Maximum-Weighted λ-Colorable Subgraph: Revisiting And Applications

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Abstract. Given a vertex-weighted graph G and a positive integer λ, a subset F of the vertices is said to be λ-colorable if F can be partitioned into at most λ independent subsets. This problem of seeking a λ-colorable F with maximum total weight is known as **Maximum-Weighted λ-Colorable Subgraph (λ-MWCS)**. This problem is a generalization of the classical problem **Maximum-Weighted Independent Set** (**MWIS**) and has broader applications in wireless networks. All existing approximation algorithms for **λ-MWCS** have approximation bound strictly increasing with λ. It remains open whether the problem can be approximated with the same factor as the problem **MWIS**. In this paper, we present new approximation algorithms for **λ-MWCS**. For certain range of λ, the approximation bounds of our algorithms are the same as those for **MWIS**, and for a larger range of λ, the approximation bounds of our algorithms are strictly smaller than the best-known ones in the literature. In addition, we give an exact polynomial-time algorithm for **λ-MWCS** in co-comparability graphs. We also present a number of applications of our algorithms in wireless networking.
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1 Introduction

Consider an undirected graph $G = (V, E)$ and a positive integer λ. A subset I of V is said to be *independent* if any pair of vertices in I are non-adjacent. A subset F of the vertices is said to be λ-*colorable* if F can be partitioned into at most λ independent subsets (or equivalently, the subgraph of G induced by F is λ-colorable). Suppose that each vertex v has a positive weight $w(v)$. The weight of each subset $F \subseteq V$ is defined to be $w(F) := \sum_{v \in F} w(v)$. The problem of seeking a λ-colorable subset F of V with maximum weight is known as **Maximum-Weighted λ-Colorable Subgraph (λ-MWCS)** [3, 12, 13]. The special case of this problem with $\lambda = 1$ is the classical problem **Maximum-Weighted Independent Set** (**MWIS**). In general, the problem **λ-MWCS** may be strictly harder than **MWIS**. Indeed, when restricted the class of split graphs, Yannakakis and Gavril [12] showed that this problem is NP-hard, but the problem **MWIS** can be solved in polynomial time. On the other hand, by a result in [3], the existence of a μ-approximation algorithm for **MWIS** always implies a greedy $1/\left[1-(1-1/(\mu\lambda))^{\lambda}\right]$-approximation algorithm for **λ-MWCS**, which repeatedly takes an μ-approximate weighted independent set in the remaining graph for λ times. Note that the approximation bound $1/\left[1-(1-1/(\lambda\mu))^{\lambda}\right]$ strictly increases with λ.

Various approximation algorithms for **MWIS** have been developed in [1, 9, 13]. For each $v \in V$, $N(v)$ denotes the set of neighbors of v in G. Let $\prec$ be an ordering of V. For any $u, v \in V$, both $v \prec u$ and $u \succ v$ represent that v appears before u in the ordering $\prec$. For any $v \in V$ and any $U \subseteq V$, we use $U_{\prec v}$ (respectively, $U_{\succ v}$) to denote the set of $u \in U$ satisfying that $u \prec v$ (respectively, $u \succ v$); in addition, $U_{\preceq v}$ denotes $\{v\} \cup U_{\prec v}$, and $U_{\succeq v}$ denotes $\{v\} \cup U_{\succ v}$. The *backward local independence number* (BLIN) of G in $\prec$ is defined to be the maximum number of non-adjacent vertices in $N(v) \cap V_{\prec v}$ for all $v \in V$. An *orientation* of G is a digraph obtained from G by imposing an orientation on each edge of G. Suppose that D is an orientation of G. For each $v \in V$, $N_D^{in}(v)$ (resp., $N_D^{out}(v)$) denotes the set of in-neighbors (resp., out-neighbors) of v in D. The *inward local independence number* (ILIN) of D is defined to be the maximum number of non-adjacent vertices in $N_D^{in}(v)$ for all $v \in V$. Then, the following algorithmic results have be established:

- Given an ordering of V with BLIN β, there is a β-approximation algorithm for **MWIS** [1, 13].
- Given an orientation of G with ILIN γ, there is a 2γ-approximation algorithm for **MWIS** [9].

