Advanced Counting Techniques

Chapter 8

Chapter Summary

- Applications of Recurrence Relations
- Solving Linear Recurrence Relations
  - Homogeneous Recurrence Relations
  - Nonhomogeneous Recurrence Relations
- Divide-and-Conquer Algorithms and Recurrence Relations
- Generating Functions
- Inclusion-Exclusion
- Applications of Inclusion-Exclusion

Applications of Recurrence Relations

Section 8.1

Recurrence Relations
(recalling definitions from Chapter 2)

Definition: A *recurrence relation* for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer.

- A sequence is called a *solution* of a recurrence relation if its terms satisfy the recurrence relation.
- The *initial conditions* for a sequence specify the terms that precede the first term where the recurrence relation takes effect.

Rabbits and the Fiobonacci Numbers

Example: A young pair of rabbits (one of each gender) is placed on an island. A pair of rabbits does not breed until they are 2 months old. After they are 2 months old, each pair of rabbits produces another pair each month. Find a recurrence relation for the number of pairs of rabbits on the island after n months, assuming that rabbits never die.

This is the original problem considered by Leonardo Pisano (Fibonacci) in the thirteenth century.

Rabbits and the Fiobonacci Numbers (*cont.*)

Reproducing pairs (at least two months old)	Young pairs (less than two months old)	Month	Reproducing pairs	Young pairs	Total pairs
		1	0	1	1
		2	0	1	1
		3	1	1	2
		4	1	2	3
		5	2	3	5
		6	3	5	8

Modeling the Population Growth of Rabbits on an Island

Rabbits and the Fibonacci Numbers (*cont.*)

Solution: Let f_n be the the number of pairs of rabbits after n months.

- There are is $f_1 = 1$ pairs of rabbits on the island at the end of the first month.
- We also have $f_2 = 1$ because the pair does not breed during the first month.
- To find the number of pairs on the island after n months, add the number on the island after the previous month, f_{n-1}, and the number of newborn pairs, which equals f_{n-2}, because each newborn pair comes from a pair at least two months old.

Consequently the sequence $\{f_n\}$ satisfies the recurrence relation $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$ with the initial conditions $f_1 = 1$ and $f_2 = 1$. The number of pairs of rabbits on the island after n months is given by the nth Fibonacci number.

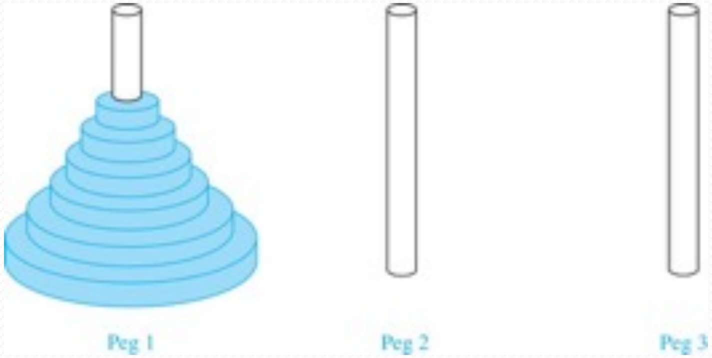
The Tower of Hanoi

In the late nineteenth century, the French mathematician Édouard Lucas invented a puzzle consisting of three pegs on a board with disks of different sizes. Initially all of the disks are on the first peg in order of size, with the largest on the bottom.

Rules: You are allowed to move the disks one at a time from one peg to another as long as a larger disk is never placed on a smaller.

Goal: Using allowable moves, end up with all the disks on the second peg in order of size with largest on the bottom.

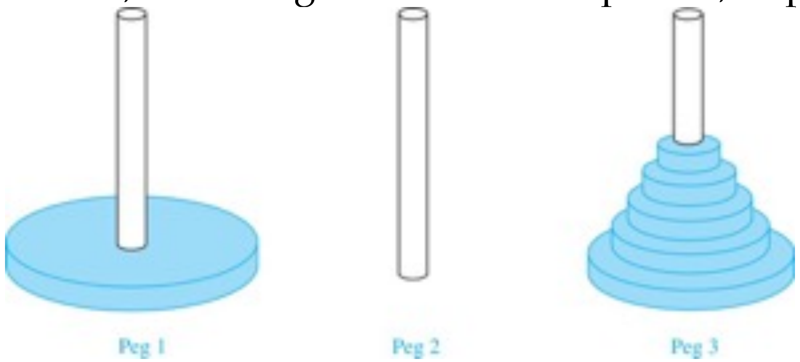
The Tower of Hanoi (*continued*)



The Initial Position in the Tower of Hanoi Puzzle

The Tower of Hanoi (*continued*)

Solution: Let $\{H_n\}$ denote the number of moves needed to solve the Tower of Hanoi Puzzle with n disks. Set up a recurrence relation for the sequence $\{H_n\}$. Begin with n disks on peg 1. We can transfer the top $n-1$ disks, following the rules of the puzzle, to peg 3 using H_{n-1} moves.



First, we use 1 move to transfer the largest disk to the second peg. Then we transfer the $n-1$ disks from peg 3 to peg 2 using H_{n-1} additional moves. This can not be done in fewer steps. Hence,

$$H_n = 2H_{n-1} + 1.$$

The initial condition is $H_1 = 1$ since a single disk can be transferred from peg 1 to peg 2 in one move.

