# CDS in Plane Geometric Networks: Short, Small, And Sparse

Peng-Jun Wan

# Outline

- Three Parameters of CDS
- Dominators
- Basic Set of Connectors
- The First Improvement
- The Second Improvement
- The Third Improvement

# Three Parameters of A Virtual Backbone (CDS)

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$s \in V$: a fixed node
$R$: radius of $G$ w.r.t. $s$.

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$U$: a CDS of $G$ containing $s$

- Size: $|U|$
- Radius: $Rad(G[U], s)$
- Sparsity: $\Delta(G[U])$

# Virtual Backbone (CDS): Small, Short, Sparse

Our objective is to construct a CDS $U$ of $G$ with $s \in U$ such that

1. $U$ is small: $|U| = \Theta(poly(R))$
2. $U$ is short: $Rad(G[U], s) = \Theta(R)$
3. $U$ is sparse: $\Delta(G[U])$ is small, preferably bounded by a constant.

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- Phase 2: Augment $I$ with a set $C$ of "connectors" to form a CDS

# Roadmap

- Three Parameters of CDS
- **Dominators**
- Basic Set of Connectors
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# Sparsity of Dominators

- Each node is adjacent to at most 5 dominators.

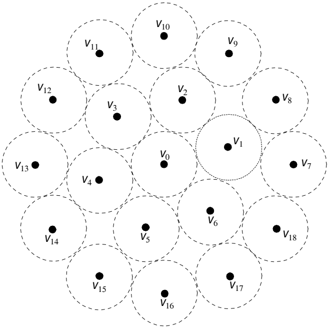
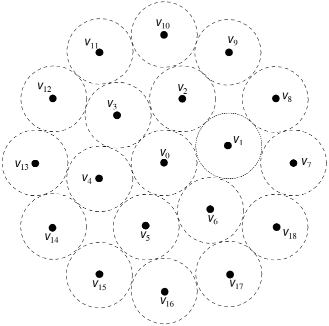



Figure: The layout of 18 dominators in the annulus of radii $1 + 2\varepsilon$ and $(1 + 2\varepsilon)\sqrt{2 + \sqrt{3}}$ centered at a node $v_0$.

# Sparsity of Dominators

- Each node is adjacent to at most 5 dominators.
- The annulus of radii one and two centered at each node contains at most 18 dominators (Bateman and Erdös)

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# Groemer Inequality

**Theorem**

*Suppose that $S$ is a compact convex set and $U$ is a set of points with mutual distances at least one. Then*

$$|U \cap S| \leq \frac{area\,(S)}{\sqrt{3}/2} + \frac{peri\,(S)}{2} + 1,$$

*where $area\,(S)$ and $peri\,(S)$ are the area and perimeter of $S$ respectively.*

# Packing in Disks and Half-Disks

## Corollary

*Suppose that $S$ (respectively, $S'$) is a disk (respectively, half-disk) of radius $r$, and $U$ is a set of points with mutual distances at least one. Then*

$$|U \cap S| \leq \frac{2\pi}{\sqrt{3}} r^2 + \pi r + 1,$$
$$|U \cap S'| \leq \frac{\pi}{\sqrt{3}} r^2 + \left(\frac{\pi}{2} + 1\right) r + 1.$$

# An Upper Bound on The Number of Dominators

Since all dominators lie in the disk of radius $R$ centered at $s$, we have

$$|I| \leq \left( \frac{2\pi}{\sqrt{3}} R^2 + \pi R + 1 \right).$$

# Roadmap

- Three Parameters of CDS
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- **Basic Set of Connectors**
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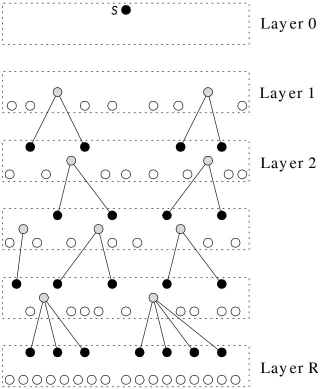
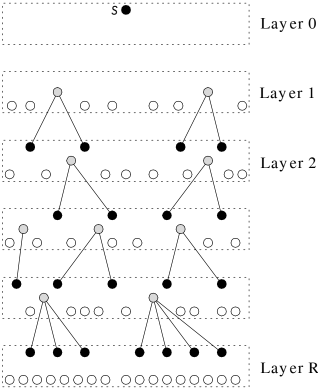
# Basic Set of Connectors

- $\forall u \in I \setminus \{s\}$, $p(u) \leftarrow$ the least-ID neighbor of $u$ in the layer above $u$.



