Preliminaries on Graphs

Peng-Jun Wan

wan@cs.iit.edu

Graphs in Wireless Networks

- Communication Topology
- Conflict/Interference Topology
- Abstract representation of a network in a common language
- Two types graphs
  - undirected graph or graph
  - directed graph of digraph

References

- T.H. Cormen, C.E. Leiserson, R.L. Rivest, and C. Stein, *Introduction to Algorithms* (2nd ed.). MIT Press and McGraw-Hill, 2001. (Chapter 5)
- R. Diestel, *Graph Theory* (3rd ed.). Springer-Verlag, Graduate Texts in Mathematics, Volume 173, July 2005. (Chapter 1)
- B. Korte and J. Vygen, *Combinatorial Optimization: Theory and Algorithms* (4th ed.). Algorithms and Combinatorics 21, Springer, Berlin Heidelberg New York, 2008. (Chapter 2)
- A. Schrijver, *Combinatorial Optimization: Polyhedra and Efficiency*. Algorithms and Combinatorics 24, Springer, Berlin, 2003. (Chapter 3)

Roadmap

- **Undirected Graph**
- Directed Graph (Digraph)
- Weighted Graph/Digraph

(Undirected) Graph $G = (V, E)$

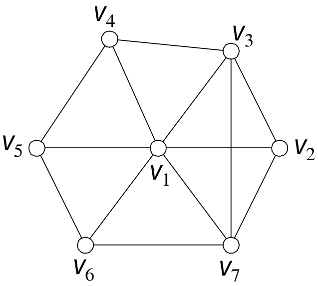


Figure: An example of gaph $G = (V, E)$.

- V: vertices/nodes; E: edges

(Undirected) Graph $G = (V, E)$

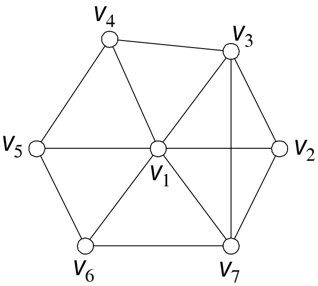
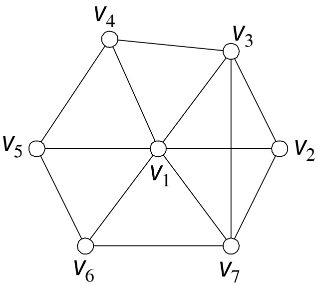


Figure: An example of gaph $G = (V, E)$.

- V: vertices/nodes; E: edges
- $e = uv := \{u, v\} \in E$:
    - e *joins*/*connects*/is *incident on* u and v
    - u and v are *ends* of/*incident with* e
    - u and v are *adjacent*
    - u is a *neighbor* of v, and vice versa.

(Undirected) Graph $G = (V, E)$



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  - u is a *neighbor* of v, and vice versa.
- A pair of edges: *adjacent*, *disjoint*

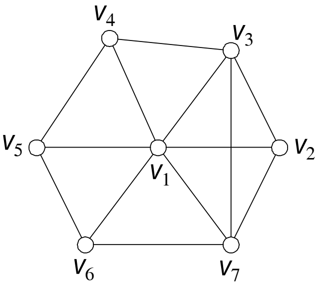
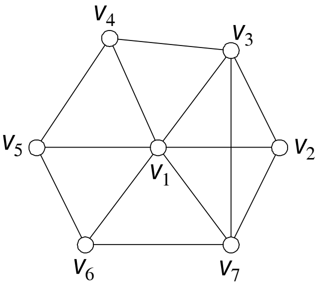


Figure:
$\Delta(G) = \deg(v_1) = 6$ and
$\delta(G) = \deg(v_4) = 3$.

