pr [B_{2k}(C_k(\mathcal{X}_{2k})) = 1 \mid \mathcal{X}_{2k} = x] \, dx = \sum_{l=1}^{k} \left( \frac{n^2}{2} \right)^k \int_{\Gamma_l(C_k)} \Pr [B_{2k}(C_k(\mathcal{X}_{2k})) = 1 \mid \mathcal{X}_{2k} = x] \, dx.
\]
By Lemma 32,

\[
\left( \frac{n^2}{2} \right)^k \int_{\Gamma_k(C_k)} \Pr [B_{2k} (C_k (X_{2k})) = 1 \mid X_{2k} = x] \, dx = \left( \frac{n^2}{2} \right)^k \int_{\Gamma_k(C_k)} e^{-\nu(C_k(x))} \, dx \sim \left( \frac{\beta_0^2}{2} e^{-\xi} \right)^k.
\]

For any \(1 \leq l < k\), by Lemma 30 and 31,

\[
\left( \frac{n^2}{2} \right)^k \int_{\Gamma_l(C_k)} \Pr [B_{2k} (C_k (X_{2k})) = 1 \mid X_{2k} = x] \, dx \leq \left( \frac{n^2}{2} \right)^k \int_{\Gamma_l(C_k)} \chi (C_k (x)) \, dx \sim o(1).
\]

Thus, the asymptotic equality (3.4) is true for any \(k \geq 2\).

Next, we prove the asymptotic equality (3.3) for any \(\tau \in T_m\) with \(k^* \leq m < 2k\).

Since such \(\tau\) does not exist for \(k = 1\), we assume that \(k \geq 2\). Let \(F\) be any maximal spanning forest of \(\tau\). Then, the maximum degree of \(F\) is at least two and the minimum degree of \(F\) is at least one. By Lemma 33, we have

\[
n^m \mathbb{E} [B_m (\tau (X_m))] \leq n^m \mathbb{E} [B_m (F (X_m))]
\]

\[
= n^m \int_{\Gamma(F)} \Pr [B_m (F (X_m)) = 1 \mid X_m = x] \, dx
\]

\[
\leq n^m \int_{\Gamma(F)} \chi (F (x)) \, dx \sim o(1).
\]

So, the asymptotic equality (3.3) is true for any \(\tau \in T_m\) with \(k^* \leq m < 2k\). Therefore, Lemma 37 is proved.

Thus, the proof of Theorem 28 is complete.
3.4 Conclusion

The RNG is one of the widely used geometric structures in localized topology control and geographic routing in wireless ad hoc networks and can be constructed by distributed and localized algorithms. If all nodes have the same transmission radii, the maximal length of RNG edges is the smallest transmission radius for constructing the RNG by using only 1-hop neighbor information. All those applications of RNG require the maximum transmission radius of the networking nodes be no shorter than the longest edge in the RNG. In this chapter, we assume a wireless ad hoc networks is represented by a Poisson point process with mean \( n \) on a unit-area disk. We derived the precise asymptotic probability distribution of the maximum edge length in the RNG. Specifically, we showed that the probability of the event that the maximum edge length of the RNG at most \( \beta_0 \sqrt{\frac{\ln n + \xi}{\pi n}} \) is asymptotically equal to \( e^{-\frac{\sqrt{2}}{3} e^{-\xi}} \), where \( \beta_0 = 1/\sqrt{\frac{2}{3} - \frac{\sqrt{2}}{2\pi}} \).
CHAPTER 4
ASYMPTOTIC DISTRIBUTION OF THE CRITICAL TRANSMISSION RADIUS FOR GREEDY FORWARD ROUTING

Greedy forward routing (abbreviated by GFR) in wireless ad hoc networks is a localized geographic routing in which each node discards a packet if none of its neighbors is closer to the destination of the packet than itself, or otherwise forwards the packet to the neighbor closest to the destination of the packet. If all nodes have the same transmission radii, the critical transmission radius for GFR is the smallest transmission radius which ensures that packets can be delivered between any source-destination pairs. In this chapter, we consider a random wireless ad hoc network represented by a Poisson point process over a unit-area disk with mean $n$. Let $\phi_n$ denote its critical transmission radius for greedy forward routing. Little is known about the asymptotics of $\phi_n$ until the study in 2006 by Wan et al. [34]. Denote by $\sigma$ the constant $\frac{2\pi}{3} - \frac{\sqrt{3}}{2}$. It was proved in [34] that for any constant $\varepsilon > 0$, it is asymptotically almost sure (abbreviated by a.a.s.) that

$$(1 - \varepsilon) \sqrt{\frac{\ln n}{\sigma n}} \leq \phi_n \leq (1 + \varepsilon) \sqrt{\frac{\ln n}{\sigma n}}.$$ 

However, the precise asymptotic probability distribution of $\phi_n$ remains open. In this chapter, we settle this open problem. Specifically, for any constant $c$, the asymptotic probability of $\phi_n \leq \sqrt{\frac{\ln n + c}{\sigma n}}$ is exactly $\exp \left( - \left( \frac{1}{\frac{\pi}{2} - \frac{\pi}{2}} - \frac{\pi}{2\sigma} \right) e^{-c} \right)$.

4.1 Introduction

In GFR, each node discards a packet if none of its neighbors is closer to the destination of the packet than itself, or otherwise forwards the packet to the neighbor closest to the destination of the packet. Therefore, each node only need to maintain the locations of its one-hop neighbors and each packet should contain the location of the destination node. Thus, it can be implemented in a localized and memoryless manner. There are
many variants of GFR. For example, in [32] and [40], the shortest projected distance to the
destination on the straight line joining the current node and the destination node is con-
sidered as the greedy metrics. In [32], packets are allowed to be sent backward if there
is no forwarding neighbor. In [40], only the nodes whose Voronoi cells intersect with the
source-destination line segment are eligible. Some other variants of GFR can be found
in [2], [4], [5], [8], [12], [14], [15], [17], [18], [19], [20], [29], [31].

Due to the existence of local minima where none of neighbors is closer to the des-
tination than the current node, a packet may be discarded before it reaches its destination.
To ensure that every packet can reach its destination, all nodes should have sufficiently
large transmission radii to avoid the existence of local minima. The critical transmission
radius of a planar node set \( V \) for greedy forward routing is the smallest transmission radius
by \( V \) which ensures successful delivery of any packet from any source node in \( V \) to any
destination node in \( V \). The critical transmission radius of GFR is explicitly given by

\[
\phi(V) = \max_\{u \in V\} \max_\{v \in V\} \min\{\|wu\| : \|wu\| \leq \|uv\|, \ w \in V\}.
\]

In the definition, \((u, v)\) is a source-destination pair and \( w \) is a node that is closer to \( v \) than
\( u \). If the transmission radius is not less than \( \|wu\| \), \( w \) might be the one to relay packets
from \( u \) to \( v \). Therefore, for each pair \((u, v)\), the minimum of \( \|wu\| \) over all nodes on the
disk \( B(v, \|uv\|) \) guarantees there exists one node that can route packets from \( u \) to \( v \), and
the maximum of the minimum over all \((u, v)\) pairs guarantees the existence of relay nodes
between any source-destination pair. Clearly, if the transmission radius is at least \( \phi(V) \),
packets can be delivered between any source-destination pairs. On the other hand, if the
transmission radius is less than \( \phi(V) \), there must exist some source-destination pair, e.g.
the pair \((u, v)\) that gives the value \( \phi(V) \), such that packets cannot be delivered. Therefore,
the value \( \phi(V) \) is the critical transmission radius for GFR that guarantees the delivery of
any packets between any source-destination pair of nodes among the set \( V \).
The analytic work of GFR can date back to 1978 by Kleinrock and Silvester [17] (1978). They studies in [17] the optimal transmission radius to maximize the capacity. Although using a very large transmission radius gives a high degree of connectivity, there will be much interference and a corresponding loss of channel throughput. They analyzed this tradeoff and found that there is a transmission radius under which one can achieve a throughput proportional to the square root of the number of nodes in the network. They also investigated the expected progress of packets in one hop. Takagi and Kleinrock [32] (1984) studied the optimal transmission radius to maximize the expected progress of packets based on most forward and least backward routing strategy in which every node delivers each packet to the neighbor (not including itself) with the shortest projected distance to the destination on the straight line joining the current node. However, the deliverability of packets is not considered in [17] and [32]. In the last two decades, there is no significant progress. Recently, Xing et al. [40] (2004) showed that in a fully covered homogeneous wireless sensor network, if the transmission radius is larger than 2 times of the sensing radius, the deliverability of packets can be guaranteed between any source-destination pair by greedy forwarding schemes in which a packet is sent to the neighbor either with the shortest Euclidean distance to the destination [8] [14] or with the shortest projected distance to the destination on the straight line joining the current node and the destination node [32] and by bounded Voronoi greedy forwarding scheme in which only those nodes whose Voronoi cells intersect with the line segment between the source and destination are eligible to relay the packet.