# Basic Set of Connectors

- $\forall u \in I \setminus \{s\}$, $p(u) \leftarrow$ the least-ID neighbor of $u$ in the layer above $u$.
- output $C = \{p(u) : u \in I \setminus \{s\}\}$.



# Size And Radius

$$|C| \leq |I \setminus \{s\}| = |I| - 1.$$

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Hence,

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Hence,

$$|I \cup C| \leq 2|I| - 1 \leq \frac{4\pi}{\sqrt{3}} R^2 + 2\pi R + 1.$$

In addition,

$$Rad(G[I \cup C], s) \leq 2(R-1).$$

# Sparsity

- Each dominatee is adj. to at most 4 dominators in the layer below itself.

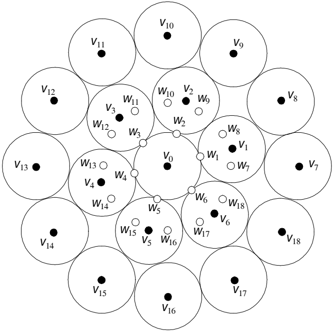


Figure: The node $v_0$ is $s$. It is adjacent to 16 connectors.

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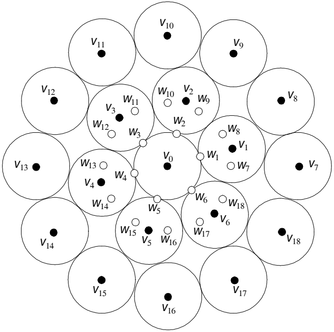


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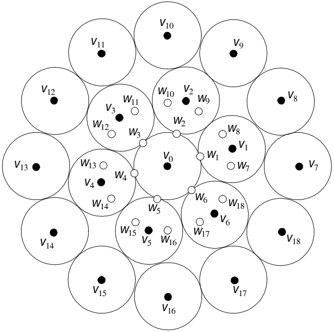
# Sparsity

- Each dominatee is adj. to at most 4 dominators in the layer below itself.
- $s$ is adj. to at most 18 connectors
- Each other dominator is adj. to at most 17 connectors in the same or the next layer.

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# Roadmap

# Minimal Cover

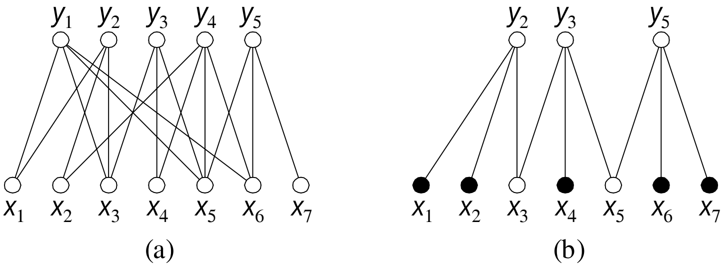
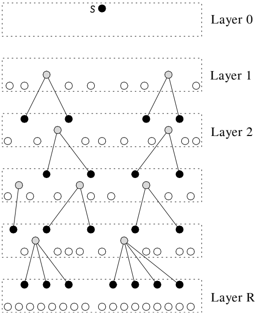



Figure: (a) $X = \{x_i : 1 \leq i \leq 7\}$ is covered by $Y = \{y_i : 1 \leq i \leq 5\}$. (b) $\{y_2, y_3, y_4\}$ is a minimal cover of $X$. The black nodes are private neighbors.

# The First Improvement on Connectors

for each $0 \leq i \leq R$, $I_i \leftarrow$ {dominators of depth $i$ in $G$};
$C \leftarrow \emptyset$;
for each $1 \leq i \leq R - 1$,
  $C_i \leftarrow$ a minimal cover of $I_{i+1}$ in $\{p(u) : u \in I_{i+1}\}$;
  $C \leftarrow C \cup C_i$;
output $C$.