By plugging the above approximation algorithms for MWIS into the greedy approximation framework for **λ-MWCS** proposed in [3], the following algorithmic results can be obtained:

- Given an ordering of V with BLIN β, there is an $1/\left[1-(1-1/(\beta\lambda))^{\lambda}\right]$-approximation algorithm for **λ-MWCS**.
- Given an orientation of G with ILIN γ, there is an $1/\left[1-(1-1/(2\gamma\lambda))^{\lambda}\right]$-approximation algorithm for **λ-MWCS**.

Moreover, given an ordering with BLIN β, Ye and Borodin [13] extends β-approximation algorithm for **MWIS** directly to a$\left(\beta+1-\frac{1}{\lambda}\right)$-approximation algorithm for **λ-MWCS**. While the approximation bound is relatively larger, the algorithm is simpler and more efficient in implementation. All those approximation bounds are greater than the respective approximation bounds for **MWIS**. A natural open question is whether and when the same approximation bound for **MWIS** can be achieved for **λ-MWCS**. A more general open question is whether and when a better approximation bound may be achieved than those best-known approximation bounds.

Motivated by the above two open questions, this paper develops two new approximation algorithms for λ-**MWCS**.

- Given an ordering of V with BLIN β, the algorithm developed in Section 3 is not only simpler than that proposed in [13], but also achieves a strictly better approximation bound
$$\max\left\{\beta, \left(1-\frac{1}{\lambda}\right)\beta+1\right\}.$$
In particular, if $\lambda \leq \beta$, the approximation bound is β, which is also the best-known approximation bound for the problem **MWIS.** If $\lambda < 2\beta$, this approximation bound is strictly smaller than the best-known approximation bound $1/\left[1-(1-1/(\beta\lambda))^{\lambda}\right]$.
- Given an orientation with ILIN γ, the algorithm developed in Section 4 achieve an approximation bound
$$\max\left\{2\gamma, 2\gamma\left(1-\frac{1}{\lambda}\right)+1\right\}.$$
Again if $\lambda \leq 2\gamma$, the approximation bound is 2γ, which is also the best-known approximation bound for the problem **MWIS.** If $\lambda < 4\gamma$, this approximation bound is strictly smaller than the best-known approximation bound $1/\left[1-(1-1/(2\gamma\lambda))^{\lambda}\right]$.

The above two algorithms are not only of theoretical interest, but have significant implications for the practical applications where λ is not too large compared to the BLIN or ILIN.

An ordering $\prec$ of V is said to be *cocomparable* if the following transitivity of independence is satisfied: for any triple of vertices $v \prec v' \prec v''$ with $vv' \notin E$ and $v'v'' \notin E$, we have $vv'' \notin E$. If there is a cocomparable ordering of V, then G is called a *cocomparability graph*. When restricted to comparability graphs, the problem **MWIS** is solvable in polynomial time (e.g., [9]), and consequently the greedy approximation framework for λ-**MWCS** proposed in [3] can achieve an approximation bound $1/\left[1-(1-1/(\lambda))^{\lambda}\right]$. In Section 5, we give an exact polynomial-time algorithm for λ-**MWCS** in cocomparability graphs. This algorithm generalizes the classic result by Frank [6] on the unweighted variant of λ-**MWCS** in cocomparability graphs.

The remainder of this paper is organized as follows. In Section 2 we introduce two basic algorithmic ingredients to be used in the two subsequent sections. In Section 3, we develop an ordering-based selection algorithm. In Section 4, we present an orientation-based selection algorithm. In Section 5, we give an exact polynomial-time algorithm when the graph G is a co-comparability graphs. In Section 5, we apply these algorithms to the problem of seeking a maximum-weighted wireless communication requests which can transmit at the same time over multiple channels.

2 Preliminaries

Motivated by the NP-completeness of the feasibility test, we introduce a tractable and strong type of feasibility. Let $\prec$ be an ordering of V. A set $F \subseteq V$ is said to be *inductively feasible* in $\prec$ if for each $v \in F$, $|N(v) \cap F_{\prec v}| < \lambda$. An inductively feasible set F is always λ-colorable; indeed, a coloring of F can be greedily produced as follows:

- The first vertex v in F receives the first color.
- For each subsequent vertex $v \in F$ in the ordering $\prec$, it receives the first color which is not used by any vertex in $N(v) \cap F_{\prec v}$. This is always possible because the number of colors that have been assigned to $N(v) \cap F_{\prec v}$ is at most $|N(v) \cap F_{\prec v}| < \lambda$.