In this chapter, we assume a wireless ad hoc network is represented by a Poisson point process over the unit-area disk $\mathbb{D}$ with density $n$, which is denoted by $\mathcal{P}_n$, and all nodes have the same maximal transmission radius which is a function of $n$. Denote by $\phi_n$ the critical transmission radius of $\mathcal{P}_n$ for greedy forward routing. Little is known about the asymptotics of $\phi_n$ until the study in 2006 by Wan et al. [34]. Let $\sigma$ be the area of the lune
of two points with unit distance, i.e.

\[ \sigma = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}. \]

It was proved in [34] that for any constant \( \varepsilon > 0 \), it is asymptotically almost sure (abbreviated by a.a.s.) that

\[ (1 - \varepsilon) \sqrt{\frac{\ln n}{\sigma n}} \leq \phi_n \leq (1 + \varepsilon) \sqrt{\frac{\ln n}{\sigma n}}. \] (4.1)

However, the asymptotic probability distribution of \( \phi_n \) was still unresolved. Recently, we derived tighter asymptotic bounds of \( \phi_n \) in [37]. Fix a constant \( c \) and let

\[ r_n = \sqrt{\frac{\ln n + c}{\sigma n}}. \]

We proved in [37] that the asymptotic probability of \( \phi_n \leq r_n \) is at least \( 1 - \left( \frac{1}{(\sigma/\pi - 1/3) - \pi/2\sigma} \right) e^{-c} \) and at most \( e^{-\frac{\sigma}{2\sigma}} e^{-c} \). As an immediate consequence, for any positive sequence \( (c_n : n \geq 1) \) with \( c_n = o(\ln n) \) and \( c_n \to \infty \), it is a.a.s. that

\[ \sqrt{\frac{\ln n - c_n}{\sigma n}} < \phi_n \leq \sqrt{\frac{\ln n + c_n}{\sigma n}}. \] (4.2)

We remark that the above inequalities are stronger than those given in equation (4.1). But the precise asymptotic probability distribution of \( \phi_n \) remains open.

In this chapter, we derive the precise asymptotic probability distribution of \( \phi_n \). Let

\[ \mu = \left( \frac{1}{\sigma/\pi - 1/3} - \frac{\pi}{2\sigma} \right) e^{-c}. \]

The main result of this chapter is stated in the following theorem.
**Theorem 38** \( \lim_{n \to \infty} \Pr [\phi_n \leq r_n] = e^{-\mu}. \)

Our proof for the above theorem only uses two basic tools from the probability theory: the Palm theory for Poisson processes and the Brun’s sieve (see Theorem 34 and Theorem 35 in Chapter 3). The innovative part of our proof is the subtle partition of the event \( \phi_n \leq r_n \) into a number of special events whose asymptotic probabilities can be computed easily. We believe our approach is quite general and can be applied to obtain the critical transmission radii of other variants of greedy forward routing proposed in [2], [8], [12], [14], [17], [18], [19].

The remaining of this chapter is organized as follows. The proof for Theorem 38 is presented in Section 4.3. Some preliminary results and integral ingredients to be used in the proof for Theorem 38 are established in Section 4.2. The proof for a technical lemma presented in Section 4.2 on the limits of two relevant integrals is postponed to Section 4.4. We summarize the chapter in Section 4.5.

**4.2 Preliminary and Integral Ingredients**

We use the same notations for lunes and quasi-lunes as in Section 2.2 in Chapter 2. For any pair of points \( u \) and \( v \), \( L(u, v) \) denotes the lune of \( u \) and \( v \). It is easy to verify that

\[
|L(u, v)| = \sigma \|uv\|^2,
\]

where \( \sigma \) is the area of the lune of two points with unit distance.

Let \( r \) be a positive number. For any \( r > 0 \) and any two points \( u \) and \( v \) on the plane, define

\[
L_r(u, v) = B(u, r) \cap B(v, \|uv\|),
\]
\[ \nu_r (u, v) = \min \{ |L_r (u, v) \cap \mathbb{D}|, |L_r (v, u) \cap \mathbb{D}| \} . \]

When \( 0 < r \leq 2 \| uv \| \), the set \( L_r (u, v) \) is the quasi-lune of the ordered pair \( (u, v) \) defined in Chapter 2. Its area \( |L_r (u, v)| \) is given by Lemma 1 in Section 2.2 of Chapter 2. When \( r > 2 \| uv \| \), clearly, \( L_r (u, v) \) is the disk \( B(v, \| uv \|) \) and its area \( |L_r (u, v)| = \pi \| uv \|^2 \).

A geometric graph is a graph on a finite planar set whose edges are line segments. Let \( V \) be a finite planar set. We use \( K(V) \) to denote the complete geometric graph over \( V \). Suppose that \( r \) is a positive number. We use \( GFR_r (V) \) to denote the geometric graph on \( V \) in which there is an edge between two nodes \( u \) and \( v \) if and only if \( \| uv \| > r \) and either \( L_r (u, v) \cap V = \emptyset \) or \( L_r (v, u) \cap V = \emptyset \). The \( r \)-disk graph of the set \( V \) is a geometric graph over \( V \) which consists of all edges \( uv \) satisfying that \( \| uv \| \leq r \).

Let \( H \) be a nonempty geometric graph. We use \( V(H) \) and \( E(H) \) to denote the vertex set and edge set of \( H \) respectively. An orientation of a graph \( H \) is a digraph obtained from \( H \) by orienting each edge of \( H \) into an arc. Clearly, if \( H \) has \( k \) edges, then \( H \) has \( 2^k \) orientations. Suppose that \( r \) is a positive number. Let \( \chi_r (H) \) be the indicator for \( H \subseteq GFR_r (V(H)) \). If \( \chi_r (H) = 1 \), we use \( D(H) \) to denote the set of orientations \( H' \) of \( H \) satisfying that \( L_r (u, v) \cap V(H) = \emptyset \) for each arc \( (u, v) \) of \( H' \), and define

\[ \nu_r (H) = \min_{H' \in D(H)} \left| \left( \bigcup_{(u,v) \in E(H')} L_r (u, v) \right) \cap \mathbb{D} \right| . \]

Suppose that \( n \) is a positive number. For pair of points \( u \) and \( v \), define

\[ g_n (u, v) = e^{-n|L_{rn}(u,v)\cap\mathbb{D}|} + e^{-n|L_{rn}(v,u)\cap\mathbb{D}|} - e^{-n(|L_{rn}(u,v)\cup L_{rn}(v,u))\cap\mathbb{D}|} . \]
For any nonempty geometric graph $H$, define

$$f_n (H) = \begin{cases} 0, & \text{if } \chi_{r_n} (H) = 0; \\ e^{-n\nu_{r_n}(H)}, & \text{if } \chi_{r_n} (H) = 1. \end{cases}$$

$$g_n (H) = \prod_{(u,v) \in E(H)} g_n (u,v).$$

Let $\varepsilon = 0.001$. Fix a sequence $(c_n)$ of real numbers satisfying that $c_n > c$, $c_n = o (\ln \ln n)$ and $c_n \to \infty$. Let

$$R_n = \sqrt{\frac{\ln n + c_n}{\sigma n}},$$
$$R'_n = \left(1 + \frac{\varepsilon}{2}\right) \sqrt{\frac{\ln n}{\sigma n}}.$$

Then, for sufficiently large $n$, we have $r_n < R_n < R'_n < (1 + \varepsilon) r_n$. Define

$$\Omega = \{ (x_1, x_2) \in \mathbb{D}^2 : r_n < ||x_1 x_2|| \leq R_n \},$$
$$\Omega' = \{ (x_1, x_2) \in \mathbb{D}^2 : R_n < ||x_1 x_2|| \leq R'_n \}.$$

The following lemma gives three asymptotic equalities which will be used in later sections.

