CDS in Plane Geometric Networks: Short, Small, And Sparse

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Outline

- Three Parameters of CDS
- Dominators
- Basic Set of Connectors
- The First Improvement
- The Second Improvement
- The Third Improvement
Three Parameters of A Virtual Backbone (CDS)

\[ G = (V, E): \text{a (connected) UDG} \]
\[ s \in V: \text{a fixed node} \]
\[ R: \text{radius of } G \text{ w.r.t. } s. \]
Three Parameters of A Virtual Backbone (CDS)

$G = (V, E)$: a (connected) UDG

$s \in V$: a fixed node

$R$: radius of $G$ w.r.t. $s$.

$U$: a CDS of $G$ containing $s$

- Size: $|U|$
Three Parameters of A Virtual Backbone (CDS)

\[ G = (V, E): \text{a (connected) UDG} \]
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\[ R: \text{radius of } G \text{ w.r.t. } s. \]
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- **Size:** \( |U| \)
- **Radius:** \( \text{Rad}(G[U], s) \)
Three Parameters of A Virtual Backbone (CDS)

\[ G = (V, E): \text{a (connected) UDG} \]
\[ s \in V: \text{a fixed node} \]
\[ R: \text{radius of } G \text{ w.r.t. } s. \]
\[ U: \text{a CDS of } G \text{ containing } s \]

- **Size:** \(|U|\)
- **Radius:** \( \text{Rad}(G[U], s) \)
- **Sparsity:** \( \Delta(G[U]) \)
Our objective is to construct a CDS $U$ of $G$ with $s \in U$ such that

1. $U$ is small: $|U| = \Theta (\text{poly} (R))$
2. $U$ is short: $\text{Rad} (G [U], s) = \Theta (R)$
3. $U$ is sparse: $\Delta (G [U])$ is small, preferably bounded by a constant.
2-Phased Algorithm

- Phase 1: First-fit selection of an MIS $I$ in the BFS ordering w.r.t. $s$
2-Phased Algorithm

- Phase 1: First-fit selection of an MIS $I$ in the BFS ordering w.r.t. $s$
- Phase 2: Augment $I$ with a set $C$ of “connectors” to form a CDS
Roadmap

- Three Parameters of CDS
- **Dominator**s
- Basic Set of Connectors
- The First Improvement
- The Second Improvement
- The Third Improvement
Sparsity of Dominators

- Each node is adjacent to at most 5 dominators.

Figure: The layout of 18 dominators in the annulus of radii $1 + 2\varepsilon$ and $(1 + 2\varepsilon) \sqrt{2 + \sqrt{3}}$ centered at a node $v_0$. 
Sparsity of Dominators

- Each node is adjacent to at most 5 dominators.
- The annulus of radii one and two centered at each node contains at most 18 dominators (Bateman and Erdös)

**Figure:** The layout of 18 dominators in the annulus of radii $1 + 2\varepsilon$ and $(1 + 2\varepsilon) \sqrt{2 + \sqrt{3}}$ centered at a node $v_0$. 
Theorem

Suppose that $S$ is a compact convex set and $U$ is a set of points with mutual distances at least one. Then

$$|U \cap S| \leq \frac{\text{area}(S)}{\sqrt{3}/2} + \frac{\text{peri}(S)}{2} + 1,$$

where $\text{area}(S)$ and $\text{peri}(S)$ are the area and perimeter of $S$ respectively.
Corollary

Suppose that $S$ (respectively, $S'$) is a disk (respectively, half-disk) of radius $r$, and $U$ is a set of points with mutual distances at least one. Then

\[ |U \cap S| \leq \frac{2\pi}{\sqrt{3}} r^2 + \pi r + 1, \]

\[ |U \cap S'| \leq \frac{\pi}{\sqrt{3}} r^2 + \left( \frac{\pi}{2} + 1 \right) r + 1. \]
Since all dominators lie in the disk of radius $R$ centered at $s$, we have

$$|I| \leq \left( \frac{2\pi}{\sqrt{3}} R^2 + \pi R + 1 \right).$$
Roadmap

- Three Parameters of CDS
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• \( \forall u \in I \setminus \{s\}, p(u) \leftarrow \) the least-ID neighbor of \( u \) in the layer above \( u \).
Basic Set of Connectors

- $\forall u \in I \setminus \{s\}, p(u)$ ← the least-ID neighbor of $u$ in the layer above $u$.
- output $C = \{p(u) : u \in I \setminus \{s\}\}$. 

