

Preliminaries on Graphs

(Draft)

Abstract

This appendix is not meant as a rush course in graph theory, but rather as a reference guide and to settle notation and terminology. To promote readability of the book, nonstandard notation and terminology will be, besides below in this appendix, also explained on the spot in the main chapters.

1 Undirected graphs

A (simple) *undirected graph*, or just *graph*, is a pair $G = (V, E)$, where V is a finite set and E is a set of *unordered* pairs from V . The elements of V are called the *vertices* (sometimes the *nodes*), and the elements of E are called the *edges*. Figure 1(a) is a pictorial representation of a graph on seven vertices. Vertices are represented by circles in the figure, and arcs are represented by lines or curves. By convention, we use uv as shorthand notation for the edge $\{u, v\}$. We say that an edge $e = uv$ *joins* (sometimes *connects*, or *covers*, or be *incident on*) u and v . In this case, u and v are the *ends* of e and *incident with* e , u and v are *adjacent*, v is a *neighbor* of u (and vice versa). Two edges are said to be *adjacent* if share a common end; otherwise, they are called *disjoint*. For any $U \subseteq V$, a vertex v is called a *neighbor* of U if $v \notin U$ and v has a neighbor in U . If $U \subseteq V$ and both ends of an edge e belong to U , then we say that U *spans* e . If at least one end of e belongs to U , then U is said to be *incident with* e . An edge joining a vertex in a set S and a vertex in a set T is said to *join* S and T . A set F of edges is said to *cover* a vertex v if v is covered by at least one edge in F , and to *miss* v otherwise.

For a vertex v , $N_G(v)$ denotes the set of neighbors of v and $\delta_G(v)$ denotes the set of edges incident on v . Here and below, notation with the subscript omitted is used if the graph is clear from the context. The *degree* $\deg_G(v)$ of a vertex v is $|\delta_G(v)|$, the number of edges incident on v . A vertex of degree 0 is called *isolated*. The maximum (respectively, minimum) degree of the vertices of G is denoted by $\Delta(G)$ (respectively, $\delta(G)$) and is called the *maximum* (respectively, *minimum*) *degree* of G . In Figure 1(a), $\Delta(G) = \deg(v_1) = 6$ and $\delta(G) = \deg(v_4) = 3$.

For any $U \subseteq V$, $N_G(U)$ denotes the set of neighbors of U and $\delta_G(U)$ denotes the set of edges joining U and $V \setminus U$. A subset F of E is called a *cut* if $F = \delta_G(U)$ for some $U \subseteq V$. If $s \in U$ and $t \notin U$, then $\delta_G(U)$ is called an $s - t$ *cut*. If $S \subseteq U$ and $T \subseteq V \setminus U$, $\delta_G(U)$ is called an $S - T$ *cut*.

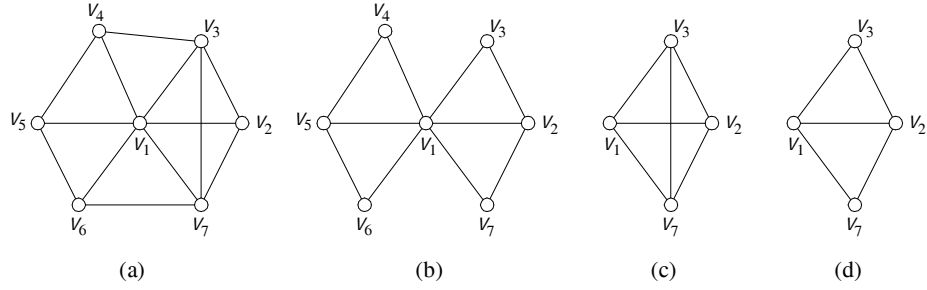


Figure 1: (a). A graph G ; (b) a spanning but not induced subgraph of G ; (c) an induced but not spanning subgraph of G ; (d) a subgraph which is nether spanning nor induced.