**Lemma 39** The following asymptotic equalities are true:

$$\frac{n^2}{2} \int_{\Omega} e^{-n\nu_{r_n}(x_1, x_2)} dx_1 dx_2 \sim \frac{e^{-c}}{2(\sigma/\pi - 1/3)}, \quad (4.3)$$

$$\frac{n^2}{2} \int_{\Omega} g_n (x_1, x_2) dx_1 dx_2 \sim \mu, \quad (4.4)$$
\[ \frac{n^2}{2} \int_{\Omega'} g_n(x_1, x_2) \, dx_1 \, dx_2 = o(1). \] (4.5)

**Proof** For each \((x_1, x_2) \in \Omega \cup \Omega',\) let \(\rho = \rho(x_1, x_2)\) be the half-length of \(x_1 x_2,\) and \(z = z(x_1, x_2)\) be the midpoint of \(x_1 x_2.\) We split the union of \(\Omega \cup \Omega'\) into three subregions as follows. Let \(\Omega_1\) be the set of points \((x_1, x_2) \in \Omega\) satisfying that \(z \in D_{\sqrt{3}/4}(0).\) Let \(\Omega_2\) be the set of points \((x_1, x_2) \in \Omega'\) satisfying that \(z \in D_{\sqrt{3}/4}(0).\) Let \(\Omega_3 = (\Omega \cup \Omega') \setminus (\Omega_1 \cup \Omega_2).\) Then, \(\Omega_1, \Omega_2\) and \(\Omega_3\) form a partition of the union \(\Omega \cup \Omega'.\) In addition, for each \((x_1, x_2) \in \Omega_1 \cup \Omega_2,\) we have \(L(x_1, x_2) \subset D.\) For any \(\beta > \frac{\pi}{\sigma},\) let

\[ \gamma(\beta) = \frac{\sigma}{\pi} - \frac{1}{\beta}. \]

We will prove the following six asymptotic equalities, from which the three asymptotic equalities (4.3), (4.4) and (4.5) follow immediately:

\[ \frac{n^2}{2} \int_{\Omega_1} e^{-n|L_{rn}(x_1, x_2)|} \, dx_1 \, dx_2 \sim \frac{1}{2\gamma(3)} e^{-c}, \] (4.6)

\[ \frac{n^2}{2} \int_{\Omega_1} e^{-n|L_{rn}(x_1, x_2) \cup L_{rn}(x_2, x_1)|} \, dx_1 \, dx_2 \sim \frac{\pi}{2\sigma} e^{-c}, \] (4.7)

\[ \frac{n^2}{2} \int_{\Omega_2} e^{-n|L_{rn}(x_1, x_2)|} \, dx_1 \, dx_2 = o(1), \] (4.8)

\[ \frac{n^2}{2} \int_{\Omega_2} e^{-n|L_{rn}(x_1, x_2) \cup L_{rn}(x_2, x_1)|} \, dx_1 \, dx_2 = o(1), \] (4.9)

\[ \frac{n^2}{2} \int_{\Omega_3} e^{-n|L_{rn}(x_1, x_2) \cap D|} \, dx_1 \, dx_2 = o(1), \] (4.10)

\[ \frac{n^2}{2} \int_{\Omega_3} e^{-n|L_{rn}(x_1, x_2) \cup L_{rn}(x_2, x_1) \cap D|} \, dx_1 \, dx_2 = o(1). \] (4.11)

Among these six asymptotic equalities, the two asymptotic equalities (4.9) and (4.11) are implied by the two asymptotic equalities (4.8) and (4.10) respectively, and thus will not be
further proved. We will frequently replace the integral variables $x_1$ and $x_2$ by the midpoint $z$, half-length $\rho$ and the slope of the line $x_1 x_2$. The Jacobian determinant of this variable change is $4\rho$.

We first claim that for any $\beta > \frac{\pi}{\sigma}$,

$$4\pi n^2 e^{-\frac{\pi n^2 z^2}{\beta}} \int_{\mathbb{R}^n} e^{-4\gamma(\beta) n \pi \rho^2} \rho d\rho \sim \frac{1}{2\gamma(\beta)} e^{-c},$$  \hspace{1cm} (4.12)

$$4\pi n^2 e^{-\frac{\pi n^2 z^2}{\beta}} \int_{\frac{R_0}{2}}^{\frac{R_0}{2}} e^{-4\gamma(\beta) n \pi \rho^2} \rho d\rho = o(1).$$  \hspace{1cm} (4.13)

Indeed,

$$4\pi n^2 e^{-\frac{\pi n^2 z^2}{\beta}} \int_{\frac{R_0}{2}}^{\frac{R_0}{2}} e^{-4\gamma(\beta) n \pi \rho^2} \rho d\rho = \frac{n}{2\gamma(\beta)} e^{-\frac{\pi n^2 z^2}{\beta}} \int_{\frac{R_0}{2}}^{\frac{R_0}{2}} e^{-4\gamma(\beta) n \pi \rho^2} d\left(4\gamma(\beta) n \pi \rho^2\right)$$

$$= \frac{n}{2\gamma(\beta)} e^{-\frac{\pi n^2 z^2}{\beta}} \left(e^{-\gamma(\beta) n \pi R_0^2} - e^{-\gamma(\beta) n \pi R_n^2}\right)$$

$$= \frac{n}{2\gamma(\beta)} e^{-\frac{\pi n^2 z^2}{\beta}} e^{-\gamma(\beta) n \pi R_0^2} \left(1 - e^{-\gamma(\beta) n \pi (R_n^2 - r_n^2)}\right)$$

$$\sim \frac{n}{2\gamma(\beta)} e^{-\frac{\pi n^2 z^2}{\beta}}$$

$$= \frac{1}{2\gamma(\beta)} e^{-c}.$$

So the asymptotic equality (4.12) holds. Similarly, we can show that

$$4\pi n^2 e^{-\frac{\pi n^2 z^2}{\beta}} \int_{\frac{R_0}{2}}^{\frac{R_0}{2}} e^{-4\gamma(\beta) n \pi \rho^2} \rho d\rho \sim \frac{1}{2\gamma(\beta)} e^{-c},$$

which together with the asymptotic equality (4.12) implies the asymptotic equality (4.13).
Now, we prove the asymptotic equality (4.6). By Lemma 6, for any \((x_1, x_2) \in \Omega_1,\)

\[
|L_{r_n}(x_1, x_2)| = |L(x_1, x_2)| - |L(x_1, x_2) \setminus L_{r_n}(x_1, x_2)|
\]

\[
= \frac{\pi \|x_1x_2\|^2}{\beta_0} - \frac{\pi}{3} \left(\|x_1x_2\|^2 - r_n^2\right) - o\left(\frac{1}{n}\right)
\]

\[
= \gamma (3) \pi \|x_1x_2\|^2 + \frac{\pi}{3} r_n^2 - o\left(\frac{1}{n}\right).
\]