The Tower of Hanoi (*continued*)

- We can use an iterative approach to solve this recurrence relation by repeatedly expressing H_n in terms of the previous terms of the sequence.

$$
\begin{aligned}
H_n &= 2H_{n-1} + 1 \\
&= 2(2H_{n-2} + 1) + 1 = 2^2 H_{n-2} + 2 + 1 \\
&= 2^2(2H_{n-3} + 1) + 2 + 1 = 2^3 H_{n-3} + 2^2 + 2 + 1 \\
&\vdots \\
&= 2^{n-1}H_1 + 2^{n-2} + 2^{n-3} + \ldots + 2 + 1 \\
&= 2^{n-1} + 2^{n-2} + 2^{n-3} + \ldots + 2 + 1 \quad \text{because } H_1 = 1 \\
&= 2^n - 1 \quad \text{using the formula for the sum of the terms of a geometric series}
\end{aligned}
$$

  - There was a myth created with the puzzle. Monks in a tower in Hanoi are transferring 64 gold disks from one peg to another following the rules of the puzzle. They move one disk each day. When the puzzle is finished, the world will end.
  - Using this formula for the 64 gold disks of the myth,

    $$2^{64} - 1 = 18{,}446{,}744{,}073{,}709{,}551{,}615$$

    days are needed to solve the puzzle, which is more than 500 billion years.
  - Reve's puzzle (proposed in 1907 by Henry Dudeney) is similar but has 4 pegs. There is a well-known unsettled conjecture for the the minimum number of moves needed to solve this puzzle. (*see Exercises* 38-45)

Counting Bit Strings

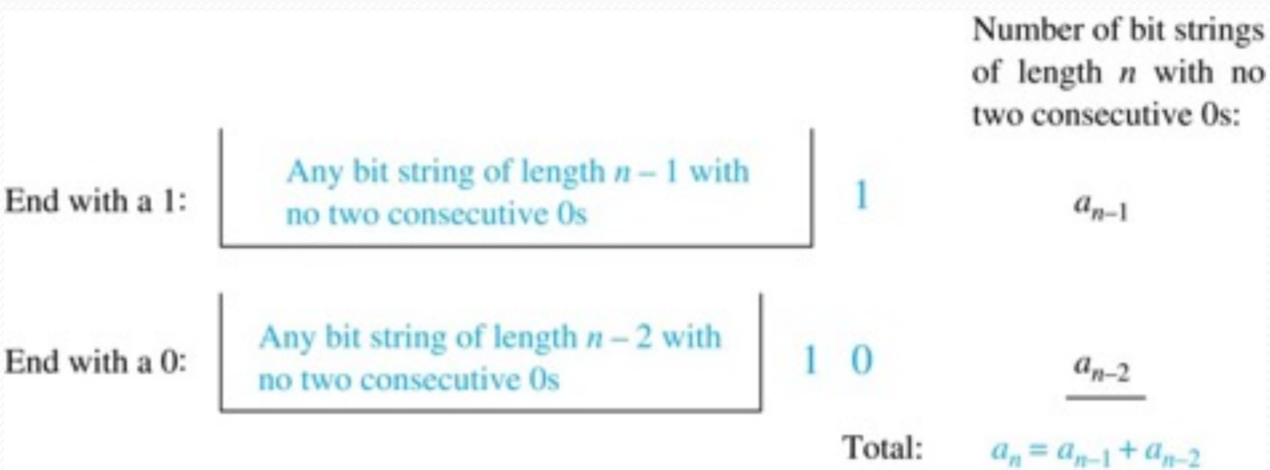
Example 3: Find a recurrence relation and give initial conditions for the number of bit strings of length n without two consecutive 0s. How many such bit strings are there of length five?

Solution: Let a_n denote the number of bit strings of length n without two consecutive 0s. To obtain a recurrence relation for $\{a_n\}$ note that the number of bit strings of length n that do not have two consecutive 0s is the number of bit strings ending with a 0 plus the number of such bit strings ending with a 1.

Now assume that $n \geq 3$.

- The bit strings of length n ending with 1 without two consecutive 0s are the bit strings of length n −1 with no two consecutive 0s with a 1 at the end. Hence, there are a_{n-1} such bit strings.
- The bit strings of length n ending with 0 without two consecutive 0s are the bit strings of length n −2 with no two consecutive 0s with 10 at the end. Hence, there are a_{n-2} such bit strings.

We conclude that $a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$.



Bit Strings (*continued*)

The initial conditions are:

- $a_1 = 2$, since both the bit strings 0 and 1 do not have consecutive 0s.
- $a_2 = 3$, since the bit strings 01, 10, and 11 do not have consecutive 0s, while 00 does.

To obtain a_5, we use the recurrence relation three times to find that:

- $a_3 = a_2 + a_1 = 3 + 2 = 5$
- $a_4 = a_3 + a_2 = 5 + 3 = 8$
- $a_5 = a_4 + a_3 = 8 + 5 = 13$

Note that $\{a_n\}$ satisfies the same recurrence relation as the Fibonacci sequence. Since $a_1 = f_3$ and $a_2 = f_4$, we conclude that $a_n = f_{n+2}$.

Counting the Ways to Parenthesize a Product

Example: Find a recurrence relation for C_n, the number of ways to parenthesize the product of $n + 1$ numbers, $x_0 \cdot x_1 \cdot x_2 \cdot \cdots \cdot x_n$, to specify the order of multiplication.
For example, $C_3 = 5$, since all the possible ways to parenthesize 4 numbers are

$((x_0 \cdot x_1) \cdot x_2) \cdot x_3$, $(x_0 \cdot (x_1 \cdot x_2)) \cdot x_3$, $(x_0 \cdot x_1) \cdot (x_2 \cdot x_3)$, $x_0 \cdot ((x_1 \cdot x_2) \cdot x_3)$, $x_0 \cdot (x_1 \cdot (x_2 \cdot x_3))$

Solution: Note that however parentheses are inserted in $x_0 \cdot x_1 \cdot x_2 \cdot \cdots \cdot x_n$, one "$\cdot$" operator remains outside all parentheses. This final operator appears between two of the n + 1 numbers, say x_k and x_{k+1}. Since there are C_k ways to insert parentheses in the product $x_0 \cdot x_1 \cdot x_2 \cdot \cdots \cdot x_k$ and C_{n-k-1} ways to insert parentheses in the product $x_{k+1} \cdot x_{k+2} \cdot \cdots \cdot x_n$, we have

ß

$$C_n = C_0C_{n-1} + C_1C_{n-2} + \cdots + C_{n-2}C_1 + C_{n-1}C_0 = \sum_{k=0}^{n-1} C_kC_{n-k-1}$$

The initial conditions are $C_0 = 1$ and $C_1 = 1$.