Such coloring is referred to as the *greedy coloring* of F in $\prec$.

For any subset $S \subseteq V$, an inductively feasible subset F of S can be computed as follows:

- Initially, F is empty.
- For each $v \in S$ in the ordering $\prec$, v is added to F if and only if $|N(v) \cap F_{\prec v}| < \lambda$.

The set F computed in this greedy manner is referred to as the *maximal inductively feasible subset* of S in $\prec$. It is maximal in the sense that for each $v \in S \setminus F$, $|N(v) \cap F_{\prec v}| \geq \lambda$. However, how to select the "candidate" subset S is very essential. In fact, when $S = V$ or $S = \emptyset$ at the two opposite extremes, the maximal inductively feasible subset of S in $\prec$ is almost surely to have poor performance in general. This motivates us to utilize variants of the local-ratio scheme [1–3, 5, 9, 13], which is equivalent to the primal-dual scheme [4], for selecting candidate set S properly.

Given an ordering $\prec$ of V, the local-ratio scheme computes a candidate set S in the following *greedy* manner:

- S is initially empty.
- For each $v \in V$ in the *reverse* order of $\prec$, a *discounted* weight $\overline{w}(v)$ of v is computed by
$$\overline{w}(v) = w(v) - \frac{1}{\lambda}\overline{w}(N(v) \cap S);$$
and if $\overline{w}(v) > 0$, v is added to S.

The final set S computed in this greedy manner is referred to as the *greedy candidate subset* of V in $\prec$. For each $v \in V$, its original weight and its discounted weight have the following relation:

$$w(v) = \overline{w}(v) + \frac{1}{\lambda}\overline{w}(N(v) \cap S_{\succ v}).$$

In addition, the maximal inductively compatible subset F of S in $\prec$ has the following property

Lemma 1. *The maximal inductively feasible subset F of S in $\prec$ satisfies that* $w(F) \geq \overline{w}(S)$.

Proof. By the greedy selection of F, for each $v \in S \setminus F$, $|N(v) \cap F_{\prec v}| \geq \lambda$. Thus,

$$\begin{aligned} w(F) &= \sum_{v \in F} w(v) = \sum_{v \in F} \overline{w}(v) + \frac{1}{\lambda} \sum_{v \in F} \overline{w}(N(v) \cap S_{\succ v}) \\ &= \sum_{v \in F} \overline{w}(v) + \frac{1}{\lambda} \sum_{v \in F} \sum_{u \in N(v) \cap S_{\succ v}} \overline{w}(u) = \overline{w}(F) + \frac{1}{\lambda} \sum_{v \in S} \overline{w}(v) |N(v) \cap F_{\prec v}| \\ &\geq \overline{w}(F) + \frac{1}{\lambda} \sum_{v \in S \setminus F} \overline{w}(v) |N(v) \cap F_{\prec v}| \geq \overline{w}(F) + \sum_{v \in S \setminus F} \overline{w}(v) = \overline{w}(S). \end{aligned}$$

So, the lemma holds.

Next, we proceed to compute an ordering $\prec$ such that the discounted weight of the greedy candidate subset S in $\prec$ is close to the original weight of an optimal solution.

3 Ordering-Based Selection

Suppose that $\prec$ is an ordering of V with BLIN β. Denote

$$\beta_\lambda := \max\left\{\beta, \beta\left(1 - \frac{1}{\lambda}\right) + 1\right\}.$$

Then, the discounted weight of the greedy candidate subset S in $\prec$ and the weight of a maximum-weight feasible subset O are related as follows.

Lemma 2. *The greedy candidate subset S of V in $\prec$ satisfies that* $\overline{w}(S) \geq w(O)/\beta_\lambda$.

Proof. Consider any $v \in V$. We show that

$$|O \cap \{v\}| + \frac{1}{\lambda} |N(v) \cap O_{\prec v}| \leq \beta_\lambda.$$

We consider two cases.