\[ 
\begin{align*}
\text{Layer 0} & \quad s
\end{align*}
\] 
\[ 
\begin{align*}
\text{Layer 1} & \quad \quad \\
\text{Layer 2} & \quad \quad \\
\text{Layer R} & \quad \quad \\
\end{align*}
\]
\[ |C| \leq |I \setminus \{s\}| = |I| - 1. \]
Size And Radius

\[ |C| \leq |I \setminus \{s\}| = |I| - 1. \]

Hence,

\[ |I \cup C| \leq 2|I| - 1 \leq \frac{4\pi}{\sqrt{3}} R^2 + 2\pi R + 1. \]
\[ |C| \leq |I \setminus \{s\}| = |I| - 1. \]

Hence,
\[ |I \cup C| \leq 2|I| - 1 \leq \frac{4\pi}{\sqrt{3}} R^2 + 2\pi R + 1. \]

In addition,
\[ \operatorname{Rad} (G[I \cup C], s) \leq 2(R - 1). \]
Sparsity

- Each dominatee is adj. to at most 4 dominators in the layer below itself.

**Figure:** The node $v_0$ is $s$. It is adjacent to 16 connectors.
Sparsity

- Each dominatee is adj. to at most 4 dominators in the layer below itself.
- $s$ is adj. to at most 18 connectors.

Figure: The node $v_0$ is $s$. It is adjacent to 16 connectors.
Sparsity

- Each dominatee is adj. to at most 4 dominators in the layer below itself.
- s is adj. to at most 18 connectors
- Each other dominator is adj. to at most 17 connectors in the same or the next layer.

Figure: The node $v_0$ is $s$. It is adjacent to 16 connectors.
Roadmap

- Three Parameters of CDS
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- Basic Set of Connectors
- **The First Improvement**
- The Second Improvement
- The Third Improvement
Figure: (a) $X = \{x_i : 1 \leq i \leq 7\}$ is covered by $Y = \{y_i : 1 \leq i \leq 5\}$. (b) $\{y_2, y_3, y_4\}$ is a minimal cover of $X$. The black nodes are private neighbors.
for each $0 \leq i \leq R$, $l_i \leftarrow \{\text{dominators of depth } i \text{ in } G\}$;
$C \leftarrow \emptyset$;
for each $1 \leq i \leq R - 1$,
$C_i \leftarrow \text{a minimal cover of } l_{i+1} \text{ in } \{p(u) : u \in l_{i+1}\}$;
$C \leftarrow C \cup C_i$;
output $C$. 

\[\text{Layer 0} \quad \text{Layer 1} \quad \text{Layer 2} \quad \text{Layer R}\]
\[ |C| \leq |I| - 1, \]
\[ \text{Rad} \left( G [I \cup C], s \right) \leq 2(R - 1). \]

**Lemma**

For each \( 2 \leq i < R - 1 \), each dominator in \( I_i \) is adjacent to at most 12 connectors in \( C_i \) and at most 11 connectors in \( C_{i+1} \). In addition, \( |C_0| \leq 12 \).
An Equilateral Triangle Property

Figure: The two circles have unit radius, and $1 \leq \|uv\| \leq 2$. Then, both $\triangle pvx$ and $\triangle qvy$ are equilateral.
Lemma

Consider three nodes $u, v$ and $w$ satisfying that $1 < \|uw\| \leq \|uv\| \leq 2$ and $\|vw\| > 1$. If $\overline{vw} \leq 2 \arcsin \frac{1}{4} \approx 28.955^\circ$, then $B(u) \cap B(v) \subseteq B(w)$.

Figure: If $\theta \leq 2 \arcsin \frac{1}{4}$, then $\|uy\| \geq \|uv\|$, and hence $w \in ux \subset \triangle upq$. 
A Geometric Lemma on Angle Separation

\[ \| uy \| \geq \| uv \| \iff \widehat{uy} \geq \widehat{uv} = \widehat{uxy} \iff \widehat{xvy} \geq \theta. \]

\[ \| vz \| = \| uv \| \sin \theta \leq 2 \sin \theta = 4 \sin \frac{\theta}{2} \cos \frac{\theta}{2} \leq \cos \frac{\theta}{2} \]

\[ \Rightarrow \widehat{xvy} = 2 \arccos \| vz \| \geq 2 \arccos \left( \cos \frac{\theta}{2} \right) = \theta. \]

Figure: If \( \theta \leq 2 \arcsin \frac{1}{4} \), then \( \| uy \| \geq \| uv \| \).
Proof of Sparsity

Each dominator $u \in I_i$ is adj. to at most 12 connectors in $C_i$:

Figure: $w_1, w_2, \cdots, w_k$ are the connectors in $C_i$ adjacent to $u$. Each $v_j$ is a private dominator neighbor of $w_j$ in $I_{i+1}$.
Proof of Sparsity

Each dominator \( u \in I_i \) is adj. to at most 12 connectors in \( C_i \):

\[
\begin{align*}
\text{Figure: } w_1, w_2, \cdots, w_k \text{ are the connectors in } C_i \text{ adjacent to } u. \text{ Each } v_j \text{ is a private dominator neighbor of } w_j \text{ in } I_{i+1}.
\end{align*}
\]

If \( k \geq 13 \), then there exist two dominators \( v_{j'} \) and \( v_{j''} \) s.t. \( \angle v_{j'} uv_{j''} \leq \frac{2\pi}{13} \).