The sequence $\{C_n\}$ is the sequence of **Catalan Numbers**. This recurrence relation can be solved using the method of generating functions; see Exercise 41 in Section 8.4.

Solving Linear Recurrence Relations

Section 8.2

Section Summary

- Linear Homogeneous Recurrence Relations
- Solving Linear Homogeneous Recurrence Relations with Constant Coefficients.
- Solving Linear Nonhomogeneous Recurrence Relations with Constant Coefficients.

Linear Homogeneous Recurrence Relations

Definition: A *linear homogeneous recurrence relation of degree* k *with constant coefficients* is a recurrence relation of the form $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots.. + c_k a_{n-k}$, where $c_1, c_2, \ldots., c_k$ are real numbers, and $c_k \neq 0$

• it is *linear* because the right-hand side is a sum of the previous terms of the sequence each multiplied by a function of n.
• it is *homogeneous* because no terms occur that are not multiples of the a_js. Each coefficient is a constant.
• the *degree* is k because a_n is expressed in terms of the previous k terms of the sequence.

By strong induction, a sequence satisfying such a recurrence relation is uniquely determined by the recurrence relation and the k initial conditions $a_0 = C_1$, $a_0 = C_1, \ldots, a_{k-1} = C_{k-1}$.

Examples of Linear Homogeneous Recurrence Relations

- $P_n = (1.11)P_{n-1}$ linear homogeneous recurrence relation of degree one
- $f_n = f_{n-1} + f_{n-2}$ linear homogeneous recurrence relation of degree two
- $a_n = a_{n-1} + a_{n-2}^2$ not linear
- $H_n = 2H_{n-1} + 1$ not homogeneous
- $B_n = nB_{n-1}$ coefficients are not constants

Solving Linear Homogeneous Recurrence Relations

- The basic approach is to look for solutions of the form $a_n = r^n$, where r is a constant.
- Note that $a_n = r^n$ is a solution to the recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2} + \cdots + c_k a_{n-k}$ if and only if $r^n = c_1r^{n-1} + c_2r^{n-2} + \cdots + c_k r^{n-k}$.
- Algebraic manipulation yields the *characteristic equation*:

  $$r^k - c_1r^{k-1} - c_2r^{k-2} - \cdots - c_{k-1}r - c_k = 0$$

- The sequence $\{a_n\}$ with $a_n = r^n$ is a solution if and only if r is a solution to the characteristic equation.
- The solutions to the characteristic equation are called the *characteristic roots* of the recurrence relation. The roots are used to give an explicit formula for all the solutions of the recurrence relation.

Solving Linear Homogeneous Recurrence Relations of Degree Two

Theorem 1: Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1 and r_2. Then the sequence $\{a_n\}$ is a solution to the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if

$$a_n = \alpha r_1^n + \alpha_2 r_2^n$$

for $n = 0,1,2,\ldots$, where α_1 and α_2 are constants.

Using Theorem 1

Example: What is the solution to the recurrence relation

$$a_n = a_{n-1} + 2a_{n-2} \text{ with } a_0 = 2 \text{ and } a_1 = 7?$$

Solution: The characteristic equation is $r^2 - r - 2 = 0$.
Its roots are $r = 2$ and $r = -1$. Therefore, $\{a_n\}$ is a solution to the recurrence relation if and only if $a_n = \alpha_1 2^n + \alpha_2(-1)^n$, for some constants α_1 and α_2.

To find the constants α_1 and α_2, note that

$$a_0 = 2 = \alpha_1 + \alpha_2 \text{ and } a_1 = 7 = \alpha_1 2 + \alpha_2(-1).$$

Solving these equations, we find that $\alpha_1 = 3$ and $\alpha_2 = -1$.

Hence, the solution is the sequence $\{a_n\}$ with $a_n = 3 \cdot 2^n - (-1)^n$.

An Explicit Formula for the Fibonacci Numbers

We can use Theorem 1 to find an explicit formula for the Fibonacci numbers. The sequence of Fibonacci numbers satisfies the recurrence relation $f_n = f_{n-1} + f_{n-2}$ with the initial conditions: $f_0 = 0$ and $f_1 = 1$.

Solution: The roots of the characteristic equation $r^2 - r - 1 = 0$ are

$$r_1 = \frac{1+\sqrt{5}}{2}$$

$$r_2 = \frac{1-\sqrt{5}}{2}$$

Fibonacci Numbers (*continued*)

Therefore by Theorem 1

$$f_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$$

for some constants α_1 and α_2.

Using the initial conditions $f_0 = 0$ and $f_1 = 1$, we have

$$f_0 = \alpha_1 + \alpha_2 = 0$$

$$f_1 = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right) = 1 .$$

Solving, we obtain $\alpha_1 = \frac{1}{\sqrt{5}}$, $\alpha_2 = -\frac{1}{\sqrt{5}}$.

Hence, $$f_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n$$

The Solution when there is a Repeated Root

Theorem 2: Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has one repeated root r_0. Then the sequence $\{a_n\}$ is a solution to the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if

$$a_n = \alpha r_0^n + \alpha_2 n r_0^n$$

for $n = 0,1,2,\ldots$, where α_1 and α_2 are constants.

Using Theorem 2

Example: What is the solution to the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$ with $a_0 = 1$ and $a_1 = 6$?

Solution: The characteristic equation is $r^2 - 6r + 9 = 0$.
The only root is $r = 3$. Therefore, $\{a_n\}$ is a solution to the recurrence relation if and only if

$$a_n = \alpha_1 3^n + \alpha_2 n(3)^n$$

where α_1 and α_2 are constants.