- For $v \in V$,
  - $\delta_G(v)$: the set of edges incident on v.
  - $\deg_G(v) = |\delta_G(v)|$: degree of v
  - $N_G(v)$: (open) neighborhood of v
  - $N_G[v] = N_G(v) \cup \{v\}$: closed neighborhood of v



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  - $N_G(v)$: (open) neighborhood of v
  - $N_G[v] = N_G(v) \cup \{v\}$: closed neighborhood of v
- $\Delta(G)$: maximum degree of G
- $\delta(G)$: minimum degree of G

Basic Terms and Notations

- For $U \subseteq V$,
  - $\delta_G(U)$: the set of edges joining U and $V \setminus U$
  - $N_G(U)$: (open) neighborhood of U
  - $N_G[U] = N_G(U) \cup U$: closed neighborhood of U

Subgraph

$G' = (V', E') \subseteq G = (V, E)$: if $V' \subseteq V$ and $E' \subseteq E$.

- *Spanning*: if $V' = V$
- *Induced*: if E' consists of all edges of G spanned by V'. In this case, G' is denoted by $G[V']$

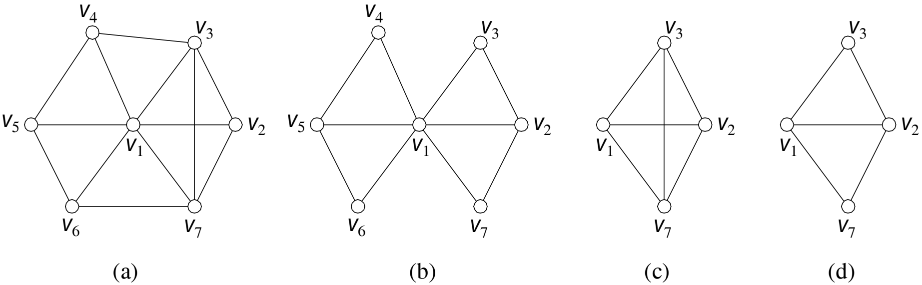


Figure: (a). A graph G; (b) an spanning but not induced subgraph of G; (c) an induced but not spanning subgraph of G; (d) a subgraph which is nether spanning nor induced.

Walk, Path And Circuit

Let $P = \langle v_0, v_1, \cdots, v_k \rangle$ be a sequence of vertices such that $v_{i-1}v_i \in E$ for $1 \leq i \leq k$. Then, P is a

- *walk*: if all edges in P are distinct
- *path*: if all vertices in P are distinct
- *circuit*: if $v_0 = v_k$ and $v_1, \cdots, v_k$ are all distinct

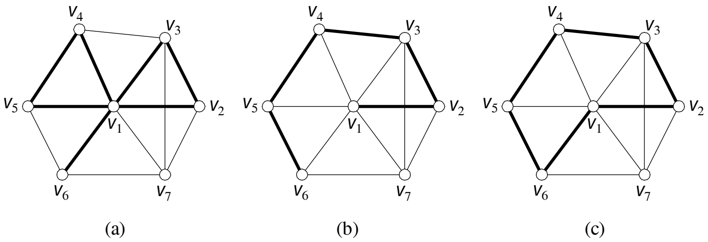


Figure: (a) A walk; (b) a path; and (c) a circuit.

Walk, Path And Circuit

- *Length* of a walk/path/circuit: the number of edges
- *Hamiltonian path*: spanning path
- *Hamiltonian circuit*: spanning circuit
- *Hamiltonian graph*: a graph with Hamiltonian circuit

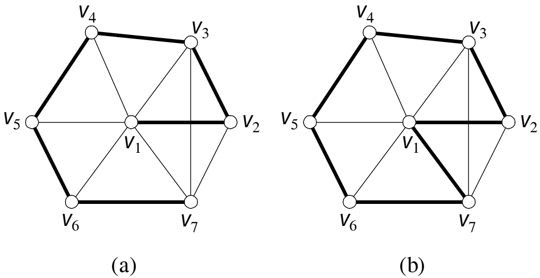


Figure: (a) A Hamiltonian path; and (b) a Hamiltonian circuit.