Case 1: $v \notin O$. Then

$$|O \cap \{v\}| + \frac{1}{\lambda} |N(v) \cap O_{\prec v}| = \frac{1}{\lambda} |N(v) \cap O_{\prec v}| \leq \frac{1}{\lambda} \lambda \beta = \beta \leq \beta_\lambda.$$

Case 2: $v \in O$. Then

$$\begin{aligned} &|O \cap \{v\}| + \frac{1}{\lambda} |N(v) \cap O_{\prec v}| \\ &= 1 + \frac{1}{\lambda} |N(v) \cap O_{\prec v}| \leq 1 + \frac{1}{\lambda} (\lambda - 1) \beta = \beta\left(1 - \frac{1}{\lambda}\right) + 1 \leq \beta_\lambda. \end{aligned}$$

Using the relations between the original weights and the discounted weights, we have

$$\begin{aligned}
w(O) &= \sum_{v\in O} w(v) \\
&= \sum_{v\in O} \overline{w}(v) + \frac{1}{\lambda}\sum_{v\in O} \overline{w}\left(N(v)\cap S_{\succ v}\right) \\
&\geq \sum_{v\in S\cap O} \overline{w}(v) + \frac{1}{\lambda}\sum_{v\in O} \overline{w}\left(N(v)\cap S_{\succ v}\right) \\
&= \sum_{v\in S\cap O} \overline{w}(v) + \frac{1}{\lambda}\sum_{u\in O} \overline{w}\left(N(u)\cap S_{\succ v}\right) \\
&= \sum_{v\in S\cap O} \overline{w}(v) + \frac{1}{\lambda}\sum_{v\in S} \overline{w}(v)\left|N(v)\cap O_{\prec v}\right| \\
&= \sum_{v\in S} \overline{w}(v)\left[\left|O\cap\{v\}\right| + \frac{1}{\lambda}\left|N(v)\cap O_{\prec v}\right|\right] \\
&\leq \beta_\lambda \sum_{v\in S} \overline{w}(v) = \beta_\lambda \overline{w}(S).
\end{aligned}$$

So, the lemma holds.

Motivated by the above lemma and Lemma 1, we propose the following ordering-based approximation algorithm, which runs in three steps:

1. Compute the greedy candidate subset S of V in $\prec$.
2. Compute the maximal inductively feasible subset F of S in $\prec$.
3. Compute the greedy coloring of F in $\prec$.

All the three steps are greedy in nature and have very simple implementations. The approximation bound of this algorithm follows immediately from Lemma 1 and Lemma 2.

Theorem 1. *F has an approximate bound β_λ.*

Note that when $\lambda \leq \beta$, $\beta_\lambda = \beta$. For $\lambda > \beta$, $\beta_\lambda = \beta\left(1-\frac{1}{\lambda}\right)+1$ and we compare it against the best-known approximation bound. The comparison is a based on the following algebraic inequalities.

Lemma 3. *For any positive integer μ, if $1 < \lambda < 2\mu$, then*

$$\left[1-\left(1-\frac{1}{\mu\lambda}\right)^{\lambda}\right]^{-1} > \mu\left(1-\frac{1}{\lambda}\right)+1;$$

if $\lambda \geq 2\mu$ then

$$\left[1-\left(1-\frac{1}{\mu\lambda}\right)^{\lambda}\right]^{-1} < \mu\left(1-\frac{1}{\lambda}\right)+1.$$

The proof of the above lemma is quite lengthy, and is omitted in this paper. From the above lemma, we conclude that when $\lambda < 2\mu$, β_λ is smaller than the best-known approximation bound $\left[1-\left(1-\frac{1}{\beta\lambda}\right)^{\lambda}\right]^{-1}$.