To find the constants α_1 and α_2, note that

$$a_0 = 1 = \alpha_1 \quad \text{and} \quad a_1 = 6 = \alpha_1 \cdot 3 + \alpha_2 \cdot 3.$$

Solving, we find that $\alpha_1 = 1$ and $\alpha_2 = 1$.
Hence,

$$a_n = 3^n + n3^n .$$

Solving Linear Homogeneous Recurrence Relations of Arbitrary Degree

This theorem can be used to solve linear homogeneous recurrence relations with constant coefficients of any degree when the characteristic equation has distinct roots.

Theorem 3: Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - \cdots - c_k = 0$$

has k distinct roots $r_1, r_2, \ldots, r_k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots\ldots + c_k a_{n-k}$$

if and only if

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^m + \cdots + \alpha_k r_k^n$$

for $n = 0, 1, 2, \ldots$, where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are constants.

The General Case with Repeated Roots Allowed

Theorem 4: Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - \cdots - c_k = 0$$

has t distinct roots $r_1, r_2, \ldots, r_t$ with multiplicities $m_1, m_2, \ldots, m_t$, respectively so that $m_i \geq 1$ for $i = 0, 1, 2, \ldots, t$ and $m_1 + m_2 + \ldots + m_t = k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots.. + c_k a_{n-k}$$

if and only if

$$\begin{aligned} a_n = {} & (\alpha_{1,0} + \alpha_{1,1} n + \cdots + \alpha_{1,m_1-1} n^{m_1-1}) r_1^n \\ & + (\alpha_{2,0} + \alpha_{2,1} n + \cdots + \alpha_{2,m_2-1} n^{m_2-1}) r_2^n \\ & + \cdots + (\alpha_{t,0} + \alpha_{t,1} n + \cdots + \alpha_{t,m_t-1} n^{m_t-1}) r_t^n \end{aligned}$$

for $n = 0, 1, 2, \ldots$, where $\alpha_{i,j}$ are constants for $1 \leq i \leq t$ and $0 \leq j \leq m_{i-1}$.

Linear Nonhomogeneous Recurrence Relations with Constant Coefficients

Definition: A *linear nonhomogeneous recurrence relation with constant coefficients* is a recurrence relation of the form:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots.. + c_k a_{n-k} + F(n),$$

where $c_1, c_2, \ldots., c_k$ are real numbers, and $F(n)$ is a function not identically zero depending only on n.

The recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots.. + c_k a_{n-k},$$

is called the associated homogeneous recurrence relation.

Linear Nonhomogeneous Recurrence Relations with Constant Coefficients (*cont.*)

The following are linear nonhomogeneous recurrence relations with constant coefficients:

$a_n = a_{n-1} + 2^n$,

$a_n = a_{n-1} + a_{n-2} + n^2 + n + 1$,

$a_n = 3a_{n-1} + n3^n$,

$a_n = a_{n-1} + a_{n-2} + a_{n-3} + n!$

where the following are the associated linear homogeneous recurrence relations, respectively:

$a_n = a_{n-1}$,

$a_n = a_{n-1} + a_{n-2}$,

$a_n = 3a_{n-1}$,

$a_n = a_{n-1} + a_{n-2} + a_{n-3}$

Solving Linear Nonhomogeneous Recurrence Relations with Constant Coefficients

Theorem 5: If $\{a_n^{(p)}\}$ is a particular solution of the nonhomogeneous linear recurrence relation with constant coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

then every solution is of the form $\{a_n^{(p)} + a_n^{(h)}\}$, where $\{a_n^{(h)}\}$ is a solution of the associated homogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}.$$

Solving Linear Nonhomogeneous Recurrence Relations with Constant Coefficients (*continued*)

Example: Find all solutions of the recurrence relation $a_n = 3a_{n-1} + 2n$.
What is the solution with $a_1 = 3$?

Solution: The associated linear homogeneous equation is $a_n = 3a_{n-1}$.
Its solutions are $a_n^{(h)} = \alpha 3^n$, where α is a constant.

Because $F(n)= 2n$ is a polynomial in n of degree one, to find a particular solution we might try a linear function in n, say $p_n = cn + d$, where c and d are constants. Suppose that $p_n = cn + d$ is such a solution.
Then $a_n = 3a_{n-1} + 2n$ becomes $cn + d = 3(c(n-1) + d)+ 2n$.

Simplifying yields $(2 + 2c)n + (2d - 3c) = 0$. It follows that $cn + d$ is a solution if and only if $2 + 2c = 0$ and $2d - 3c = 0$. Therefore, $cn + d$ is a solution if and only if $c = -1$ and $d = -3/2$.
Consequently, $a_n^{(p)} = -n - 3/2$ is a particular solution.

By Theorem 5, all solutions are of the form $a_n = a_n^{(p)} + a_n^{(h)} = -n - 3/2 + \alpha 3^n$, where α is a constant.

To find the solution with $a_1 = 3$, let $n = 1$ in the above formula for the general solution.
Then $3 = -1 - 3/2 + 3\alpha$, and $\alpha = 11/6$. Hence, the solution is $a_n = -n - 3/2 + (11/6)3^n$.

Divide-and-Conquer Algorithms and Recurrence Relations

Section 8.3

Section Summary

- Divide-and-Conquer Algorithms and Recurrence Relations
- Examples
  - Binary Search
  - Merge Sort
  - Fast Multiplication of Integers
- Master Theorem
- Closest Pair of Points (*not covered yet in these slides*)

Divide-and-Conquer Algorithmic Paradigm

Definition: A *divide-and-conquer algorithm* works by first *dividing* a problem into one or more instances of the same problem of smaller size and then *conquering* the problem using the solutions of the smaller problems to find a solution of the original problem.

Examples:

- Binary search, covered in Chapters 3 and 5: It works by comparing the element to be located to the middle element. The original list is then split into two lists and the search continues recursively in the appropriate sublist.
- Merge sort, covered in Chapter 5: A list is split into two approximately equal sized sublists, each recursively sorted by merge sort. Sorting is done by successively merging pairs of lists.