Connectivity, Distance, Diameter and Radius

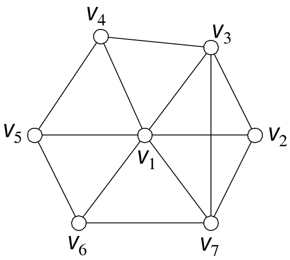


Figure: $diam(G) = 2$, $rad(G) = 1$ and v_1 is the graph center of G.

- G is *connected* if there is a path in G between any pair of nodes.
- *Distance* between u and v: $dist_G(u, v)$
- *Diameter* of G: $diam(G) = \max_{u,v \in V} dist_G(u, v)$
- *Radius* of G: $rad(G) = \min_{u \in V} \max_{v \in V} dist_G(u, v)$
  - *graph center*
  - $rad(G) \leq diam(G) \leq 2rad(G)$.

Connected Components

- A maximal connected nonempty subgraph of G is called a *connected component*, or just a *component*, of G.
- Each component is an induced subgraph, and hence is often identified with the set of its vertices.
- Each vertex and each edge of G belong to exactly one component.

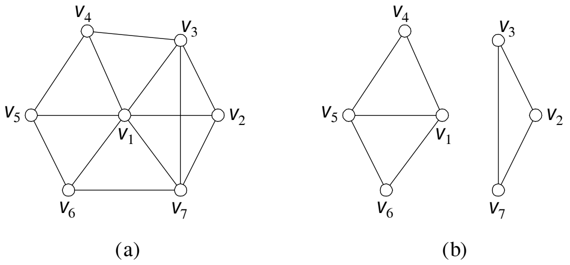


Figure: (a) A connected graph; and (b) a disconnected graph with two connected components.

Trees

- *Tree*: a connected graph without a circuit
  - minimal connected graph
- For any tree (V, E), $|E| = |V| - 1$
- *Leaf*: a vertex of degree 1
- Special trees: *star*, *spider*
- *Subtree*: connected subgraph of a tree

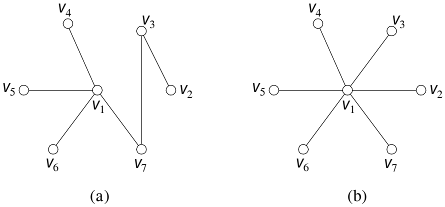


Figure: (a) A tree which is a spider, (b) a tree which is a star.

Forests

- *Forest*: a graph without a circuit
- Each component is a tree
- Any forest (V, E) has $|V| - |E|$ tree components

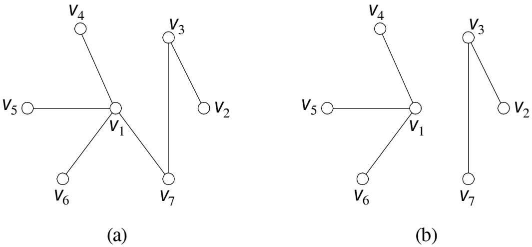


Figure: (a) A forest, which is a tree, (b) a forest, consisting of two trees.

Independent Set

- Independent set (IS): pairwise non-adjacent set of vertices
- Maximum IS: NP-hard in general
  - Independence number $\alpha(G)$
- Maximal IS
  - first-fit selection

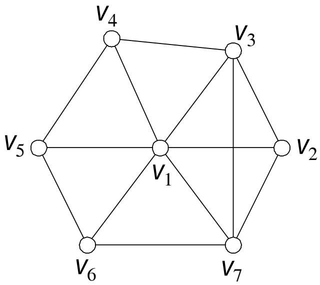


Figure: $\{v_2, v_4, v_6\}$ is a maximum IS, while both $\{v_1\}$ and $\{v_2, v_5\}$ are maximal IS.