Thus,

\[
\frac{n^2}{2} \int_{\Omega_1} e^{-n|L_{r_n}(x_1, x_2)|} dx_1 dx_2
\]

\[
= \frac{n^2}{2} \int_{\Omega_1} e^{-n \left(4\gamma (3) \pi \rho^2 + \frac{\pi}{3} r_n^2\right) + o(1)} dx_1 dx_2
\]

\[
\sim \frac{n^2}{2} e^{-\frac{\pi}{3} r_n^2} \int_{\Omega_1} e^{-4\gamma (3) n \pi \rho^2} dx_1 dx_2
\]

\[
= 4\pi n^2 e^{-\frac{\pi}{3} r_n^2} \int_{\frac{r_n}{2}}^{\frac{R_n}{2}} e^{-4\gamma (3) n \pi \rho^2} \rho d\rho \int_{D_\gamma \tau_n(0)} dz
\]

\[
\sim 4\pi n^2 e^{-\frac{\pi}{3} r_n^2} \int_{\frac{r_n}{2}}^{\frac{R_n}{2}} e^{-4\gamma (3) n \pi \rho^2} \rho d\rho
\]

\[
\sim \frac{1}{2\gamma (3)} e^{-c},
\]

where the last step follows from the the asymptotic equality (4.12). So, the asymptotic equality (4.6) holds.

Next, we prove the asymptotic equality (4.7). By Lemma 6, for any \((x_1, x_2) \in \Omega_1,\)

\[
|L_{r_n}(x_1, x_2) \cup L_{r_n}(x_2, x_1)| = |L(x_1, x_2)| - o\left(\frac{1}{n}\right).
\]
Thus,

\[
\frac{n^2}{2} \int_{\Omega_1} e^{-n|L_{r_n}(x_1, x_2) \cup L_{r_n}(x_2, x_1)|} dx_1 dx_2 \\
= \frac{n^2}{2} \int_{\Omega_1} e^{-n|L(x_1, x_2)| + o(1)} dx_1 dx_2 \\
\sim \frac{n^2}{2} \int_{\Omega_1} e^{-n|L(x_1, x_2)|} dx_1 dx_2 \\
= \frac{n^2}{2} \int_{\Omega_1} e^{-n\|x_1 x_2\|^2} dx_1 dx_2 \\
= 4\pi n^2 \int_{\frac{r_n}{2}}^{\frac{R_n}{2}} e^{-4n\rho^2\sigma} \rho d\rho \int_{D_\sqrt{\sigma\rho}(0)} dz \\
\sim 4\pi n^2 \int_{\frac{r_n}{2}}^{\frac{R_n}{2}} e^{-4n\rho^2\sigma} \rho d\rho \\
= -\frac{\pi}{2\sigma} ne^{-4n\rho^2\sigma} \frac{R_n}{2} \\
\sim \frac{\pi}{2\sigma} e^{-c}.
\]

So, the asymptotic equality (4.7) holds.

We proceed to prove the asymptotic equality (4.8). When \( n \) is sufficiently large, \( R'_n \leq 1.1 r_n \). By Lemma 4, for any \((x_1, x_2) \in \Omega_2\),

\[
|L_{r_n}(x_1, x_2)| \\
= |L(x_1, x_2)| - |L(x_1, x_2) \setminus L_{r_n}(x_1, x_2)| \\
\geq \frac{\pi \|x_1 x_2\|^2}{\pi} - \frac{\pi}{2.7} (\|x_1 x_2\|^2 - r_n^2) \\
= \gamma (2.7) \pi \|x_1 x_2\|^2 + \frac{\pi r_n^2}{2.7}.
\]
Thus,

$$\frac{n^2}{2} \int_{\Omega_2} e^{-n|L_{rn}(x_1,x_2)|} dx_1 dx_2$$

$$\leq \frac{n^2}{2} \int_{\Omega_2} e^{-n\left(\gamma(2.7)\pi\|x_1x_2\|^2 + \frac{\pi r_n^2}{2}\right)} dx_1 dx_2$$

$$= 4\pi n^2 e^{-\frac{n\pi^2 r_n^2}{2}} \int_{\mathbb{R}^n} e^{-4\gamma(2.7)\pi n \rho^2} \rho d\rho \int_D dz$$

$$\sim 4\pi n^2 e^{-\frac{n\pi^2 r_n^2}{2}} \int_{\mathbb{R}^n} e^{-4\gamma(2.7)\pi n \rho^2}$$

$$= o(1),$$

where the last step follows from the the asymptotic equality (4.13). So, the asymptotic equality (4.8) holds.

Finally, we prove the asymptotic equality (4.10). For any \((x_1, x_2) \in \Omega_3\), let \(t = t(x_1, x_2)\) be the distance between the midpoint \(z\) of \(x_1 x_2\) and \(\partial \mathbb{D}\). By Lemma 3, we have

$$|L_{rn}(x_1, x_2) \cap \mathbb{D}| \geq \frac{1}{2} r_n^2 \sigma + \frac{1}{4} r_n t$$

and

$$\rho \theta (z, \rho) \leq 2\pi t.$$

Thus,

$$\frac{n^2}{2} \int_{\Omega_3} e^{-n|L_{rn}(x_1,x_2)\cap\mathbb{D}|} dx_1 dx_2$$

$$\leq \frac{n^2}{2} \int_{\Omega_3} e^{-n\left(\frac{1}{2} \pi r_n^2 + \frac{1}{4} r_n t\right)} dx_1 dx_2$$
\[\begin{align*}
&= 2n^2 e^{-\frac{nr^2}{2}} \int_0^{\frac{\pi}{2}} d\rho \int_{\mathbb{D}_{\sqrt{3}r_n}(1) \setminus \mathbb{D}_{\sqrt{3}r_n}(2)} e^{-\frac{1}{4nrr^2} \rho \theta (z, \rho)} dz \\
&\leq 4\pi n^2 e^{-\frac{nr^2}{2}} \int_0^{\frac{\pi}{2}} d\rho \int_{\mathbb{D}} e^{-\frac{1}{4nrr^2} \rho t} dt dz \\
&= O(1) n^2 e^{-\frac{nr^2}{2}} \int_0^{\frac{\pi}{2}} d\rho \int_0^{1/\sqrt{\pi}} e^{-\frac{1}{4nrr^2} td} dt \\
&\leq O(1) R_n e^{-\frac{nr^2}{2}} \int_0^{\infty} e^{-\frac{1}{4nrr^2} td} dt \\
&= O(1) R_n e^{-\frac{nr^2}{2}} \int_0^{\infty} e^{-\frac{1}{4nrr^2} td} dt \\
&= O(1) R_n e^{-\frac{nr^2}{2}} R_n^{-2} \\
&= O(1) \frac{r_n}{\sqrt{n}} \\
&= O(1) \sqrt{\frac{1}{\ln n}} \\
&= o(1).
\end{align*}\]

So, the asymptotic equality (4.10) holds.