Subgraphs

A graph $G' = (V', E')$ is called a *subgraph* of a graph $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. If G' is a subgraph of G , we say that G *contains* G' . If $G' \neq G$, then G' is called a *proper subgraph* of G . If $V' = V$, then G' is called a *spanning subgraph* of G . If E' consists of all edges of G spanned by V' , G' is called an *induced subgraph*, or *the subgraph induced by V'* . We use $G[V']$ to denote subgraph of G induced by V' . In Figure 1, all the three graphs shown in (b), (c) and (d) are subgraphs of the graph in (a): (b) is spanning but not induced; (c) is induced but not spanning; and (d) is nether spanning nor induced.

Paths and circuits

Let

$$P = \langle v_0, v_1, \dots, v_k \rangle$$

be a sequence of vertices in $G = (V, E)$ such that $v_{i-1}v_i \in E$ for $1 \leq i \leq k$. Then, P is a *walk* if all edges in P are distinct, a *path* if all vertices in P are distinct, and a *circuit* if $v_0 = v_k$ and v_1, \dots, v_k are all distinct. By definition, both path and circuit and walks. The walk P is also identified with a subgraph with vertices v_0, v_1, \dots, v_k and edges $v_0v_1, v_1v_2, \dots, v_{k-1}v_k$. The length of P is k , the number of edges. Figure 2(a) is an example of walk.

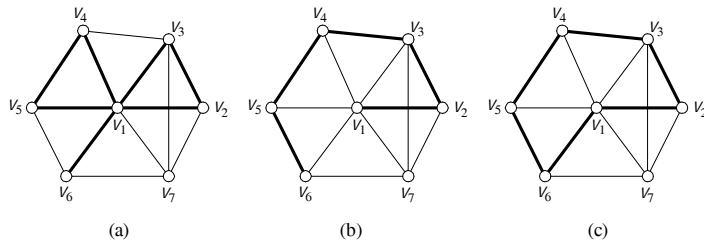


Figure 2: (a) A walk; (b) a path; and (c) a circuit.

The walk P is called a *path* if v_0, v_1, \dots, v_k are all distinct. The vertex v_0 is called the starting (or

first) vertex of P and the vertex v_k the end (or last) vertex of P . Sometimes, both v_0 and v_k are called the end vertices, or just the ends of P . The vertices v_1, \dots, v_{k-1} are called the internal vertices of P . P is also referred to as an $s - t$ path. A *subpath* of P is a contiguous subsequence of its vertices. The minimum length of a path from u to v is called the *distance* of u and v , which is denoted by $dist_G(u, v)$. The maximum distance over all vertices u, v of G is called the *diameter* of G , which is denoted by $diam(G)$. A spanning path in G is called a *Hamiltonian path*.

It is called a *circuit* if $v_k = v_0$ and v_1, \dots, v_{k-1} are all distinct. A spanning circuit in G is called a *Hamiltonian circuit*. A graph containing a Hamiltonian circuit is called *Hamiltonian*.

Connectivity and components

A graph $G = (V, E)$ is *connected* if for any two vertices u and v there is a path from u to v . A maximal connected nonempty subgraph of G is called a *connected component*, or just a *component*, of G . Here “maximal” is taken with respect to taking subgraphs. Each component is an induced subgraph, and each vertex and each edge of G belong to exactly one component. We often identify a component with the set of its vertices. Then the components are precisely the equivalence classes of the equivalence relation \sim on V defined by: $u \sim v$ if and only if there exists a path from u to v . A set of vertices U is called *connected* if the subgraph induced by U is connected. A vertex v is called an *articulation* if $G - v$ has more connected components than G . An edge e is called a *bridge* if $G - e$ has more connected components than G .

Trees and forests

A *tree* is a connected graph without a circuit. So for any tree (V, E) , $|E| = |V| - 1$. A vertex of degree 1 in a tree is called a *leaf*. Any tree with at least one edge has at least two leaves. A connected subgraph of a tree T is called a *subtree* of T . A *star* is a tree where at most one vertex is not a leaf, and a *spider* is a tree where at most one vertex has degree at most three. The graph in Figure 3(a) is a spider, and the graph in Figure 3(b) is a star. A graph without a circuit is called a *forest*. Each component of a forest is a tree. Any forest (V, E) has $|E| - |V|$ tree components.

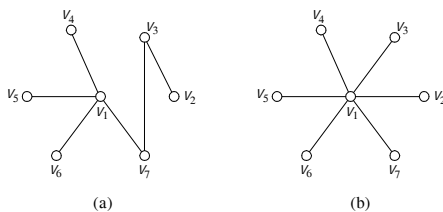


Figure 3: (a) A tree which is a spider, (b) a tree which is a star.