Divide-and-Conquer Recurrence Relations

- Suppose that a recursive algorithm divides a problem of size n into a subproblems.
- Assume each subproblem is of size n/b.
- Suppose $g(n)$ extra operations are needed in the conquer step.
- Then $f(n)$ represents the number of operations to solve a problem of size n satisisfies the following recurrence relation:

$$f(n) = af(n/b) + g(n)$$

- This is called a *divide-and-conquer recurrence relation.*

Example: Binary Search

- Binary search reduces the search for an element in a sequence of size n to the search in a sequence of size $n/2$. Two comparisons are needed to implement this reduction;
  - one to decide whether to search the upper or lower half of the sequence and
  - the other to determine if the sequence has elements.
- Hence, if $f(n)$ is the number of comparisons required to search for an element in a sequence of size n, then

$$f(n) = f(n/2) + 2$$

when n is even.

Example: Merge Sort

- The merge sort algorithm splits a list of n (assuming n is even) items to be sorted into two lists with $n/2$ items. It uses fewer than n comparisons to merge the two sorted lists.
- Hence, the number of comparisons required to sort a sequence of size n, is no more than than $M(n)$ where

$$M(n) = 2M(n/2) + n.$$

Example: Fast Multiplication of Integers

- An algorithm for the fast multiplication of two $2n$-bit integers (assuming n is even) first splits each of the $2n$-bit integers into two blocks, each of n bits.
- Suppose that a and b are integers with binary expansions of length $2n$. Let
  $a = (a_{2n-1}a_{2n-2} \ldots a_1a_0)_2$ and $b = (b_{2n-1}b_{2n-2} \ldots b_1b_0)_2$.
- Let $a = 2^nA_1 + A_0$, $b = 2^nB_1 + B_0$, where
  $A_1 = (a_{2n-1} \ldots a_{n+1}a_n)_2$, $A_0 = (a_{n-1} \ldots a_1a_0)_2$,
  $B_1 = (b_{2n-1} \ldots b_{n+1}b_n)_2$, $B_0 = (b_{n-1} \ldots b_1b_0)_2$.
- The algorithm is based on the fact that ab can be rewritten as:
  $ab = (2^{2n} + 2^n)A_1B_1 + 2^n (A_1 - A_0)(B_0 - B_1) + (2^n + 1)A_0B_0$.
- This identity shows that the multiplication of two $2n$-bit integers can be carried out using three multiplications of n-bit integers, together with additions, subtractions, and shifts.
- Hence, if $f(n)$ is the total number of operations needed to multiply two n-bit integers, then

$$f(2n) = 3f(n) + Cn$$

where Cn represents the total number of bit operations; the additions, subtractions and shifts that are a constant multiple of n-bit operations.

Estimating the Size of Divide-and-Conquer Functions

Theorem 1: Let f be an increasing function that satisfies the recurrence relation

$$f(n) = af(n/b) + cn^d$$

whenever n is divisible by b, where $a \geq 1$, b is an integer greater than 1, and c is a positive real number. Then

$$f(n) \text{ is } \begin{cases} O(n^{\log_b a}) & \text{if } a > 1 \\ O(\log n) & \text{if } a = 1. \end{cases}$$

Furthermore, when $n = b^k$ and $a \neq 1$, where k is a positive integer,

$$f(n) = C_1 n^{\log_b a} + C_2$$

where $C_1 = f(1) + c/(a-1)$ and $C_1 = -c/(a-1)$.

Complexity of Binary Search

Binary Search Example: Give a big-O estimate for the number of comparisons used by a binary search.

Solution: Since the number of comparisons used by binary search is $f(n) = f(n/2) + 2$ where n is even, by Theorem 1, it follows that $f(n)$ is $O(\log n)$.

Estimating the Size of Divide-and-conquer Functions (*continued*)

Theorem 2. Master Theorem: Let f be an increasing function that satisfies the recurrence relation

$$f(n) = af(n/b) + cn^d$$

whenever $n = b^k$, where k is a positive integer greater than 1, and c and d are real numbers with c positive and d nonnegative. Then

$$f(n) \text{ is } \begin{cases} O(n^d) & \text{if } a < b^d, \\ O(n^d \log n) & \text{if } a = b^d, \\ O(n^{\log_b a}) & \text{if } a > b^d. \end{cases}$$

Complexity of Merge Sort

Merge Sort Example: Give a big-O estimate for the number of comparisons used by merge sort.

Solution: Since the number of comparisons used by merge sort to sort a list of n elements is less than $M(n)$ where $M(n) = 2M(n/2) + n$, by the master theorem $M(n)$ is $O(n \log n)$.

Complexity of Fast Integer Multiplication Algorithm

Integer Multiplication Example: Give a big-O estimate for the number of bit operations used needed to multiply two n-bit integers using the fast multiplication algorithm.

Solution: We have shown that $f(n) = 3f(n/2) + Cn$, when n is even, where $f(n)$ is the number of bit operations needed to multiply two n-bit integers. Hence by the master theorem with $a = 3$, $b = 2$, $c = C$, and $d = 0$ (so that we have the case where $a > b^d$), it follows that $f(n)$ is $O(n^{\log 3})$.

Note that $\log 3 \approx 1.6$. Therefore the fast multiplication algorithm is a substantial improvement over the conventional algorithm that uses $O(n^2)$ bit operations.

Generating Functions

Section 8.4

Section Summary

- Generating Functions
- Counting Problems and Generating Functions
- Useful Generating Functions
- Solving Recurrence Relations Using Generating Functions (*not yet covered in the slides*)
- Proving Identities Using Generating Functions (*not yet covered in the slides*)

Generating Functions

Definition: The *generating function for the sequence* $a_0, a_1, \ldots, a_k, \ldots$ of real numbers is the infinite series

$$G(x) = a_0 + a_1 x + \cdots + a_k x^k + \cdots = \sum_{k=0}^{\infty} a_k x^k.$$

Examples:

- The sequence $\{a_k\}$ with $a_k = 3$ has the generating function $\sum_{k=0}^{\infty} 3x^k.$
- The sequence $\{a_k\}$ with $a_k = k + 1$ has the generating function has the generating function $\sum_{k=0}^{\infty} (k+1)x^k.$
- The sequence $\{a_k\}$ with $a_k = 2^k$ has the generating function $\sum_{k=0}^{\infty} 2^k x^k.$

Generating Functions for Finite Sequences

- Generating functions for finite sequences of real numbers can be defined by extending a finite sequence $a_0, a_1, \ldots, a_n$ into an infinite sequence by setting $a_{n+1} = 0$, $a_{n+2} = 0$, and so on.
- The generating function $G(x)$ of this infinite sequence $\{a_n\}$ is a polynomial of degree n because no terms of the form $a_j x^j$ with $j > n$ occur, that is,

$$G(x) = a_0 + a_1 x + \cdots + a_n x^n.$$

Generating Functions for Finite Sequences (continued)

Example: What is the generating function for the sequence 1,1,1,1,1,1?