# Size, Radius And Sparsity

$$|C| \leq |I| - 1,$$
$$Rad\left(G\left[I \cup C\right], s\right) \leq 2\left(R - 1\right).$$

**Lemma**

*For each $2 \leq i < R - 1$, each dominator in $I_i$ is adjacent to at most 12 connectors in $C_i$ and at most 11 connectors in $C_{i+1}$. In addition, $|C_0| \leq 12$.*

# An Equilateral Triangle Property

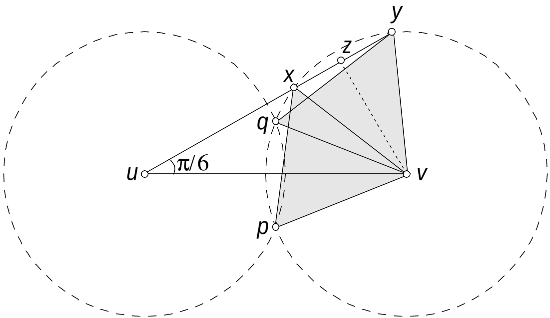



Figure: The two circles have unit radius, and $1 \leq \|uv\| \leq 2$. Then, both $\triangle pvx$ and $\triangle qvy$ are equilateral.

# A Geometric Lemma on Angle Separation

## Lemma

*Consider three nodes u, v and w satisfying that* $1 < \|uw\| \leq \|uv\| \leq 2$ *and* $\|vw\| > 1$*. If* $\widehat{vuw} \leq 2 \arcsin \frac{1}{4} \approx 28.955^{\circ}$*, then* $B(u) \cap B(v) \subseteq B(w)$*.*

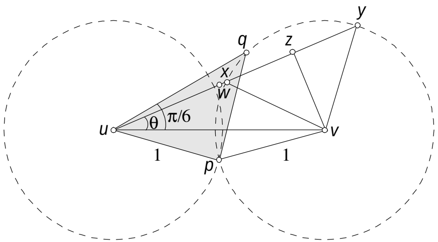



Figure: If $\theta \leq 2 \arcsin \frac{1}{4}$, then $\|uy\| \geq \|uv\|$, and hence $w \in ux \subset \triangle upq$.

# A Geometric Lemma on Angle Separation

$$\|uy\| \geq \|uv\| \Leftrightarrow \widehat{uvy} \geq \widehat{uyv} = \widehat{uxy} \Leftrightarrow \widehat{xvy} \geq \theta.$$

$$\|vz\| = \|uv\| \sin\theta \leq 2\sin\theta = 4\sin\frac{\theta}{2}\cos\frac{\theta}{2} \leq \cos\frac{\theta}{2}$$

$$\Rightarrow \widehat{xvy} = 2\arccos\|vz\| \geq 2\arccos\left(\cos\frac{\theta}{2}\right) = \theta.$$

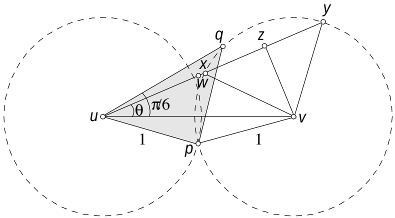



Figure: If $\theta \leq 2\arcsin\frac{1}{4}$, then $\|uy\| \geq \|uv\|$.

# Proof of Sparsity

Each dominator $u \in I_i$ is adj. to at most 12 connectors in $C_i$:

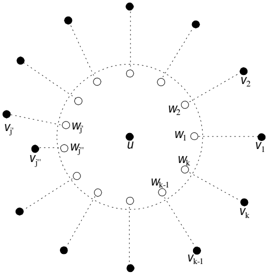



Figure: $w_1, w_2, \cdots, w_k$ are the connectors in $C_i$ adjacent to $u$. Each $v_j$ is a private dominator neighbor of $w_j$ in $I_{i+1}$.