Independent sets, Cliques, and Matchings

A subset U of V is an *independent set* of G if no two vertices in U are adjacent. If U is a independent set of G but no proper superset of U is a independent set of G , then U is called a *maximal independent*

set (MIS) of G . An independent set of the largest size is called a *maximum independent set*. The *independence number* of G , denoted by $\alpha(G)$, is the size of a maximum independent set in G . For example, in the graph shown in Figure 1, all the three subsets

$$\{v_1\}, \{v_2, v_5\}, \{v_2, v_4, v_6\}$$

are maximal, but only the last one is maximum. Thus, the independence number of the graph shown in Figure 1 is three.

A subset U of V is a *clique* of G if any pair of vertices in U are adjacent. If U is a clique of G but no proper superset of U is a clique of G , then U is called a *maximal clique* of G . A clique of the largest size is called a *maximum clique*. The *clique number* of G , denoted by $\omega(G)$, is the size of a maximum clique in G . For example, in the graph shown in Figure 1, both subsets

$$\{v_1, v_3, v_4\}, \{v_1, v_2, v_3, v_7\}$$

are maximal, but only the latter one is maximum. Thus, the clique number of the graph shown in Figure 1 is four.

A vertex coloring of G is an assignment of colors to all vertices in V satisfying that adjacent vertices are assigned with distinct colors. The *chromatic number* of G , denoted by $\chi(G)$, is the smallest number of colors required by any vertex coloring of G . A vertex coloring corresponds to a partition of V into independent sets, and hence

$$\chi(G) \geq \frac{|V|}{\alpha(G)}.$$

Since all nodes in a clique must receive distinct colors, we have

$$\chi(G) \geq \omega(G).$$

Figure 4 presents a coloring of the graph shown in Figure 1. In this example, four colors are used. Since the clique number of the graph is also four, the coloring in Figure 4 is minimal.

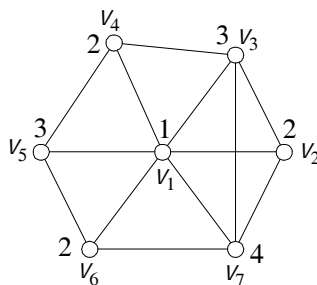


Figure 4: An example of coloring.

A subset M of E is called a *matching* if any two edges in M are disjoint, an *edge cover* if each vertex of G is covered by at least one edge in M , and a *perfect matching* if it is both a matching and an edge cover. So a perfect matching M satisfies $|M| = |V|/2$. Given a matching M in a graph $G = (V, E)$, we will say that a vertex u is matched to a vertex v , or u is the mate of v , if $uv \in M$. A subset U of V is called *matchable* if the subgraph $G[U]$ of G induced by U has a perfect matching.

Dominating Set and Connected Dominating Set

A subset U of V is a *dominating set* (DS) of G if any vertex not in U has a neighbor in U . In Figure 5(a), the set of black nodes form a DS. A dominating set of the smallest size is called a *minimum independent set*. The *domination number* of G , denoted by $\gamma(G)$, is the size of a minimum DS in G . Clearly, any maximal IS is a DS, which implies that $\gamma(G) \leq \alpha(G)$.

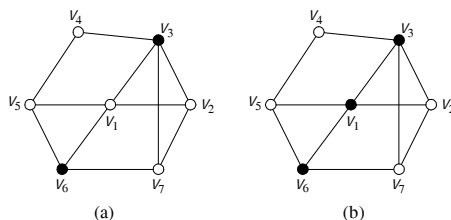


Figure 5: (a) The black nodes form a DS; (b) the black nodes form a CDS.

A subset U of V is a *connected dominating set* (CDS) of G if U is a DS of G and $G[U]$ is connected. In Figure 5(b), the set of black nodes form a CDS. A CDS of the smallest size is called a *minimum CDS*. The *connected domination number* of G , denoted by $\gamma_c(G)$, is the size of a minimum CDS in G .