Solution: The generating function of 1,1,1,1,1,1 is

$1 + x + x^2 + x^3 + x^4 + x^5$.

By Theorem 1 of Section 2.4, we have

$(x^6 - 1)/(x - 1) = 1 + x + x^2 + x^3 + x^4 + x^5$

when $x \neq 1$.

Consequently $G(x) = (x^6 - 1)/(x - 1)$ is the generating function of the sequence.

$1 - x \quad \sum_{k=0}$	
$\frac{1}{1-ax} = \sum_{k=0}^{\infty} a^k x^k = 1 + ax + a^2x^2 + \cdots$	a^k
$\frac{1}{1-x^r} = \sum_{k=0}^{\infty} x^{rk} = 1 + x^r + x^{2r} + \cdots$	1 if $r \mid k$; 0 otherwise
$\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} (k+1)x^k = 1 + 2x + 3x^2 + \cdots$	$k+1$
$\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)x^k = 1 + C(n,1)x + C(n+1,2)x^2 + \cdots$	$C(n+k-1,k) = C(n+k-1,n-1)$
$\frac{1}{(1+x)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)(-1)^k x^k = 1 - C(n,1)x + C(n+1,2)x^2 - \cdots$	$(-1)^k C(n+k-1,k) = (-1)^k C(n+k-1,n-1)$
$\frac{1}{(1-ax)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)a^k x^k = 1 + C(n,1)ax + C(n+1,2)a^2x^2 + \cdots$	$C(n+k-1,k)a^k = C(n+k-1,n-1)a^k$
$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$	$1/k!$
$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$	$(-1)^{k+1}/k$

Note: The series for the last two generating functions can be found in most calculus books when power series are discussed.

Counting Problems and Generating Functions

Example: Find the number of solutions of

$$e_1 + e_2 + e_3 = 17,$$

where e_1, e_2, and e_3 are nonnegative integers with $2 \le e_1 \le 5$, $3 \le e_2 \le 6$, and $4 \le e_3 \le 7$.

Solution: The number of solutions is the coefficient of x^{17} in the expansion of

$$(x^2 + x^3 + x^4 + x^5)\,(x^3 + x^4 + x^5 + x^6)\,(x^4 + x^5 + x^6 + x^7).$$

This follows because a term equal to is obtained in the product by picking a term in the first sum x^{e_1}, a term in the second sum x^{e_2}, and a term in the third sum x^{e_3}, where $e_1 + e_2 + e_3 = 17$.

There are three solutions since the coefficient of x^{17} in the product is 3.

Counting Problems and Generating Functions (*continued*)

Example: Use generating functions to find the number of k-combinations of a set with n elements, i.e., $C(n,k)$.

Solution: Each of the n elements in the set contributes the term $(1 + x)$ to the generating function $f(x) = \sum_{k=0}^{n} a^k x^k.$

Hence $f(x) = (1 + x)^n$ where $f(x)$ is the generating function for $\{a^k\}$, where a^k represents the number of k-combinations of a set with n elements.

By the binomial theorem, we have $$f(x) = \sum_{k=0}^{n} \binom{n}{k} x^k,$$

where $$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Hence, $$C(n,k) = \frac{n!}{k!(n-k)!}.$$

Inclusion-Exclusion

Section 8.5

Principle of Inclusion-Exclusion

- In Section 2.2, we developed the following formula for the number of elements in the union of two finite sets:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

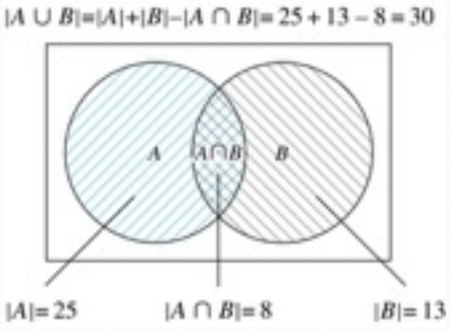
- We will generalize this formula to finite sets of any size.

Two Finite Sets

Example: In a discrete mathematics class every student is a major in computer science or mathematics or both. The number of students having computer science as a major (possibly along with mathematics) is 25; the number of students having mathematics as a major (possibly along with computer science) is 13; and the number of students majoring in both computer science and mathematics is 8. How many students are in the class?