Clique

- clique: pairwise adjacent
- Maximum clique: NP-hard in general
  - clique number $\omega(G)$
- Maximal clique
  - first-fit selection

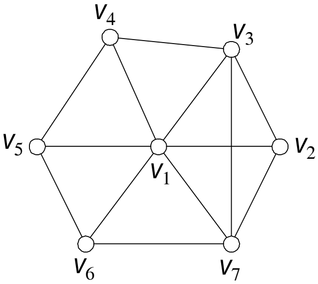
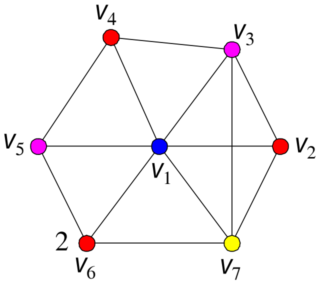


Figure: $\{v_1, v_2, v_3, v_7\}$ is a maximum clique, while $\{v_1, v_3, v_4\}$ is a maximal clique.

Vertex Coloring

- Vertex coloring: adj. nodes receive distinct colors
  - partition of vertices into IS's
- Minimum Vertex Coloring: NP-hard in general
  - chromatic number $\chi(G)$

$$\chi(G) \geq \max\left\{\frac{|V|}{\alpha(G)}, \omega(G)\right\}.$$



Dominating Set And Connected Dominating Set

- Dominating set (DS): a subset $U \subseteq V$ s.t. $N_G[U] = V$.
  - Any maximal IS is a DS
  - Minimum DS: NP-hard in general
  - Domination number $\gamma(G)$: size of a minimum DS
- Connected dominating set (CDS): a DS U s.t. $G[U]$ is connected
  - Minimum CDS: NP-hard in general
  - Connected domination number $\gamma_c(G)$: size of a minimum CDS

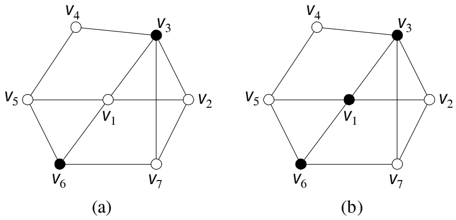


Figure: (a) The black nodes form a DS; (b) the black nodes form a CDS.

Matching

- Matching: a set of disjoint edges
- Maximum matching: solvable in polynomial time

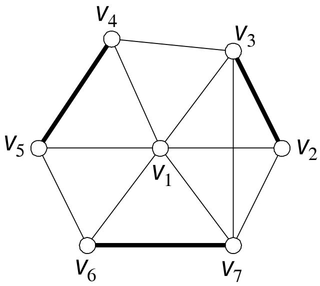


Figure: The black edges form a maximum matching.

Roadmap

- Undirected Graph
- **Directed Graph (Digraph)**
- Weighted Graph/Digraph

Directed Graph (Digraph) $D = (V, A)$

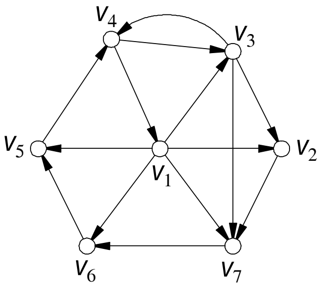


Figure: An example of digaph $D = (V, A)$.

- V: vertices/nodes; A: arcs/links
- For $a = (u, v) \in A$:
    - a *joins/connects*/is *incident on* u and v
    - a *leaves* u and *enters* v.
    - u and v are *ends* of/*incident with* e
    - u is the *tail* of a, and v the *head* of a
    - u and v are *adjacent*
    - v is an *out-neighbor* of u, and u is an *in-neighbor* of v.
- A pair of arcs: adjacent, disjoint

Basic Terms and Notations

- For $v \in V$,
  - $\delta_D^{out}(v)$: the set of arcs leaving v
  - $\delta_D^{in}(v)$: the set of arcs entering v
  - $\deg_D^{out}(v) = \left|\delta_D^{out}(v)\right|$: out-degree of v
  - $\deg_D^{in}(v) = \left|\delta_D^{in}(v)\right|$: in-degree of v
  - $N_D^{out}(v), N_D^{in}(v)$; $N_D^{out}[v], N_D^{in}[v]$