A topology with numbered vertices is specified by a collection of the pairs of the indices of the numbered vertices. For any integer \( m \geq 2 \), denote \( \mathcal{T}_m \) set of topologies on \( m \) numbered vertices without isolated vertex. For any \( \tau \in \mathcal{T}_m \), and any sequence \( U \) of \( m \) planar points, \( \tau (U) \) denotes the geometric graph on \( U \) with topology \( \tau \). For any \( \tau \in \mathcal{T}_m \), we denote by \( \Gamma (\tau) \) the set of \( x \in \mathbb{D}^m \) satisfying that all edges of \( \tau (x) \) have length in \((r_n, R_n]\). Note that for each \( x \in \Gamma (\tau) \), the \( \sqrt{3}R_n \)-disk graph on the midpoints of the edges in any connected component of \( \tau (x) \) is connected. Thus, the \( \sqrt{3}R_n \)-disk graph on the midpoints of the edges in \( \tau (x) \) has no more connected components than \( \tau (x) \) itself. For any positive integer \( l \) no more than the number of connected components of \( \tau \), we denote by \( \Gamma_l (\tau) \) the set of \( x \in \Gamma (\tau) \) such that the \( \sqrt{3}R_n \)-disk graph on the midpoints of the edges in \( \tau (x) \) has \( l \) connected components.
Lemma 40 Suppose that $2 < m \leq 2k$ and $\tau$ is a forest in $T_m$ with $k$ edges. Then, for any positive integer $l \leq \min\{m - k, k - 1\}$,

$$n^m \int_{\Gamma_l(\tau)} f_n(\tau(x)) \, dx = o(1),$$

$$n^m \int_{\Gamma_l(\tau)} g_n(\tau(x)) \, dx = o(1).$$

The proof for Lemma 40 is postponed to Section 4.4. Lemma 40 implies the following two corollaries.

**Corollary 41** Suppose that $\tau \in T_{2k}$ is a perfect matching for some $k \geq 2$. Then,

$$n^{2k} \int_{\Gamma_{k}(\tau)} g_n(\tau(x)) \, dx \sim (2\mu)^k.$$

**Proof** We denote by $C_k$ the perfect matching of $2k$ numbered vertices $v_1, v_2, \ldots, v_{2k}$ which consists of $k$ edges $v_{2i-1}v_{2i}$ for $1 \leq i \leq k$. By symmetry, we only have to prove the lemma holds for $\tau = C_k$. Note that

$$\Gamma_k(C_k) = \Gamma(C_k) \setminus \bigcup_{i=1}^{k-1} \Gamma_i(C_k).$$

Since $\Gamma(C_k) = \Omega^k$, we have

$$n^{2k} \int_{\Gamma(C_k)} g_n(C_k(x)) \, dx$$

$$= \prod_{i=1}^{k} \left( n^2 \int_{\Omega} g_n(x_{2i-1}, x_{2i}) \, dx_{2i-1} dx_{2i} \right)$$

$$\sim (2\mu)^k,$$
By Lemma 40 for each $1 \leq l \leq k - 1$,

$$n^{2k} \int_{\Gamma_l(C_k(x))} g_n(C_k(x)) \, dx = o(1).$$

Thus, the corollary holds.

**Corollary 42** Suppose that $\tau \in T_m$ is not a perfect matching. Then,

$$n^m \int_{\Gamma(\tau)} f_n(\tau(x)) \, dx = o(1).$$

**Proof** Clearly, $m > 2$. Let $\tau'$ be a maximal spanning forest of $\tau$. Then, $\tau' \in T_m$ and $\tau'$ is not a perfect matching. In addition, $\Gamma(\tau') \supseteq \Gamma(\tau)$, and for any $x \in \Gamma(\tau)$,

$$f_n(\tau'(x)) \geq f_n(\tau(x)).$$

Hence,

$$\int_{\Gamma(\tau)} f_n(\tau(x)) \, dx \leq \int_{\Gamma(\tau')} f_n(\tau'(x)) \, dx \leq \int_{\Gamma(\tau')} f_n(\tau'(x)) \, dx.$$

Thus, it’s sufficient to show that

$$n^m \int_{\Gamma(\tau')} f_n(\tau'(x)) \, dx = o(1).$$

Let $k$ be the number of edges in $\tau'$. Then, $\tau$ has $m - k$ tree components and hence

$$\Gamma(\tau') = \bigcup_{l=1}^{m-k} \Gamma_l(\tau').$$

Since $\tau'$ is not a perfect matching, we have $m < 2k$, which implies $m - k \leq k - 1$. By
Lemma 40, for any $1 \leq l \leq m - k$,

$$n^m \int_{\Gamma_l(\tau')} f_n(\tau'(x)) dx = o(1).$$

Therefore,

$$n^m \int_{\Gamma(\tau')} f_n(\tau'(x)) dx = \sum_{i=1}^{m-k} n^m \int_{\Gamma_i(\tau')} f_n(\tau'(x)) dx = o(1).$$

For any geometric graph $H$ and any $n > 0$, let $B_n(H)$ be the indicator for $H \subseteq GFR_{r_n}(V(H) \cup P_n)$ and all edges of $H$ have length in $(r_n, R_n]$.

**Lemma 43** For any $\tau \in T_m$, if $\tau$ is a perfect matching then

$$n^m \mathbb{E} [B_n(\tau(X_m))] \sim (2\mu)^{m/2};$$

otherwise,

$$n^m \mathbb{E} [B_n(\tau(X_m))] = o(1).$$

**Proof** Consider a topology $\tau \in T_m$. Let $k$ be the number of edges in $\tau$. Then, $k \leq m(m - 1)/2$. We claim that for any $x \in \Gamma(\tau)$,

$$\Pr [B_n(\tau(x)) = 1] \leq 2^k f_n(\tau(x)).$$

Indeed, if $\chi_{r_n}(\tau(x)) = 0$, then

$$\Pr [B_n(\tau(x)) = 1] = 0$$

and hence the claim holds trivially. Now, assume that $\chi_{r_n}(\tau(x)) = 1$. Then,

$$1 \leq \text{card}(D(\tau(x))) \leq 2^k,$$
and

\[
\Pr [B_n (\tau (x)) = 1] \\
\leq \sum_{H' \in D(\tau(x))} e^{-n|\cup_{(u,v) \in E(H')} L_{\tau_n}(u,v)|} \\
\leq \text{card} (D (\tau (x))) e^{-n \varepsilon_n(\tau(x))} \\
= \text{card} (D (\tau (x))) f_n (\tau (x)) \\
\leq 2^k f_n (\tau (x)).
\]

Hence, the claim also holds if \( \chi_{\tau_n} (\tau (x)) \). It’s easy to verify that if \( \tau \) is a perfect matching (i.e., \( m = 2k \)) then for any \( x \in \Gamma_k (\tau) \),

\[
\Pr [B_n (\tau (x)) = 1] = g_n (\tau (x)).
\]

Clearly,

\[
E [B_n (\tau (X_m))] = \int_{\Gamma(\tau)} \Pr [B_n (\tau (x)) = 1] dx.
\]

If \( \tau \) is not a perfect matching, then by Corollary 42, we have

\[
n^m E [B_n (\tau (X_m))] \leq 2^k n^m \int_{\Gamma(\tau)} f_n (\tau (x)) dx = o (1).
\]

In the next, we assume that \( \tau \) is a perfect matching. Let \( m = 2k \). For \( k = 1 \), \( \Gamma (\tau) = \Omega \) and hence by Lemma 39 we have

\[
n^2 E [B (\tau (X_2))] \\
= n^2 \int_{\Omega} \Pr [B (\tau (x)) = 1] dx \\
= n^2 \int_{\Omega} g_n (x_1, x_2) dx_1 dx_2
\]
\[ \sim 2\mu. \]

So, the lemma holds for \( k = 1 \). So, we further assume that \( k \geq 2 \). Note that

\[
n^{2k} \mathbb{E} [B_n (\tau (X_{2k}))]
\]

\[
= n^{2k} \int_{\Gamma(\tau)} \Pr [B_n (\tau (x)) = 1] \, dx
\]

\[
= \sum_{l=1}^{k} n^{2k} \int_{\Gamma_l(\tau)} \Pr [B_n (\tau (x)) = 1] \, dx.
\]

By Corollary 41 we have

\[
n^{2k} \int_{\Gamma_k(\tau)} \Pr [B_n (\tau (x)) = 1] \, dx
\]

\[
= n^{2k} \int_{\Gamma_k(\tau)} g_n (\tau (x)) \, dx \sim (2\mu)^k,
\]

and for any \( 1 \leq l < k \),

\[
n^{2k} \int_{\Gamma_l(\tau)} \Pr [B_n (\tau (x)) = 1] \, dx
\]

\[
\leq n^{2k} \int_{\Gamma_l(\tau)} 2^k f_n (\tau (x)) \, dx = o(1).
\]

Thus,

\[
n^{2k} \mathbb{E} [B_n (\tau (X_{2k}))] \sim (2\mu)^k.
\]

So, the lemma holds in this case.