Solution: $|A \cup B| = |A| + |B| - |A \cap B|$

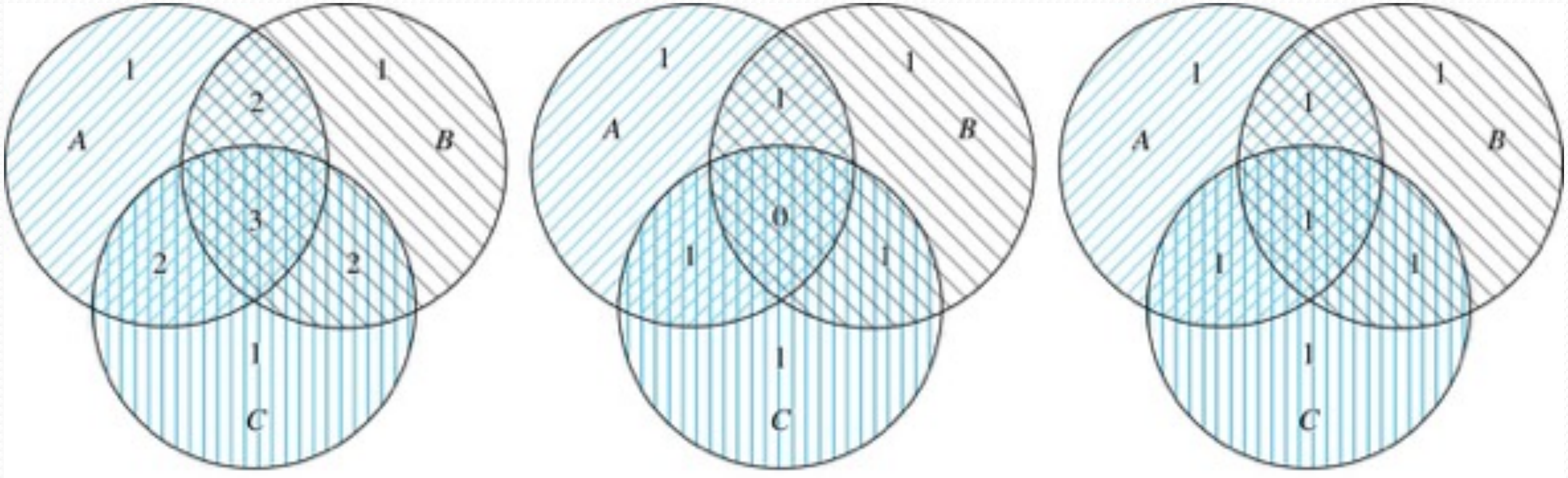
$$= 25 + 13 - 8 = 30$$



Three Finite Sets

$$|A \cup B \cup C| =$$

$$|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$



(a) Count of elements by $|A|+|B|+|C|$

(b) Count of elements by $|A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C|$

(c) Count of elements by $|A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C|+|A\cap B\cap C|$

Three Finite Sets Continued

Example: A total of 1232 students have taken a course in Spanish, 879 have taken a course in French, and 114 have taken a course in Russian. Further, 103 have taken courses in both Spanish and French, 23 have taken courses in both Spanish and Russian, and 14 have taken courses in both French and Russian. If 2092 students have taken a course in at least one of Spanish French and Russian, how many students have taken a course in all 3 languages.

Solution: Let S be the set of students who have taken a course in Spanish, F the set of students who have taken a course in French, and R the set of students who have taken a course in Russian. Then, we have

$|S| = 1232$, $|F| = 879$, $|R| = 114$, $|S \cap F| = 103$, $|S \cap R| = 23$, $|F \cap R| = 14$, and $|S \cup F \cup R| = 23$.

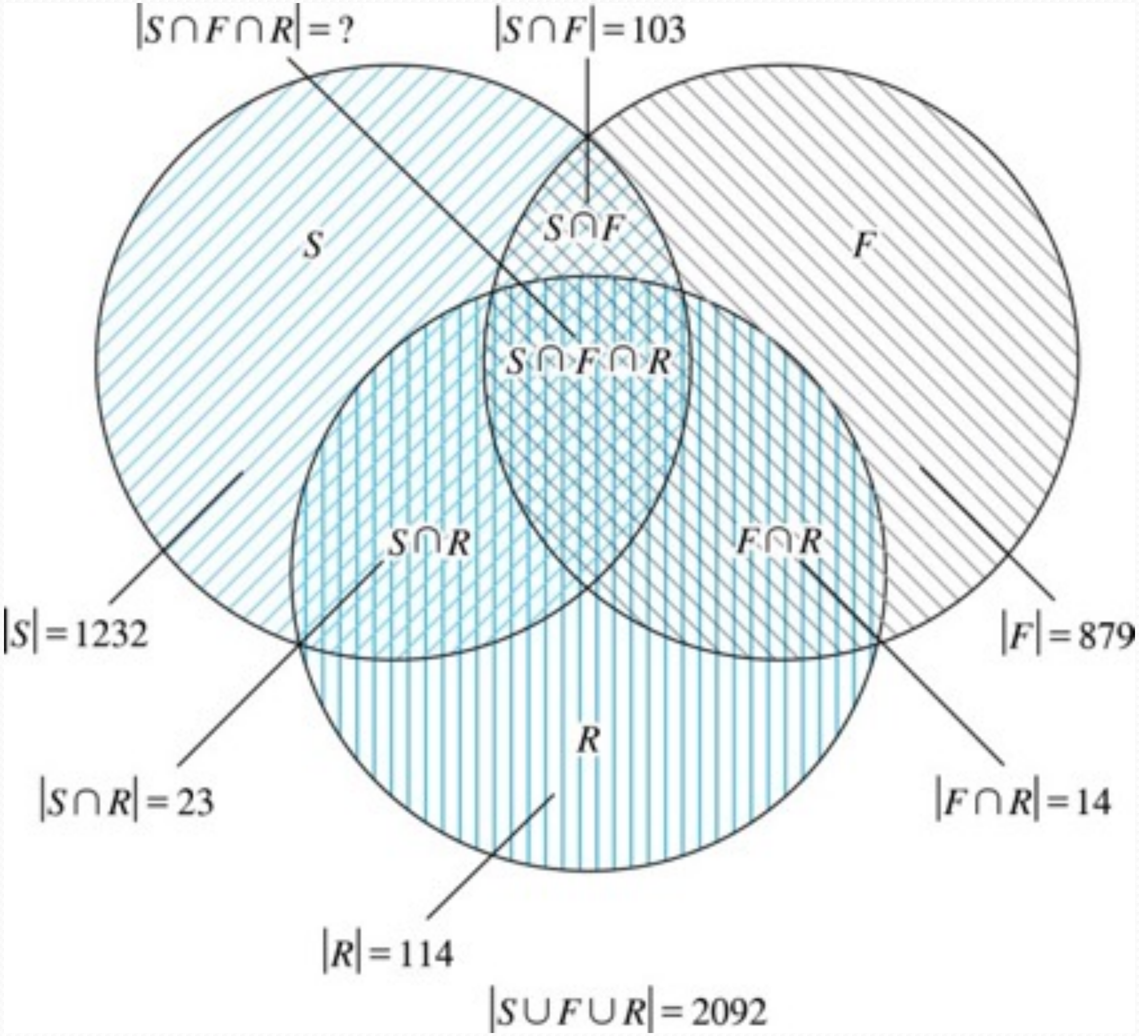
Using the equation

$$|S \cup F \cup R| = |S| + |F| + |R| - |S \cap F| - |S \cap R| - |F \cap R| + |S \cap F \cap R|,$$

we obtain $2092 = 1232 + 879 + 114 - 103 - 23 - 14 + |S \cap F \cap R|$.