Basic Terms and Notations

- $\Delta^{out}(D)$: maximum out-degree of D
- $\Delta^{in}(D)$: maximum in-degree of D
- $\delta^{out}(D)$: minimum out-degree of D
- $\delta^{in}(D)$: minimum in-degree of D

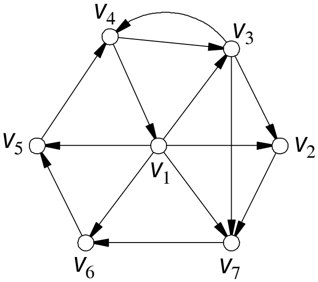


Figure: $\Delta_D^{out} = \deg_D^{out}(v_1) = 5$, $\Delta_D^{in} = \deg_D^{out}(v_7) = 3$, $\delta_D^{out} = \deg_D^{out}(v_1) = 1$, $\delta_D^{in} = \deg_D^{in}(v_2) = 2$.

Basic Terms and Notations

- For $U \subseteq V$ and an arc $a = (u, v)$,
  - a *leaves* (respectively, *enter*) U if $u \in U$ and $v \notin U$.
  - a *enters* U if $u \notin U$ and $v \in U$.
  - U *spans* a if $u \in U$ and $v \in U$.
- For $U \subseteq V$,
  - $\delta_D^{out}(U)$: the set of arcs leaving U
  - $\delta_D^{in}(U)$: the set of arcs entering U
  - $N_D^{out}(U), N_D^{in}(U)$; $N_D^{out}[U], N_D^{in}[U]$

Subgraph

$D' = (V', A') \subseteq D = (V, A)$: if $V' \subseteq V$ and $A' \subseteq A$.

- *Spanning*: if $V' = V$
- *Induced*: if A' consists of all arcs of D spanned by V'. In this case, D' is denoted by $D[V']$

Directed Walk, Path, And Circuit

Let $P = \langle v_0, v_1, \cdots, v_k \rangle$ be a sequence of vertices such that $v_{i-1}v_i \in A$ for $1 \leq i \leq k$. Then, P is a

- (directed) walk: if all arcs in P are distinct
- (directed) path: if all vertices in P are distinct
- (directed) circuit: if $v_0 = v_k$ and $v_1, \cdots, v_k$ are all distinct

Directed Walk, Path And Circuit

- Length of a (directed) walk/path/circuit: the number of arcs
- Hamiltonian path: spanning path
- Hamiltonian circuit: spanning circuit
- Hamiltonian digraph: a digraph with Hamiltonian circuit

Strong Connectivity, Distance, Diameter and Radius

- D is *strongly connected* if there is a path in D from u to v for any two distinct u and v
- Distance from u to v: $dist_D(u, v)$
- Diameter of D: $diam(D) = \max_{u,v \in V} dist_D(u, v)$

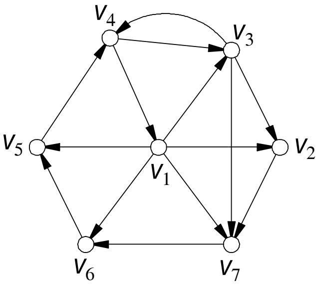


Figure: $diam(D) = dist_D(v_2, v_3) = 5.$

Strong Components

- A maximal strongly connected nonempty subgraph of D is called a *strongly connected component*, or a *strong component*, of D.
- Each strong component is an induced subgraph, and hence is often identified with the set of its vertices.
- Each vertex belongs to exactly one strong component,
- There may be arcs that belong to no strong component.

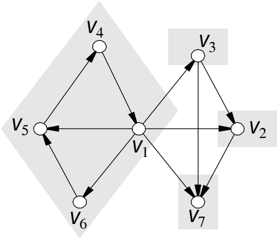


Figure: A digraph with three strong components.