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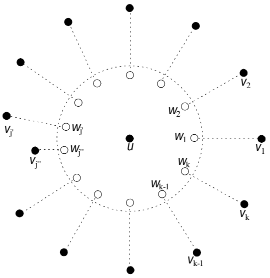


Figure: $w_1, w_2, \cdots, w_k$ are the connectors in $C_i$ adjacent to $u$. Each $v_j$ is a private dominator neighbor of $w_j$ in $I_{i+1}$.

If $k \geq 13$, then there exist two dominators $v_{j'}$ and $v_{j''}$ s.t. $\angle v_{j'} u v_{j''} \leq \frac{2\pi}{13}$. Assume by symmetry that $v_{j''}$ is closer to $u$ then $v_{j'}$. Then,

$$w_{j'} \in B(u) \cap B(v_{j'}) \subseteq B(v_{j''}).$$

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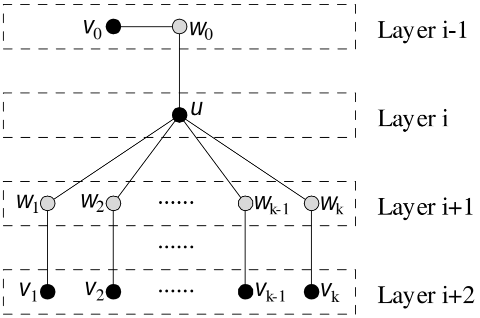

# An Example

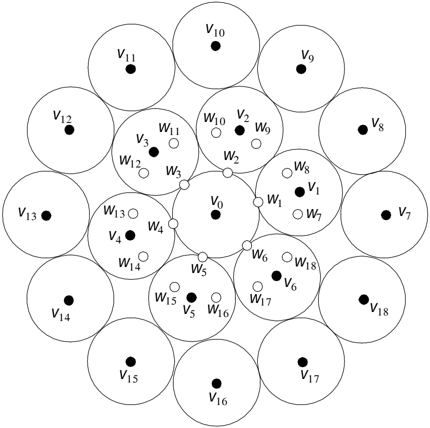


Figure: The node $v_0$ is $s$. It is adjacent to 12 connectors.

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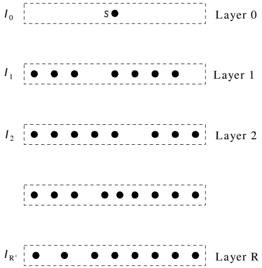
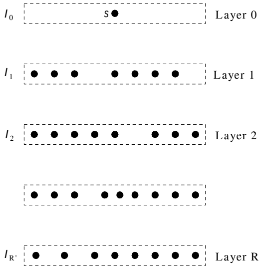
# An Auxiliary Graph on Dominators

- $G'$: the graph on dominators in which there is edge between two dominators iff they have a common neighbor in $G$.



# An Auxiliary Graph on Dominators

- $G'$: the graph on dominators in which there is edge between two dominators iff they have a common neighbor in $G$.
- $R'$: radius of $H$ w.r.t. $s$. Then, $R' \leq R - 1$.



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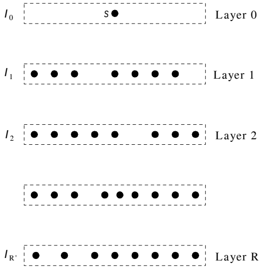
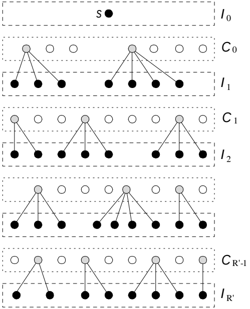
- $R'$: radius of $H$ w.r.t. $s$. Then, $R' \leq R - 1$.
- $I_i$ for $0 \leq i \leq R'$: the set of dominators of depth $i$ in $G'$.



# The Second Improvement on Connectors

$C \leftarrow \emptyset$;
for each $1 \leq i \leq R' - 1$,
  $P_i \leftarrow$ the set of nodes adj. to $I_i$ and $I_{i+1}$,
  $C_i \leftarrow$ a minimal cover in $P_i$ of $I_{i+1}$;
  $C \leftarrow C \cup C_i$
output $C$.