4 Orientation-Based Selection

Suppose that D is an orientation of G with ILIN γ. In this section, we present an approximation algorithm for **λ-MWCS** with approximation bound

$$(2\gamma)_\lambda := \max\left\{2\gamma, 2\gamma\left(1-\frac{1}{\lambda}\right)+1\right\}.$$

Consider a non-zero "opportunistic" vector $x \in [0,1]^V$. For any non-empty subset U of V, a vertex $v \in U$ is said to be a surplus vertex in U with respect to x and D if

$$x\left(N_D^{in}(v) \cap U\right) \geq x\left(N_D^{out}(v) \cap U\right).$$

It was proved in [8] that there exists at least one a surplus vertex in U with respect to x and D. Based on this fact, an ordering of V can be computed by a simple *greedy* strategy as follows.

- Initialize U to V.
- For $i = |V|$ down to 1, let v_i be a surplus bid in U with respect to x and and delete v_i from U.

Then, the ordering $\langle v_1, v_2, \cdots, v_{|V|}\rangle$ is referred to as a *surplus-preserving ordering* of V with respect to x and D. It has the following property.

Lemma 4. *Suppose that $\prec$ is a surplus-preserving ordering of V with respect to D and some non-zero $x \in [0,1]^V$. Then, the greedy candidate subset S in $\prec$ satisfies that*

$$\overline{w}(S) \geq \frac{\sum_{v\in V} w(v)\,x(v)}{\max_{v\in V}\left(x(v)+\frac{2}{\lambda}x\left(N_D^{in}(v)\right)\right)}.$$

Proof. Since $\prec$ is a surplus-preserving ordering of V with respect to x and D, for any $v \in V$ we have

$$x\left(N(v) \cap V_{\prec v}\right) \leq 2x\left(N_D^{in}(v) \cap V_{\prec v}\right) \leq 2x\left(N_D^{in}(v)\right).$$

Using the relations between the original weights and the discounted weights, we have

$$
\begin{aligned}
& \sum_{u \in V} w(v) x(v) \\
&= \sum_{v \in V} \overline{w}(v) x(v) + \frac{1}{\lambda} \sum_{v \in V} x(v) \sum_{u \in N(v) \cap S_{\succ v}} \overline{w}(u) \\
&\leq \sum_{v \in S} \overline{w}(v) x(v) + \frac{1}{\lambda} \sum_{v \in V} x(v) \sum_{u \in N(v) \cap S_{\succ v}} \overline{w}(u) \\
&= \sum_{v \in S} \overline{w}(v) x(v) + \frac{1}{\lambda} \sum_{u \in S} \overline{w}(u) \sum_{v \in N(u) \cap V_{\prec u}} x(v) \\
&= \sum_{v \in S} \overline{w}(v) x(v) + \sum_{v \in S} \overline{w}(v) x(N(v) \cap V_{\prec v}) \\
&= \sum_{v \in S} \overline{w}(v) \left(x(v) + \frac{1}{\lambda} x(N(v) \cap V_{\prec v}) \right) \\
&\leq \sum_{v \in S} \overline{w}(v) \left(x(v) + \frac{2}{\lambda} x\left(N_D^{in}(v)\right) \right) \\
&\leq \max_{v \in V} \left(x(v) + \frac{2}{\lambda} x\left(N_D^{in}(v)\right) \right) \sum_{v \in S} \overline{w}(v) \\
&= \overline{w}(S) \max_{v \in V} \left(x(v) + \frac{2}{\lambda} x\left(N_D^{in}(v)\right) \right).
\end{aligned}
$$

Therefore, the lemma holds.

Note that a non-zero $x \in [0,1]^V$ maximizing the value

$$
\frac{\sum_{v \in V} w(v) x(v)}{\max_{v \in V} \left(x(v) + \frac{2}{\lambda} x\left(N_D^{in}(v)\right) \right)}
$$

is achieved by an optimal solution to the following linear program (LP):

$$
\begin{array}{ll}
\max & \sum_{v \in V} w(v) x(v) \\
s.t. & x(v) + \frac{2}{\lambda} x\left(N_D^{in}(v)\right) \leq 1, \forall v \in V \\
 & x(v) \geq 0, \forall v \in V
\end{array}
\tag{1}
$$

The value of the above LP and the weight of a maximum-weight feasible subset O are related as follows.

Lemma 5. *The value of the LP in equation (1) is at least* $w(O) / (2\gamma)_\lambda$.

Proof. Consider any link $v \in V$. We show that

$$
|\{v\} \cap O| + \frac{2}{\lambda} \left| N_D^{in}(v) \cap O \right| \leq (2\gamma)_\lambda .
$$

We consider two cases.