For any \( x = (x_1, x_2) \in \mathbb{D}^m \), let \( B'_n (x) \) to be the indicator for \( x_1x_2 \) is void of \( P_n \) and \( R_n < \|x_1x_2\| \leq R'_n \) and either \( L_{r_n} (x_1, x_2) \cap P_n = \emptyset \) or \( L_{r_n} (x_2, x_1) \cap P_n = \emptyset \).
Lemma 44 \( n^2 \mathbb{E} [B'_n (\mathcal{A}_2)] = o (1) \).

**Proof** For any \( x = (x_1, x_2) \in \Omega' \),

\[
\Pr [B'_n (x) = 1] = g_n (x_1, x_2).
\]

Thus,

\[
\begin{align*}
    n^2 \mathbb{E} [B'_n (\mathcal{A}_2)] & = n^2 \int_{\Omega'} \Pr [B'_n (x) = 1] \, dx \\
    & = n^2 \int_{\Omega'} g_n (x_1 x_2) \, dx_1 dx_2 \\
    & = o (1),
\end{align*}
\]

where the last equality follows from Lemma 39.

### 4.3 Proof for The Main Theorem

We first give a brief overview on our approach to prove Theorem 38. Let \( M_n \) (respectively, \( M'_n \) and \( M''_n \)) denote the number of edges in \( GFR_{r_n} (\mathcal{P}_n) \) with length in \( (r_n, R_n) \) (respectively, \( (R_n, R'_n) \) and \( (R'_n, +\infty) \)). Then, \( \phi_n \leq r_n \) if and only if \( M_n + M'_n + M''_n = 0 \). In Lemma 45, we will show that \( M''_n = 0 \) is a.a.s.. In Lemma 46, we will prove that \( E [M'_n] = o (1) \), which implies that \( M'_n = 0 \) is a.a.s. by Markov’s inequality. In Lemma 48, we will prove that \( M_n \) is asymptotically Poisson with mean \( \mu \). Consequently,

\[
\lim_{n \to \infty} \Pr [\phi_n \leq r_n] = \lim_{n \to \infty} \Pr [M_n + M'_n + M''_n = 0] = \lim_{n \to \infty} \Pr [M_n = 0]
\]
We first utilize the tool of minimal scan statistics developed in [34] to prove that $M''_n = 0$ is a.a.s.

**Lemma 45** $\Pr [M''_n > 0] = o(1)$.

**Proof** For any finite point set $V \subset \mathbb{D}$ and any $r > 0$, define

$$S (V, r) = \min_{(u,v) \in \mathbb{D}^2, \|uv\| = (1 + \varepsilon/4) r} |V \cap L_r (u,v)| .$$

For any $(u,v) \in \mathbb{D}^2$ with $\|uv\| = (1 + \varepsilon/4) r$, let $v'$ be the point on $uv$ satisfying that $\|uv'\| = r$. Then, there is a positive constant $\varepsilon' > 0$ such that $|L_r (u,v)| > (1 + \varepsilon') |L (u,v')|$. Using this fact and following the same argument in the proof of Lemma 9 in [34], we can prove that $S (P_n, r_n) > \Theta (\ln n)$ is a.a.s.. In particular, $S (P_n, r_n) > 0$ a.a.s.. Next, we claim that the event $M''_n > 0$ implies the event $S (P_n, r_n) = 0$. Suppose that $M''_n > 0$. Then there are a pair of nodes \{X,Y\} $\subset P_n$ such that $\|XY\| > R'_n$ and either $L_{r_n} (X,Y) \cap P_n$ or $L_{r_n} (Y,X) \cap P_n$ is empty. By symmetry, we assume that $L_{r_n} (X,Y) \cap P_n$ is empty. Let $Y'$ be the point on the ray $XY$ satisfying that $\|XY'\| = (1 + \varepsilon/4) r_n$. When $n$ is sufficiently large, $R'_n > (1 + \varepsilon/4) r_n$ and hence $Y'$ is on the line segment $XY$. Thus $L_{r_n} (X,Y') \subset L_{r_n} (X,Y)$. So $L_{r_n} (X,Y') \cap P_n$ is empty. This implies that $S (P_n, r_n) = 0$. Therefore, our claim holds. Consequently,

$$\Pr [M''_n > 0] \leq \Pr [S (P_n, r_n) = 0] = o (1).$$

Two key techniques used in our remaining proof are the Palm theory for Poisson
processes and the Brun’s sieve. Now we apply Palm theory to show that \( E[M'_n] \) is vanishing.

**Lemma 46** \( E[M'_n] = o(1) \).

**Proof** For any pair \((U, V)\) with \( V \) being a finite planar set and \( U \) being a subset of \( V \), define \( h'(U, V) \) to be the number of edges in \( K(U) \) which have length in \((R_n, R'_n)\) and are void of \( V \). By applying the Palm theory, we have

\[
E[M'_n] = \mathbb{E}\left[ \sum_{U \subseteq P_n, |U| = 2} h'(U, P_n) \right] = \frac{n^2}{2} \mathbb{E}[h'(X_2, X_2 \cup P_n)] = \frac{n^2}{2} \mathbb{E}[B'_n(X_2)].
\]

By Lemma 44, the lemma follows.

For any positive integer \( k \), denote by \( \mathcal{H}_{n,k} \) the collection of \( k \)-edge subgraphs of \( GFR_{r_n}(P_n) \) in which all edges have length in \((r_n, R_n)\) and no vertex is isolated. Next, we apply Palm theory to compute the asymptotic average of \( \text{card} (\mathcal{H}_{n,k}) \).

**Lemma 47** For any fixed positive integer \( k \),

\[
\mathbb{E}[\text{card} (\mathcal{H}_{n,k})] \sim \frac{\mu^k}{k!}.
\]

**Proof** For any pair \((U, V)\) with \( V \) being a finite planar set and \( U \) being a subset of \( V \), define \( h(U, V) \) to be the number of \( k \)-edge subgraphs of \( GFR_{r_n}(V) \) on \( U \) in which all
edges have length in \((r_n, R_n]\) and no vertex is isolated. By applying the Palm theory, we have

\[
E[\text{card}(\mathcal{H}_{n,k})] = E\left[\sum_{m=2}^{2k} \sum_{U \subseteq \mathcal{P}_n, |U| = m} h(U, \mathcal{P}_n)\right] = \sum_{m=2}^{2k} E \left[\sum_{U \subseteq \mathcal{P}_n, |U| = m} h(U, \mathcal{P}_n)\right] = \sum_{m=2}^{2k} \frac{n^m}{m!} E[h(X_m, X_m \cup \mathcal{P}_n)].
\]

Thus, it is sufficient to show that

\[
\frac{n^m}{m!} E[h(X_m, X_m \cup \mathcal{P}_n)] \sim \begin{cases} 
0, & \text{if } 2 \leq m < 2k; \\
\frac{k^k}{k!}, & \text{if } m = 2k.
\end{cases}
\]

For any positive integer \(k\) and any \(2 \leq m \leq 2k\), denote \(\mathcal{T}_{m,k}\) the set of topologies in \(\mathcal{T}_m\) with exactly \(k\) edges. Note that any topology in \(\mathcal{T}_{2k,k}\) is a perfect matching, and

\[
card(\mathcal{T}_{2k,k}) = \frac{1}{k!} \binom{2k}{2, 2, \ldots, 2} = \frac{(2k)!}{k!2^k}.
\]

By Lemma 43, we have

\[
\frac{n^{2k}}{(2k)!} E[h(X_{2k}, X_{2k} \cup \mathcal{P}_n)] = \frac{1}{(2k)!} \sum_{\tau \in \mathcal{T}_{2k,k}} n^{2k} E[B_n(\tau(X_{2k}))] \sim \frac{1}{(2k)!} \cdot \frac{(2k)!}{k!2^k} \cdot (2\mu)^k = \frac{\mu^k}{k!}.
\]
Now suppose that $2 \leq m < 2k$. Then any topology in $\mathcal{T}_{m,k}$ is not a perfect matching. By Lemma 43,

$$n^m \mathbb{E}[h(X_m, X_m \cup P_n)] = \sum_{\tau \in \mathcal{T}_{m,k}} n^m \mathbb{E}[B_n(\tau(X_m))] = o(1).$$

Finally, we apply the Brun’s sieve together with Lemma 47 to prove $M_n$ is asymptotically Poisson.