# Size, Radius And Sparsity

$$|C| \leq |I| - 1,$$
$$Rad\left(G\left[I \cup C\right], s\right) = 2R' \leq 2\left(R - 1\right).$$

**Lemma**

*$|C_0| \leq 12$, and for each $1 \leq i < R' - 1$ each dominator in $I_i$ is adjacent to at most 11 connectors in $C_i$.*

# Roadmap

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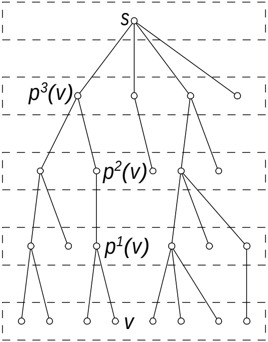
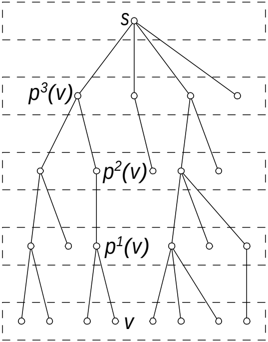
# Ancestors

- $T$: a BFS tree of $G$ rooted at $s$



# Ancestors

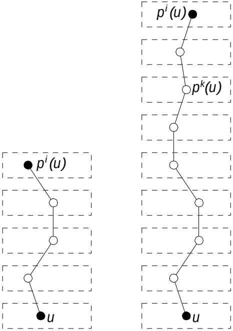
- $T$: a BFS tree of $G$ rooted at $s$
- $p^i(v)$ for $v \neq s$: the $i$-th ancestor of $v$ in $T$.



# The Third Improvement on Connectors

$k$: a positive integer parameter

| |
|---|
| $C \leftarrow \emptyset$; |
| for each dominator $u \neq s$, |
| $\quad i \leftarrow \min\{j : p^j(u) \in I\}$; |
| $\quad C \leftarrow C \cup \{p^j(u) : 1 \leq j \leq \min\{i-1, k\}\}$; |
| output $C$. |




(a) (b)

# Size

$$|C| \leq k\,(|I| - 1)\,.$$

Hence,

$$\begin{aligned}|I \cup C| &\leq |I| + k\,(|I| - 1) = (k+1)\,|I| - k \\ &\leq (k+1)\left(\frac{2\pi}{\sqrt{3}}R^2 + \pi R + 1\right) - k \\ &= (k+1)\left(\frac{2\pi}{\sqrt{3}}R^2 + \pi R\right) + 1.\end{aligned}$$

# Radius

$$Rad\left(G\left[I\cup C\right],s\right)\leq\left(1+1/k\right)R$$

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Denote $H = G\left[I \cup C\right]$. It is sufficient to show that for each dominator $u$,

$$dist_H\left(u, s\right) \leq \left(1 + \frac{1}{k}\right) dist_G\left(u, s\right).$$

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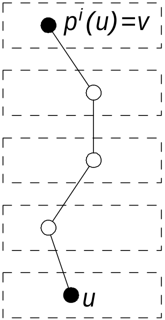
Induction on $dist_G\left(u, s\right)$: Trivial if $dist_G\left(u, s\right) = 0$. So, we assume that $dist_G\left(u, s\right) > k$ and let $i = \min\left\{j : p^j\left(u\right) \in I\right\}$.

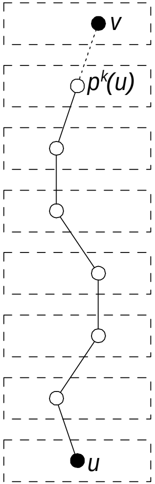
# Induction: Case 1




**Case 1**: $i \leq k+1$. Let $v = p^i(u)$. Then,

$$
\begin{aligned}
dist_H(u,s) &\leq dist_H(v,s) + i \\
&\leq \left(1+\frac{1}{k}\right) dist_G(v,s) + i \\
&< \left(1+\frac{1}{k}\right)\left(dist_G(v,s) + i\right) \\
&= \left(1+\frac{1}{k}\right) dist_G(u,s).
\end{aligned}
$$