Case 1: $v \notin O$. Then

$$|O \cap \{v\}| + \frac{2}{\lambda} \left|N_D^{in}(v) \cap O\right| = \frac{2}{\lambda} \left|N_D^{in}(v) \cap O\right| \leq \frac{2}{\lambda}\lambda\gamma \leq 2\gamma \leq (2\gamma)_\lambda .$$

Case 2: $v \in O$. Then

$$|O \cap \{v\}| + \frac{2}{\lambda} \left|N_D^{in}(v) \cap O\right| = 1 + \frac{2}{\lambda}(\lambda - 1)\gamma = 2\left(1 - \frac{1}{\lambda}\right)\gamma + 1 \leq (2\gamma)_\lambda .$$

Let y be the function on V defined by

$$y(v) = \begin{cases} \frac{1}{(2\gamma)_\lambda}, & \text{if } v \in O; \\ 0, & \text{if } v \notin O. \end{cases}$$

Then, for each $v \in V$,

$$\begin{aligned} & x(v) + \frac{2}{\lambda} x\left(N_D^{in}(v)\right) \\ = & \frac{1}{(2\gamma)_\lambda}\left[|\{v\} \cap O| + \frac{2}{\lambda}\left|N_D^{in}(v) \cap O\right|\right] \leq \frac{1}{(2\gamma)_\lambda}(2\gamma)_\lambda = 1. \end{aligned}$$

Thus, y is a feasible solution to the LP in equation (1), and consequently the value of the LP in equation (1) is at least

$$\sum_{v \in V} w(v)\, y(v) = \frac{1}{(2\gamma)_\lambda} \sum_{v \in O} w(v) = \frac{1}{(2\gamma)_\lambda} w(O) .$$

So, the lemma holds.

The above two lemmas together with Lemma 1 motivates us to propose the following orientation-based approximation algorithm, which runs in five steps

1. Compute an optimal solution x to the LP in equation (1).
2. Compute a surplus-preserving ordering $\prec$ of V with respect to D and x.
3. Compute the greedy candidate subset S of V in $\prec$.
4. Compute the maximal inductively feasible subset F of S in $\prec$.
5. Compute the greedy coloring of F in $\prec$.

The approximation bound of the output F follows immediately from the above two lemmas and Lemma 1.

Theorem 2. *The F has an approximation bound $(2\gamma)_\lambda$.*

Again when $\lambda \leq 2\gamma$, $(2\gamma)_\lambda = 2\gamma$. For $\lambda > 2\gamma$,

$$(2\gamma)_\lambda = 2\gamma\left(1 - \frac{1}{\lambda}\right) + 1.$$

By Lemma 3, when $\lambda < 4\gamma$, $(2\gamma)_\lambda$ is smaller than the best-known approximation bound $\left[1 - \left(1 - \frac{1}{2\gamma\lambda}\right)^\lambda\right]^{-1}$.

5 Exact Algorithm in Cocomparability Graphs

Suppose that $G = (V, E)$ is cocomparability graph with a cocomparability ordering $\langle v_1, v_2, \cdots, v_n \rangle$ of its vertices. In this section, we present a polynomial-time algorithm which computes a maximum-weighted λ-colorable subset F of V by a reduction to minimum-cost flow.

We first construct a flow network D. Each vertex $v_i \in V$ is replaced by two replicas x_i and y_i. Then the vertex set of D consists of all these $2n$ replicas, a source vertex s, and a sink vertex t. The arc set of D consists of the $3n$ arcs

$$\{(x_i, y_i) : 1 \leq i \leq n\} \cup \{(s, x_i) : 1 \leq i \leq n\} \cup \{(y_i, t) : 1 \leq i \leq n\}$$

and the $n(n-1)/2 - |E|$ arcs

$$\{(y_i, x_j) : 1 \leq i < j \leq n, v_i v_j \notin E\}.$$

All arcs have unit capacity. In addition, each arc (x_i, y_i) has a cost $-w(v_i)$, and each other arc has zero cost. There is an one-to-one correspondence between the s-t paths in D and the independent sets in G:

- For each path P, let I be the set of vertices in V who replications appear in the path P. Then, I is independent, and is referred to the independent set *induced by* P. In addition, the length (i.e., cost) of the path P is equal to $-w(I)$.
- For each independent set I in G sorted in the cocomparable order $v_{i_1}, v_{i_2}, \cdots, v_{i_l}$, the sequence of vertices $s, x_{i_1}, y_{i_1}, x_{i_2}, y_{i_2}, \cdots, x_{i_l}, y_{i_l}, t$ form an s-t path P in D. Then, the length (i.e., cost) of P is equal to $-w(I)$.