2 Directed graphs

A (simple) *directed graph* or *digraph* is a pair $D = (V, A)$ where V is a finite set and A is a set of *ordered* pairs from V . The elements of V are called the *vertices* (sometimes the *nodes*), and the elements of A are called the *arcs* (sometimes the *links*). Figure 6 is a pictorial representation of a digraph on seven vertices. Vertices are represented by circles in the figure, and arcs are represented by arrows. We say that an edge $a = (u, v)$ *joins* (sometimes *connects*, or *covers*, or *incident on*) u and v , leaves u , and enters v . In this case, u is the *tail* of a , and v the *head* of a , u and v are the *ends* of e and *incident with* e , u and v are *adjacent*, v is an *out-neighbor* of u , and u is an *in-neighbor* of v . Two arcs are said to be *adjacent* if share a common end; otherwise, they are called *disjoint*. For any $U \subseteq V$, a vertex v is called a *out-neighbor* (respectively, *in-neighbor*) of U if $v \notin U$ and v is an out-neighbor (respectively, in-neighbor) of some vertex in U . For $U \subseteq V$, an arc $a = (u, v)$ is said to *leave* (respectively, *enter*) U if $u \in U$ and $v \notin U$ (respectively, $u \notin U$ and $v \in U$). If $U \subseteq V$ and both ends of an arc a belong to U , then we say that U *spans* a .

$V' = V$, then D' is called a *spanning subgraph* of D . If A' consists of all arcs of D spanned by V' , D' is called an *induced subgraph*, or *the subgraph induced by V'* . We use $D[V']$ to denote subgraph of D induced by V' .

Directed paths and circuits

A *directed walk*, or just a *walk*, of *length k* (or k hops) from a vertex s to a vertex t in a digraph $D = (V, A)$ is a sequence

$$P = \langle v_0, v_1, \dots, v_k \rangle$$

of vertices such that $s = v_0$, $t = v_k$, and (v_{i-1}, v_i) for $i = 1, \dots, k$ are k distinct arcs in A . The walk P *contains* the vertices v_0, v_1, \dots, v_k and arcs $(v_0, v_1), (v_1, v_2), \dots, (v_{k-1}, v_k)$.

The walk P is called a *directed path*, or just a *path*, if v_0, v_1, \dots, v_k are all distinct. The vertex v_0 is called the starting (or first) vertex of P and the vertex v_k the end (or last) vertex of P . Sometimes, both v_0 and v_k are called the *end vertices*, or just the *ends* of P . The vertices v_1, \dots, v_{k-1} are called the internal vertices of P . P is also referred to as an $s - t$ path. A *subpath* of P is a contiguous subsequence of its vertices. The minimum length of a path from u to v is called the *distance* of u and v , which is denoted by $dist_G(u, v)$. The maximum distance over all vertices u, v of G is called the *diameter* of G , which is denoted by $diam(G)$. A spanning path in G is called a *Hamiltonian path*.

The walk P is called *closed* if $v_k = v_0$. It is called a *directed circuit*, or just a *circuit*, if $v_k = v_0$ and v_1, \dots, v_k are all distinct. A spanning circuit in G is called a *Hamiltonian circuit*. A graph containing a Hamiltonian circuit is called *Hamiltonian*.

A vertex t is called *reachable from* a vertex s if there exists a directed $s - t$ path. Similarly, a vertex s is said to *reach*, or to be *reachable to*, a vertex t if there exists a directed $s - t$ path.

Connectivity and components of digraphs

A digraph $D = (V, A)$ is called *strongly connected* if for each two vertices u and v there is a directed path from u to v . A maximal strongly connected nonempty subgraph of a digraph $D = (V, A)$ is called a *strongly connected component*, or a *strong component*, of D . Again, “maximal” is taken with respect to taking subgraphs. Figure 7 shows an example of a digraph with three strong components illustrated in shaded regions. Each strong component is an induced subgraph. Each vertex belongs to exactly one strong component, but there may be arcs that belong to no strong component. An arc (u, v) belongs to a strong component if and only if there exists a directed path in D from v to u . We sometimes identify a strong component with the set of its vertices. Then the strong components are precisely the equivalence classes of the equivalence relation \sim on V defined by: $u \sim v$ if and only if there exist a directed path from u to v and a directed path from v to u . A set of vertices U is called *strongly connected* if the subgraph induced by U is strongly connected. A vertex v is called an *articulation* if $G - v$ has more connected components than G . An edge e is called a *bridge* if $G - e$ has more connected components than G .