# Induction: Case 2




**Case 2**: $i > k+1$ and $p^k(u)$ is adjacent to some dominator $v$ at the same layer as $p^{k+1}(u)$.

$$
\begin{aligned}
dist_H(u,s) &\leq dist_H(v,s) + k + 1 \\
&\leq \left(1+\frac{1}{k}\right) dist_G(v,s) + k + 1 \\
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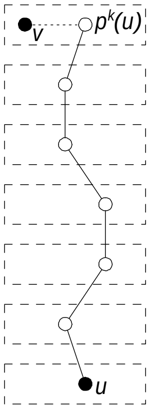
\end{aligned}
$$

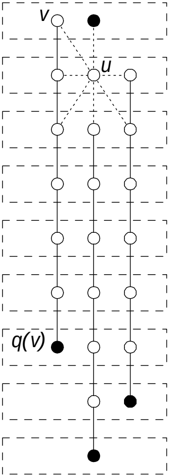
# Induction: Case 3




**Case 3**: $i > k+1$ and $p^k(u)$ is adj. to some dominator $v$ at the same layer as itself. Then,

$$\begin{aligned} dist_H(u,s) &\le dist_H(v,s) + k + 1 \\ &\le \left(1+\frac{1}{k}\right) dist_G(v,s) + (k+1) \\ &= \left(1+\frac{1}{k}\right)(dist_G(v,s)+k) \\ &= \left(1+\frac{1}{k}\right) dist_G(u,s). \end{aligned}$$

# Sparsity




$$\Delta(H) \leq 2\sqrt{3}\pi k^2 + 3\pi k + 3 + 4\pi/\sqrt{3}.$$

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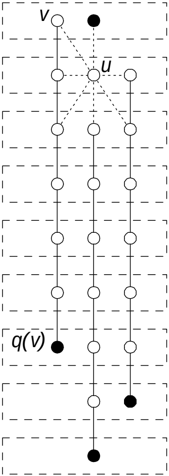


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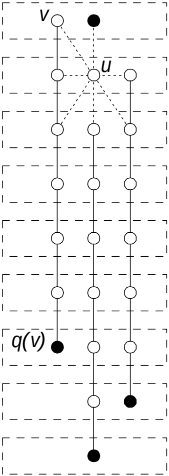
For each $v \in C$, $q(v)$ denotes the closest descendant dominator of $v$; for each $v \in I$, $q(v) = v$.

# Sparsity

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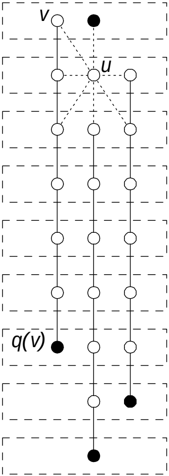
$$dist_G(v, q(v)) \le k.$$

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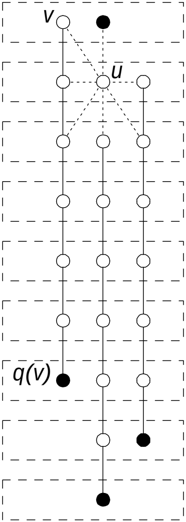
For each $v \in C$, $q(v)$ denotes the closest descendant dominator of $v$; for each $v \in I$, $q(v) = v$. Then,

$$dist_G(v, q(v)) \leq k.$$

Consider a node $u \in I \cup C$. For each $v \in N_H(u)$,

$$\begin{aligned} dist_G(u, q(v)) &\leq dist_G(u, v) + dist_G(v, q(v)) \\ &\leq k + 1. \end{aligned}$$

# Sparsity




Let $S_1(u)$ (resp. $S_2(u)$, $S_3(u)$) be the set of dominators which are at most $k-1$ (resp. $k$, $k+1$) hops away from $u$. Then,

$$
\begin{aligned}
&|N_H(u)| \\
&\leq 3|S_1(u)| + 2|S_2(u) \setminus S_1(u)| + |S_3(u) \setminus S_2(u)| \\
&= |S_1(u)| + |S_2(u)| + |S_3(u)| \\
&\leq \sum_{i=k-1}^{k+1} \left( \frac{2\pi}{\sqrt{3}} i^2 + \pi i + 1 \right) \\
&= 2\sqrt{3}\pi k^2 + 3\pi k + 3 + \frac{4\pi}{\sqrt{3}}.
\end{aligned}
$$