Since all arcs in D have unit capacity, the above correspondence implies that the minimum cost of all s-t flows of value at most λ in D is equal to the additive inverse of the maximum weight of all λ-colorable subsets of vertices in G. Based on this relation, a maximum-weighted λ-colorable subset of vertices in G can be computed as follows.

- Compute an integral minimum-cost flow s–t flow f of value at most k in D, and decompose f into s–t paths flows using the standard flow decomposition method. Since each arc has unit capacity, each path in the path flow decomposition carries exactly one unit of flow. Thus, the number of paths is at most k.
- For each path P, let I be the independent set induced by P. The collection of all these at most k independent sets is returned as the output.

6 Applications

Consider a set A of point-to-point wireless communication requests. All requests in A are assumed to be node-disjoint and can access a common set of λ channels. A subset of A is said to be feasible if they can transmit at the same time over the

λ channels. Suppose each request has a positive weight. We would like to select a maximum weighted feasible subset of A. Under a protocol interference model, the conflict relations among A is represented by a graph G on A, in which there is an edge two requests a and b in A if and only if they have conflict. Then, a subset of A is feasible if and only if it is λ-colorable in G.

The protocol interference model is classified into two communication modes:

- Unidirectional mode: For each request $a \in A$, the communication occurs in a single direction from its sender to its receiver, and the sender has an interference range, and the interference range of a is the interference range of its sender. Two requests in A conflict with each other if and only if the receiver of at least one request lies in the interference range of the other.
- Bidirectional mode: For each request $a \in A$, the communication occurs in both directions between its two endpoints, and each of its endpoint has an interference range. The interference range of a is the union of the interference ranges of its two endpoints. Two requests in A conflict with each other if and only if at least one request has an endpoint lying in the interference range of the other.

In the plane geometric variant, the interference range of an endpoint u of a request a is assumed to be a disk centered at u, whose radius is also knows as the interference radius. The following special orientations of the conflict graph and orderings of the requests have been discovered in the literature:

- Unidirectional mode: An orientation of the conflict graph introduced in [8] has ILIN at most
$$\left\lceil \pi / \arcsin \frac{c-1}{2c} \right\rceil - 1$$
under the assumption that the interference radius of each request is at least c times the distance between its sender and its receiver for some constant $c > 1$.
- Bidirectional mode: An orientation of the conflict graph defined in [10] has ILIN at most 8, and an ordering of the requests given in [8] has BLIN at most 23. In case of symmetric interference radii (i.e, the two endpoints of each request have equal interference radii), an ordering of the requests introduced in [10] has BLIN at most 8. In the bidirectional mode with uniform interference radii (i.e, all endpoints of all requests have equal interference radii), an ordering of the requests described in [7] has BLIN at most 6.

By adopting those orientations of the conflict graphs or the orderings of the requests, the approximation algorithms developed in Section 4 and Section 3 achieve constant approximation bounds when applied to the conflict graph G. The derivation of these approximation bounds are straightforward and are omitted in this paper. In addition, it is also easy to identify the range of λ over which these approximation bounds do not increase with λ, or are smaller than those achieved by the greedy approximation framework proposed in [3].

When all links have uniform interference radii, the links can be partitioned into a small constant number μ of groups such that the conflict graph of each group is a co-comparability graph [11]. By applying the algorithm presented in Section 5 to the conflict graph of each group, a maximum-weighted feasible subset of each group can be computed in polynomial time. Then, among those μ feasible subsets, the one with the largest weight is returned as the output. Such divide-and-conquer scheme achieves the approximation bound μ, which does not depend on the number λ of channels at all.

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