Solving for $|S \cap F \cap R|$ yields 7.

Illustration of Three Finite Set Example



The Principle of Inclusion-Exclusion

Theorem 1. **The Principle of Inclusion-Exclusion**:
Let A_1, A_2, …, A_n be finite sets. Then:

$$|A_1 \cup A_2 \cup \cdots \cup A_n| =$$

$$\sum_{1 \leq i \leq n} |A_i| - \sum_{1 \leq i \leq j \leq n} |A_i \cap A_j| +$$

$$\sum_{1 \leq i \leq j \leq k \leq n} |A_i \cap A_j \cap A_k| - \ldots + (-1)^{n+1} |A_1 \cap A_2 \cap \ldots \cap A_n|$$

Inequality

$$\leq$$

$$|A_1 \cup A_2 \cup \cdots \cup A_n| =$$

$$\geq$$

$$\sum_{1 \leq i \leq n} |A_i| - \sum_{1 \leq i \leq j \leq n} |A_i \cap A_j| +$$

The Principle of Inclusion-Exclusion (*continued*)

Proof: An element in the union is counted exactly once in the right-hand side of the equation. Consider an element a that is a member of r of the sets $A_1, \ldots, A_n$ where $1 \leq r \leq n$.

- It is counted $C(r,1)$ times by $\sum |A_i|$
- It is counted $C(r,2)$ times by $\sum |A_i \cap A_j|$
- In general, it is counted $C(r,m)$ times by the summation of m of the sets A_i.

The Principle of Inclusion-Exclusion (cont)

- Thus the element is counted exactly

$$C(r,1) - C(r,2) + C(r,3) - \cdots + (-1)^{r+1} C(r,r)$$

times by the right hand side of the equation.

- By Corollary 2 of Section 6.4, we have

$$C(r,0) - C(r,1) + C(r,2) - \cdots + (-1)^{r} C(r,r) = 0.$$

- Hence,

$$1 = C(r,0) = C(r,1) - C(r,2) + \cdots + (-1)^{r+1} C(r,r).$$

Applications of Inclusion-Exclusion

Section 8.6

Section Summary

- Counting Onto-Functions
- Derangements

The Number of Onto Functions

Example: How many onto functions are there from a set with six elements to a set with three elements?

Solution: Suppose that the elements in the codomain are b_1, b_2, and b_3. Let P_1, P_2, and P_3 be the properties that b_1, b_2, and b_3 are not in the range of the function, respectively. The function is onto if none of the properties P_1, P_2, and P_3 hold.

By the inclusion-exclusion principle the number of onto functions from a set with six elements to a set with three elements is

$$N - [N(P_1) + N(P_2) + N(P_3)] + [N(P_1P_2) + N(P_1P_3) + N(P_2P_3)] - N(P_1P_2P_3)$$

- Here the total number of functions from a set with six elements to one with three elements is $N = 3^6$.
- The number of functions that do not have in the range is $N(P_1) = 2^6$. Similarly, $N(P_2) = N(3_1) = 2^6$.
- Note that $N(P_1P_2) = N(P_1P_3) = N(P_2P_3) = 1$ and $N(P_1P_2P_3)= 0$.

Hence, the number of onto functions from a set with six elements to a set with three elements is:

$$3^6 - 3 \cdot 2^6 + 3 = 729 - 192 + 3 = 540$$

The Number of Onto Functions (continued)

Theorem 1: Let m and n be positive integers with $m \geq n$. Then there are

$$n^m - C(n,1)(n-1)^m + C(n,2)(n-2)^m - \cdots + (-1)^{n-1}C(n,n-1) \cdot 1^m$$

onto functions from a set with m elements to a set with n elements.

Proof follows from the principle of inclusion-exclusion (*see Exercise* 27).

Derangements

Definition: A *derangement* is a permutation of objects that leaves no object in the original position.

Example: The permutation of 21453 is a derangement of 12345 because no number is left in its original position. But 21543 is not a derangement of 12345, because 4 is in its original position.

Derangements (continued)

Theorem 2: The number of derangements of a set with *n* elements is

$$D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right].$$

Let Ai be the number of permutations that number i is in position i.

Number of dearrangements is n!-|A1 ∪ A2 ∪ A3 ∪ A4….. ∪ An|

Then |Ai|=(n-1)!,

|Ai ∩ Aj|=(n-2)!

|Ai1∩ Ai2∩ Ai3……∩ Aik|=(n-k)!

Derangements (continued)

The Hatcheck Problem: A new employee checks the hats of n people at restaurant, forgetting to put claim check numbers on the hats. When customers return for their hats, the checker gives them back hats chosen at random from the remaining hats. What is the probability that no one receives the correct hat.

Solution: The answer is the number of ways the hats can be arranged so that there is no hat in its original position divided by $n!$, the number of permutations of n hats.

Remark: It can be shown that the probability of a derangement approaches $1/e$ as n grows without bound.

$$\frac{D_n}{n!} = \left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right]$$

TABLE 1 The Probability of a Derangement.

n	2	3	4	5	6	7
$D_n/n!$	0.50000	0.33333	0.37500	0.36667	0.36806	0.36786