Assume by symmetry that \( v_{j''} \) is closer to \( u \) then \( v_{j'} \). Then,

\[
w_{j'} \in B(u) \cap B(v_{j'}) \subseteq B(v_{j''}).
\]
Proof of Sparsity

Each dominator $u \in I_i$ is adj. to at most 11 connectors in $C_{i+1}$:
Figure: The node $v_0$ is $s$. It is adjacent to 12 connectors.
Roadmap

- Three Parameters of CDS
- Dominators
- Basic Set of Connectors
- The First Improvement
- **The Second Improvement**
- The Third Improvement
An Auxiliary Graph on Dominators

- $G'$: the graph on dominators in which there is edge between two dominators iff they have a common neighbor in $G$. 

![Diagram](image-url)
An Auxiliary Graph on Dominators

- $G'$: the graph on dominators in which there is edge between two dominators iff they have a common neighbor in $G$.
- $R'$: radius of $H$ w.r.t. $s$. Then, $R' \leq R - 1$. 

![Diagram](image)
An Auxiliary Graph on Dominators

- $G'$: the graph on dominators in which there is an edge between two dominators if and only if they have a common neighbor in $G$.
- $R'$: radius of $H$ w.r.t. $s$. Then, $R' \leq R - 1$.
- $I_i$ for $0 \leq i \leq R'$: the set of dominators of depth $i$ in $G'$.
The Second Improvement on Connectors

\[ C \leftarrow \emptyset; \]
for each \( 1 \leq i \leq R' - 1 \),
\[ P_i \leftarrow \text{the set of nodes adj. to } l_i \text{ and } l_{i+1}, \]
\[ C_i \leftarrow \text{a minimal cover in } P_i \text{ of } l_{i+1}; \]
\[ C \leftarrow C \cup C_i \]
output \( C \).
\[ |C| \leq |I| - 1, \]

\[ \text{Rad} \left( G \left[ I \cup C \right], s \right) = 2R' \leq 2(R - 1). \]

**Lemma**

\[ |C_0| \leq 12, \text{ and for each } 1 \leq i < R' - 1 \text{ each dominator in } I_i \text{ is adjacent to at most 11 connectors in } C_i. \]
Roadmap

- Three Parameters of CDS
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Ancestors

- \( T \): a BFS tree of \( G \) rooted at \( s \)
- $T$: a BFS tree of $G$ rooted at $s$
- $p^i(v)$ for $v \neq s$: the $i$-th ancestor of $v$ in $T$. 

![BFS Tree Diagram]
$k$: a positive integer parameter

\[
C \leftarrow \emptyset; \\
\text{for each dominator } u \neq s, \\
i \leftarrow \min \{ j : p^i(u) \in I \}; \\
C \leftarrow C \cup \{ p^j(u) : 1 \leq j \leq \min \{ i - 1, k \} \}; \\
\text{output } C.
\]
\[ |C| \leq k (|I| - 1). \]

Hence,

\[
|I \cup C| \leq |I| + k (|I| - 1) = (k + 1)|I| - k
\]
\[
\leq (k + 1) \left( \frac{2\pi}{\sqrt{3}}R^2 + \pi R + 1 \right) - k
\]
\[
= (k + 1) \left( \frac{2\pi}{\sqrt{3}}R^2 + \pi R \right) + 1.
\]
Radius

Denote $H = G[I \cup C]$. It is sufficient to show that for each dominator $u$,

$$\text{dist}_{H}(u, s) \leq (1 + 1/k) \text{dist}_{G}(u, s).$$

Induction on $\text{dist}_{G}(u, s)$: Trivial if $\text{dist}_{G}(u, s) = 0$. So, we assume that $\text{dist}_{G}(u, s) > k$ and let $i = \min j: p_j(u) \in I$. 

$$\text{Rad}(G[I \cup C], s) \leq (1 + 1/k) R$$
Denote $H = G[I \cup C]$. It is sufficient to show that for each dominator $u$,

$$dist_H(u, s) \leq \left(1 + \frac{1}{k}\right) dist_G(u, s).$$
Radius

\[ \text{Rad}(G[l \cup C], s) \leq (1 + 1/k) R \]