**Lemma 48** $M_n$ is asymptotically Poisson with mean $\mu$.

**Proof** Let $\mathcal{E}_n$ be the set of edges of $K(P_n)$. For any edge $e \in \mathcal{E}_n$, define $\overline{B}(e)$ to be the Bernoulli random variable which equals to one if and only if $r_n < \|e\| \leq R_n$ and $e$ is an edge of $GFR_{r_n}(P_n)$. Then

$$M_n = \sum_{e \in \mathcal{E}_n} \overline{B}(e).$$

For subset $F$ of $\mathcal{E}_n$, $\prod_{e \in F} \overline{B}(e) = 1$ if and only if $F$ is the edge set of a subgraph of $GFR_{r_n}(P_n)$ in which all edges have length in $(r_n, R_n]$ and no vertex is isolated. Fix a positive integer $k$. By treating each $k$-subset $F$ of $\mathcal{E}_n$ as a $k$-edge subgraph of $K(P_n)$, we have that

$$\sum_{F \subseteq \mathcal{E}_n, |F|=k} \prod_{e \in F} \overline{B}(e) = \text{card}(\mathcal{H}_{n,k}).$$

Hence, by Lemma 47,

$$\mathbb{E}\left[\sum_{F \subseteq \mathcal{E}_n, |F|=k} \prod_{e \in F} \overline{B}(e)\right] \sim \frac{\mu^k}{k!}.$$

By Theorem 35, $M_n$ is asymptotically Poisson with mean $\mu$.

**4.4 Proof for Lemma 40**
In this section, we prove Lemma 40. For any geometric graph $H$ on a finite subset of $\mathbb{D}$, an edge $e$ of $H$ is called an outermost edge of $H$ if its midpoint is the nearest to $\partial \mathbb{D}$.

We will frequently change the integral variables using a technique introduced in [35]. Consider a tree topology on $k$ planar points $x_1, x_2, \cdots, x_k$, and assume without loss of generality that $x_{k-1}x_k$ is an edge in this tree. Let $z_{k-1}$, $\rho$, and $\omega$ be the midpoint, half-length and the slope of $x_{k-1}x_k$ respectively. We root the tree at $x_k$. For $1 \leq i \leq k-2$, let $z_i$ be the midpoint of the edge between $x_i$ and its parent in such rooted tree. Then, we replace $x_1, x_2, \cdots, x_k$ by $z_1, \cdots, z_{k-1}, \rho, \omega$. The Jacobian determinant of this change is $4^{k-1}\rho$.

**Lemma 49** Suppose that $2 < m \leq 2k$ and $\tau$ is a forest in $\mathcal{T}_m$ with $k$ edges. Then,

\[
\int_{\Gamma_1(\tau)} f_n(\tau(x)) \, dx = o(1),
\]

\[
\int_{\Gamma_1(\tau)} g_n(\tau(x)) \, dx = o(1).
\]

**Proof** Enumerate the edges of $\tau$ arbitrarily by $e_1, \cdots, e_k$. For any $x \in \Gamma_1(\tau)$, let $z_i$ denote the middle point of $e_i$ in $F(x)$ for each $1 \leq i \leq k$. For any pair of distinct integers $p$ and $q$ between 1 and $k$, let $S_{pq}$ denote the set of $x \in \Gamma_1(\tau)$ satisfying that $e_p$ is an outermost edge in $\tau(x)$ and $z_q$ is the farthest from $z_p$ among all $z_1, \cdots, z_k$. Then, it suffices to prove for any such $p$ and $q$,

\[
\int_{S_{pq}} f_n(\tau(x)) \, dx = o(1),
\]

\[
\int_{S_{pq}} g_n(\tau(x)) \, dx = o(1).
\]

Fix a pair of distinct integers $p$ and $q$ between 1 and $k$. Let $p'$ and $p''$ be the indices of the
two endpoints of the edges \( e_p \). We claim that for any \( x \in S_{pq} \),

\[
\begin{align*}
    f_n(\tau(x)) &\leq e^{-n(\nu_{\tau_n}(x_{p'}x_{p''}) + \eta_1 R_n \|z_p z_q\| - \eta_2 \varepsilon_n R_n^2)}, \\
    g_n(\tau(x)) &\leq 2^k e^{-n(\nu_{\tau_n}(x_{p'}x_{p''}) + \eta_1 R_n \|z_p z_q\| - \eta_2 \varepsilon_n R_n^2)},
\end{align*}
\]

(4.14) (4.15)

in which \( \eta_1 = 0.0026 \) and \( \eta_2 = 16 \). Indeed, if \( \chi_{\tau_n}(\tau(x)) = 0 \) then \( f_n(\tau(x)) = 0 \) and hence the inequality (4.14) holds trivially. If \( \chi_{\tau_n}(\tau(x)) = 1 \), then

\[
    f_n(\tau(x)) = e^{-n\nu_{\tau_n}(\tau(x))}
\]

and by Lemma 11 in Chapter 2,

\[
    \nu_{\tau_n}(\tau(x)) \geq \nu_{\tau_n}(x_{p'}x_{p''}) + \eta_1 R_n \|z_p z_q\| - \eta_2 \varepsilon_n R_n^2,
\]

Thus, the inequality (4.14) also holds if \( \chi_{\tau_n}(\tau(x)) = 1 \). Note that for any edge \( e \) of \( \tau(x) \),

\[
    g_n(e) \leq 2e^{-n\nu_{\tau_n}(e)}.
\]

Thus,

\[
    g_n(\tau(x)) = \prod_{e \in E(\tau(x))} g_n(e) \leq 2^k \prod_{e \in E(\tau(x))} e^{-n\nu_{\tau_n}(e)} = 2^k e^{-n \sum_{e \in E(\tau(x))} \nu_{\tau_n}(e)}.
\]

By Lemma 11 in Chapter 2, the inequality (4.15) holds. Therefore, we only need to show that

\[
    n^m \int_{S_{pq}} e^{-n(\nu_{\tau_n}(x_{p'}x_{p''}) + \eta_1 R_n \|z_p z_q\| - \eta_2 \varepsilon_n R_n^2)} dx = o(1).
\]