Weak Connectivity And Weak Components

- D is *weakly connected* if $\overline{D}$ is connected.
- A *weakly connected component*, or a *weak component*, of D is a component of $\overline{D}$.

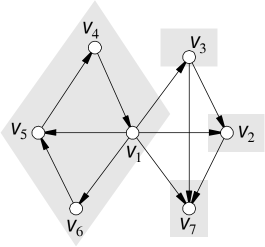


Figure: A digraph which is weakly connected, but not strongly connected.

Acyclic Digraphs and Directed Trees

- A digraph without directed circuit is called *acyclic*.
  - Any acyclic digraph with at least one vertex has at least one source and at least one sink
  - Topological sort
- A digraph D is called a *directed forest* (resp., *directed tree*) if $\overline{D}$ is a forest (resp., tree).

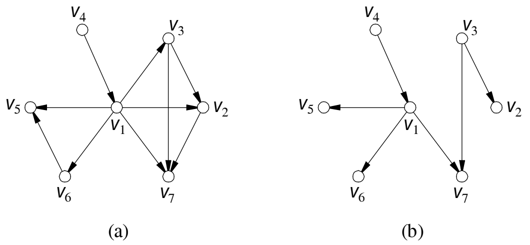
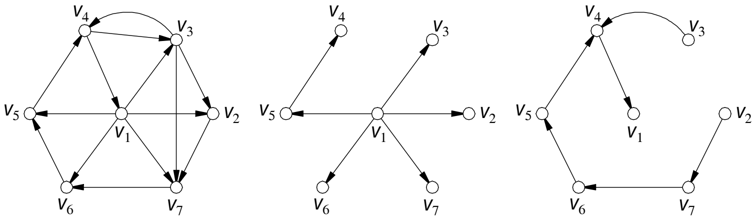


Figure: (a) An acyclic digraph, (b) a directed tree.

Arborescence And Branching

- A digraph is called an *out-arborescence* (resp., *in-arborescence*) if it is a directed forest directed tree and its maximum in-degree (resp., out-degree) is one.
  - An out-arborescence (resp. in-arborescence), has precisely one source (resp., sink), which is the called the *root*.
- An *out-branching* (resp., *in-branching*) is a collection of node-disjoint out-arborescences (resp., in-arborescences)
  - A directed forest with maximum in-degree (resp., out-degree) equals to one.



(a) (b) (c)

Dominating Set And Strongly Connected Dominating Set

- Dominating set (DS): a subset $U \subseteq V$ s.t. $N_D^{out}[U] = N_D^{in}[U] = V$.
- Strongly connected dominating set (SCDS): a DS U s.t. $D[U]$ is strongly connected

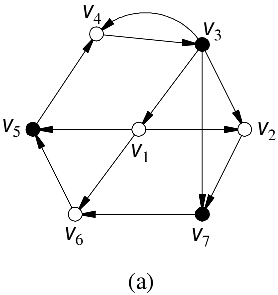


Figure: (a) The black nodes form a DS; (b) the black nodes form a SCDS.

Roadmap

- Undirected Graph
- Directed Graph
- **Weighted Graph/Digraph**

Weighted Graph/Digraph

- Edge-weighted graph
  - weigh of subgraph: total weight of edges
- Arc-weighted digraph
  - weigh of subgraph: total weight of arcs
- Vertex-weighted graph
  - weigh of subgraph or a subset of vertices: total weight of vertices

Min/Max-Weighted Subgraph

- **Shortest**/Longest **Path**
- **Min-Weighted Spanning Tree**
- **Min-Weighted Spanning Arborescence**
- Min-Weighted Strongly Connected Spanning Subgraph
- **Max-Weighted Matching**

Min/Max-Weighted Subset of Vertices

- Max-Weighted IS
- Max-Weighted Clique
- Min-Weighted DS
- Min-Weighted CDS
- Min-Weighted SCDS

Reading Assignments

- BFS tree, DFS Tree
- Shortest path, shortest path tree
- Minimum spanning tree