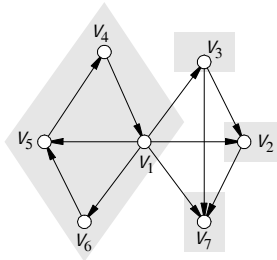


Figure 7: A digraph with three strong components.

A digraph D is called *weakly connected* if its undirected version is connected. The digraph in Figure 7 is weakly connected, but not strongly connected. A *weakly connected component*, or a *weak component*, of D is a component of its undirected version.

Acyclic digraphs and directed trees

A digraph without directed circuit is called *acyclic*. It is easy to show that an acyclic digraph has at least one source and at least one sink, provided that it has at least one vertex. A digraph is called a *directed forest* (respectively, *directed tree*) if its undirected version is a forest (respectively, tree). Figure 8 (a) shows an example of an acyclic digraph, and Figure 8 (a) shown an example of directed tree.

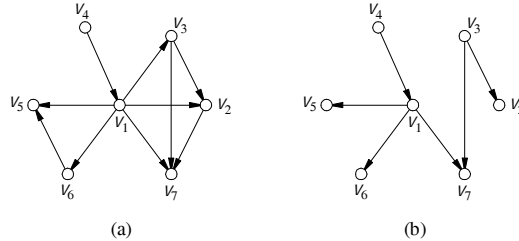


Figure 8: (a) An acyclic digraph, (b) a directed tree.

A digraph is called an *out-arborescence* (resp., *in-arborescence*) if it is a directed forest directed tree and its maximum out-degree (resp., in-degree) is one. An out-arborescence (resp. in-arborescence), has precisely one source (resp., sink), which is the called the *root*. Figure 9 illustrates these definitions. An *out-branching* (resp., *in-branching*) is a collection of node-disjoint out-arborescences (resp., in-arborescences).

Dominating set and strongly connected dominating Set

A subset U of V is a *dominating set* (DS) of D if any vertex not in U has an in-neighbor and an out-neighbor in U . In Figure 10(a), the set of black nodes form a DS. A dominating set of the smallest size is called a *minimum independent set*. The *domination number* of D , denoted by $\gamma(D)$, is the size of a minimum DS in D .

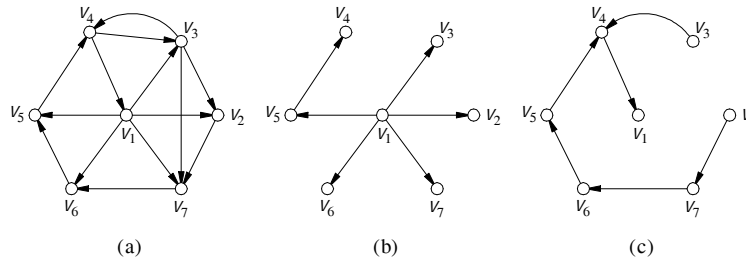


Figure 9: (a) A digraph D , (b) an out-arborescence in D rooted at v_1 , (c) (b) an in-arborescence in D rooted at v_1 .

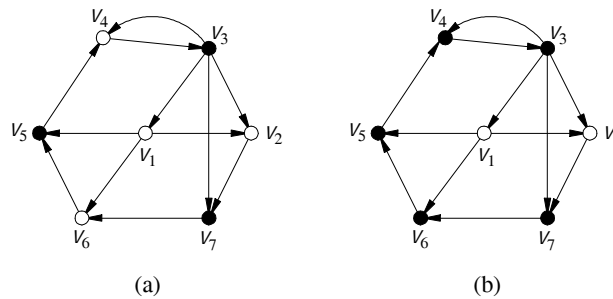


Figure 10: (a) The black nodes form a DS; (b) the black nodes form a SCDS.

A subset U of V is a *strongly connected dominating set* (SCDS) of D if U is a DS of D and $D[U]$ is strongly connected. In Figure 10(b), the set of black nodes form a SCDS. A SCDS of the smallest size is called a *minimum SCDS*. The *strong connected domination number* of D , denoted by $\gamma_c(D)$, is the size of a minimum SCDS in D .

References

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