We change the integral variables \( x_1, \ldots, x_m \) as follows. For the tree component
containing $e_p$, we replace the $x_i$’s in this tree by the midpoints of the edges in this tree except $z_p$ and $x_p'$, $x_p''$ (both of which are kept). For any other tree component, we use the method introduced at the beginning of this section: pick an arbitrary edge as the rooted edge. We replace $x_i$’s in this tree by the midpoints of all the edges in this tree together with the half-length and slope of the root edge. Such change of integration variables yields

$$
\eta^m \int_{S_{pq}} e^{-n(u_{pn}(x_p'x_p'')+c_1R_n\|z_pz_q\|-c_2\varepsilon_nR_n^2)}d\mathbf{x} \\
\leq O(1) e^{c_2n\varepsilon_nR_n^2m} \left( \int_{\mathcal{R}^2} e^{-c_1nR_n\|z_pz_q\|}dz_q \right) \\
\cdot \left( \int_{\mathcal{R}^2} \frac{\rho d\rho}{\mathcal{D}(z_p,\|z_pz_q\|)} \right)^{k-2} \\
\sim O(1) (\ln n)^{\frac{c_2\varepsilon_n}{\ln n}} n^{m-2} (R_n^2 - r_n^2)^{m-k-1} \\
\cdot \int_{\mathcal{R}^2} e^{-c_1nR_n\|z_pz_q\|} \|z_pz_q\|^{2(k-2)} dz_q \\
\leq O(1) (\ln n)^{\frac{c_2\varepsilon_n}{\ln n}} n^{m-2} (R_n^2 - r_n^2)^{m-k-1} \\
\cdot \int_0^\infty e^{-c_1nR_n\mu} \mu^{2k-3} d\mu \\
= O(1) (\ln n)^{\frac{c_2\varepsilon_n}{\ln n}} n^{m-2} (R_n^2 - r_n^2)^{m-k-1} \\
\frac{1}{(nR_n)^{2(k-1)}} \\
= O(1) (\ln n)^{\frac{c_2\varepsilon_n}{\ln n}} (nR_n^2 - nr_n^2)^{m-k-1} \frac{1}{(nR_n)^{k-1}} \\
= O(1) (\ln n)^{\frac{c_2\varepsilon_n}{\ln n}} \left( \frac{c_n}{c} \right)^{m-k-1} \frac{1}{(\ln n)^{k-1}} \\
\sim O(1) (\ln n)^{\frac{c_2\varepsilon_n}{\ln n}} \left( \frac{c_n}{c} \right)^{m-k-1} \frac{1}{(\ln n)^{k-1}} \\
= O(1) \left( \frac{c_n}{c} \right)^{m-k-1} \frac{1}{(\ln n)^{k-1}} = o(1),
$$
where the asymptotic equality follows from Lemma 39, and the last equality follows from $c_n = o\left(\ln \ln n\right)$ and $k \geq 2$.

**Lemma 50** Suppose that $2 < m \leq 2k$ and $\tau$ is a forest in $T_m$ with $k$ edges. For any integer $2 \leq l \leq \min\{m-k, k-1\}$,

$$n^m \int_{\Gamma_l(\tau)} f_n(\tau(x)) \, dx = o(1),$$

$$n^m \int_{\Gamma_l(\tau)} g_n(\tau(x)) \, dx = o(1).$$

**Proof** $\tau$ has $m-k$ tree components on $m$ numbered vertices $v_1, v_2, \cdots, v_m$. Enumerate them arbitrarily by $T_1, \cdots, T_{m-k}$. For each $1 \leq q \leq m-k$, let

$$I_q = \{1 \leq i \leq m : v_i \text{ is a vertex of } T_q\}.$$

Fix an integer $2 \leq l \leq \min\{m-k, k-1\}$. Consider any nontrivial $l$-partition $\Pi = \{Q_1, Q_2, \cdots, Q_l\}$ of $\{1, 2, \cdots, m-k\}$. It induces a partition $\Pi' = \{P_1, P_2, \cdots, P_l\}$ of $\{1, 2, \cdots, m\}$ in which $P_j = \bigcup_{q \in Q_j} I_q$ for each $1 \leq j \leq l$. Let $S(\Pi)$ denote the set of $x \in \Gamma_l(\tau)$ such that for each $1 \leq j \leq l$, the set of midpoints of the subgraph of $\tau(x)$ induced by $\{x_i : i \in P_j\}$ is a connected component of the $\sqrt{3}R_n$-disk graph on the midpoints of the edges in $\tau(x)$. Then $\Gamma_l(\tau)$ is the union of $S(\Pi)$ over all nontrivial $l$-partitions $\Pi$ of $\{1, 2, \cdots, m-k\}$. So, it is sufficient to show that for any $l$-partition $\Pi$ of $\{1, 2, \cdots, k\}$,

$$n^m \int_{S(\Pi)} f_n(\tau(x)) \, dx = o(1),$$

$$n^m \int_{S(\Pi)} g_n(\tau(x)) \, dx = o(1).$$
Now, fix an \( l \)-partition \( \Pi = \{Q_1, Q_2, \cdots, Q_l\} \) of \( \{1, 2, \cdots, m - k\} \). Let \( \Pi' = \{P_1, P_2, \cdots, P_l\} \) be the partition of \( \{1, 2, \cdots, m\} \) induced by \( \Pi \). For each \( 1 \leq j \leq l \), let \( m_j = \text{card}(P_j) \), \( \tau_j \) be the topology on \( m_j \) numbered vertices which is a subgraph of \( \tau \) induced by the subset of vertices \( \{v_i : i \in P_j\} \). Then, at least one \( m_j \geq 2 \). For any \( x = (x_1, x_2, \cdots, x_m) \in \mathbb{D}^m \) and each \( 1 \leq j \leq l \), let

\[
x^{(j)} = (x_{i_1}, x_{i_2}, \cdots, x_{i_{m_j}})
\]

where \( i_1, i_2, \cdots, i_{m_j} \) are the \( m_j \) indices in \( P_j \) in the increasing order. Clearly, for each \( x \in S(\Pi) \) and each \( 1 \leq j \leq l \), \( x^{(j)} \in \Gamma_1(\tau_j) \). Hence,

\[
S(\Pi) \subseteq \{ x \in \mathbb{D}^m : x^{(j)} \in \Gamma_1(\tau_j), 1 \leq j \leq l \}.
\]

For any \( x \in S(\Pi) \),

\[
\nu_{r_n}(\tau(x)) = \sum_{j=1}^{l} \nu_{r_n}(\tau_j(x^{(j)})),
\]

\[
\chi_{r_n}(\tau(x)) \leq \prod_{j=1}^{l} \chi_{r_n}(\tau_j(x^{(j)})),
\]

which imply

\[
f_n(\tau(x)) \leq \prod_{j=1}^{l} f_n(\tau_j(x^{(j)})).
\]

It’s obvious that for any \( x \in S(\Pi) \),

\[
g_n(\tau(x)) = \prod_{j=1}^{l} g_n(\tau_j(x^{(j)})).
\]
Thus,

\[ n^m \int_{S(\Pi)} f_n(\tau(x)) \, dx \]
\[ \leq n^m \int_{S(\Pi)} \prod_{j=1}^l f_n(\tau_j(x^{(j)})) \, dx \]
\[ \leq n^m \int_{\{x \in \mathbb{R}^m : x^{(j)} \in \Gamma_1(\tau_j), 1 \leq j \leq l\}} \prod_{j=1}^l f_n(\tau_j(x^{(j)})) \, dx \]
\[ = \prod_{j=1}^l \left( n^{m_j} \int_{\Gamma_1(\tau_j)} f_n(\tau_j(x^{(j)})) \, dx^{(j)} \right) = o(1), \]

where the last equality follows from Lemma 39, Lemma 49 and the fact that at least one \( m_j \geq 2 \). Similarly, we can show that

\[ n^m \int_{S(\Pi)} g_n(\tau(x)) \, dx = o(1). \]

So, the lemma follows.

4.5 Conclusion

Greedy forward routing is a localized and memoryless geographic routing. However, it cannot guarantee the delivery of a packet from its source to its destination if the transmission of the nodes are not large enough. The smallest transmission radius which ensures the successful delivery of any packet is referred to as the critical transmission radius \( \phi_n \). In this chapter, we derived the precise asymptotic probability distribution of \( \phi_n \) when the networking nodes are represented by a Poisson point process over a unit-area disk with mean \( n \). We showed that the asymptotic probability of the event \( \phi_n \leq \sqrt{\frac{\ln n + c}{n}} \) is equal to \( e^{-\mu} \), where \( c \) is a constant and \( \mu = \left( \frac{1}{\pi^{1/3} - 1/3} - \frac{\pi}{252} \right) e^{-c}. \)
BIBLIOGRAPHY