Denote \( H = G[l \cup C] \). It is sufficient to show that for each dominator \( u \),

\[ \text{dist}_H(u, s) \leq \left(1 + \frac{1}{k}\right) \text{dist}_G(u, s). \]

Induction on \( \text{dist}_G(u, s) \): Trivial if \( \text{dist}_G(u, s) = 0 \).
\[ \text{Rad} (G [I \cup C], s) \leq (1 + 1/k) R \]

Denote \( H = G [I \cup C] \). It is sufficient to show that for each dominator \( u \),

\[ \text{dist}_H (u, s) \leq \left( 1 + \frac{1}{k} \right) \text{dist}_G (u, s). \]

Induction on \( \text{dist}_G (u, s) \): Trivial if \( \text{dist}_G (u, s) = 0 \). So, we assume that \( \text{dist}_G (u, s) > k \) and let \( i = \min \{ j : p^j (u) \in I \} \).
Case 1: $i \leq k + 1$. Let $v = p^i(u)$. Then,

$$dist_H(u, s) \leq dist_H(v, s) + i$$

$$\leq \left(1 + \frac{1}{k}\right) dist_G(v, s) + i$$

$$\leq \left(1 + \frac{1}{k}\right) (dist_G(v, s) + i)$$

$$= \left(1 + \frac{1}{k}\right) dist_G(u, s).$$
**Induction: Case 2**

**Case 2:** $i > k + 1$ and $p^k(u)$ is adjacent to some dominator $v$ at the same layer as $p^{k+1}(u)$.

\[
\begin{align*}
\text{dist}_H(u, s) & \leq \text{dist}_H(v, s) + k + 1 \\
& \leq \left(1 + \frac{1}{k}\right) \text{dist}_G(v, s) + k + 1 \\
& < \left(1 + \frac{1}{k}\right) (\text{dist}_G(v, s) + k + 1) \\
& = \left(1 + \frac{1}{k}\right) \text{dist}_G(u, s).
\end{align*}
\]
Case 3: $i > k + 1$ and $p^k(u)$ is adj. to some dominator $v$ at the same layer as itself. Then,

\[
\text{dist}_H(u, s) \leq \text{dist}_H(v, s) + k + 1 \\
\leq \left(1 + \frac{1}{k}\right) \text{dist}_G(v, s) + (k + 1) \\
= \left(1 + \frac{1}{k}\right) (\text{dist}_G(v, s) + k) \\
= \left(1 + \frac{1}{k}\right) \text{dist}_G(u, s).
\]
\[ \Delta(H) \leq 2\sqrt{3}\pi k^2 + 3\pi k + 3 + \frac{4\pi}{\sqrt{3}}. \]
For each $v \in C$, $q(v)$ denotes the closest descendant dominator of $v$; for each $v \in I$, $q(v) = v$.

\[ \Delta(H) \leq 2\sqrt{3}\pi k^2 + 3\pi k + 3 + \frac{4\pi}{\sqrt{3}}. \]
\[ \Delta(H) \leq 2\sqrt{3}\pi k^2 + 3\pi k + 3 + 4\pi / \sqrt{3}. \]

For each \( v \in C \), \( q(v) \) denotes the closest descendant dominator of \( v \); for each \( v \in I \), \( q(v) = v \). Then,

\[ \text{dist}_G(v, q(v)) \leq k. \]
Sparsity

\[ \Delta (H) \leq 2\sqrt{3} \pi k^2 + 3\pi k + 3 + 4\pi / \sqrt{3}. \]

For each \( v \in C \), \( q(v) \) denotes the closest descendant dominator of \( v \); for each \( v \in I \), \( q(v) = v \). Then,

\[ \text{dist}_G (v, q(v)) \leq k. \]

Consider a node \( u \in I \cup C \). For each \( v \in N_H (u) \),

\[ \text{dist}_G (u, q(v)) \leq \text{dist}_G (u, v) + \text{dist}_G (v, q(v)) \leq k + 1. \]
Let $S_1(u)$ (resp. $S_2(u)$, $S_3(u)$) be the set of dominators which are at most $k - 1$ (resp. $k$, $k + 1$) hops away from $u$. Then,

$$|N_H(u)|$$

$$\leq 3|S_1(u)| + 2|S_2(u) \setminus S_1(u)| + |S_3(u) \setminus S_2(u)|$$

$$= |S_1(u)| + |S_2(u)| + |S_3(u)|$$

$$\leq \sum_{i=k-1}^{k+1} \left( \frac{2\pi}{\sqrt{3}} i^2 + \pi i + 1 \right)$$

$$= 2\sqrt{3}\pi k^2 + 3\pi k + 3 + \frac{4\pi}{\sqrt{3}}.$$