Outline

- Problem Description
- Practical Approximation Algorithms
- Polynomial-Time Approximation Scheme
- Summary
A General Specification of Multihop Wireless Networks

\((V, A, \mathcal{I})\):

- \(V\): network nodes
\((V, A, I)\):

- \(V\): network nodes
- \(A\): communication links
- \(I\): collection of independent (i.e., conflict-free) links in \(A\) implicitly given by an interference model (IM)
A General Specification of Multihop Wireless Networks

$(V, A, I)$:

- $V$: network nodes
- $A$: communication links
  - $(V, A)$: communication topology

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Multiflows in Multihop Wireless Networks
A General Specification of Multihop Wireless Networks

\[(V, A, \mathcal{I}):\]

- \(V\): network nodes
- \(A\): communication links
  - \((V, A)\): communication topology
- \(\mathcal{I}\): collection of independent (i.e. conflict-free) links in \(A\)
(V, A, I):

- V: network nodes
- A: communication links
  - (V, A): communication topology
- I: collection of independent (i.e. conflict-free) links in A
  - implicitly given by an interference model (IM)
Protocol Interference Model (IM)

Figure: (a) Communication range and interference range of each node; (b) a communication link; (c) a conflicting pair of communication links.
802.11 Interference Model (IM)

Figure: (a) Communication range and interference range of each node; (b) a communication edge; (c) a conflicting pair of communication edges.
\[ S = \{ (l_j, \lambda_j) \in \mathcal{I} \times \mathbb{R}_+ : 1 \leq j \leq k \} \]
(Fractional) Link Schedule

\[ S = \{(l_j, \lambda_j) \in \mathcal{I} \times \mathbb{R}_+ : 1 \leq j \leq k\} \]

- \( \sum_{j=1}^{k} \lambda_j \): length (or latency) of \( S \)
S = \{(I_j, \lambda_j) \in \mathcal{I} \times \mathbb{R}_+: 1 \leq j \leq k\}

- $\sum_{j=1}^{k} \lambda_j$: length (or latency) of $S$
- $|S|$: size of $S$
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If the length of \( S \leq 1 \), it determines a link capacity function

\[ c_S = \sum_{1 \leq j \leq k} \lambda_j l_j. \]
(Fractional) Link Schedule

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If the length of \( S \leq 1 \), it determines a link capacity function

\[ c_S = \sum_{1 \leq j \leq k} \lambda_j 1^{l_j}. \]

Capacity Region \( P \subset \mathbb{R}_+^A \): convex hull of \( \{1^l : l \in \mathcal{I}\} \), or equivalently

\[ P = \{c_S : S \text{ is a link schedule of length } \leq 1\}. \]
**Maximum Multiflow (MMF):** Given a set of commodities, find a link schedule \( S \) of length at most one such that the maximum multiflow subject to \( c_S \) is maximized.
(Fractional) Multiflow

1. **Maximum Multiflow (MMF)**: Given a set of commodities, find a link schedule $S$ of length at most one such that the maximum multiflow subject to $c_S$ is maximized.

2. **Maximum Concurrent Multiflow (MCMF)**: Given a set of commodities with demands, find a link schedule $S$ of length at most one such that the maximum concurrent multiflow subject to $c_S$ is maximized.
Maximum Weighted Independent Set of Links (MWISL): Given $d \in \mathbb{R}_+^A$, find an $I \in I$ such that $d(I)$ is maximized.
Related Problems

1. **Maximum Weighted Independent Set of Links (MWISL):** Given $d \in \mathbb{R}^A_+$, find an $I \in \mathcal{I}$ such that $d(I)$ is maximized.

2. **Shortest Weighted Link Schedule (SWLS):** Given $d \in \mathbb{R}^A_+$, find a shortest link schedule

$$S = \{(l_j, \lambda_j) \in \mathcal{I} \times \mathbb{R}_+ : 1 \leq j \leq k\}$$

such that $d = \sum_{1 \leq j \leq k} \lambda_j 1^{l_j}$. 

Roadmap

- Problem Description
- **Practical Approximation Algorithms**
- Polynomial-Time Approximation Scheme
- Summary
Theorem If there is a capacity sub-region $Q \subseteq P$ s.t.

1. $Q$ is a $\mu$-approximation of $P$, i.e., $P \subseteq \mu Q$,
2. $Q$ has an explicit polynomial representation, and
3. $\forall d \in Q$, a fractional link schedule of length at most one for $d$ can be computed in poly. time.

Then, both MMF and MCMF have a polynomial $\mu$-approximation.
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Then, both MMF and MCMF have a polynomial $\mu$-approximation.

Any $Q \subseteq P$ meeting the three above conditions is called a poly. $\mu$-approx. capacity subregion.
$\mathcal{F}_i$: set of $s_i-t_i$ flows

\begin{align*}
\text{max. multflow} & \quad \max \sum_{j=1}^{k} \text{val} (f_j) \\
\text{s.t.} & \quad f_j \in \mathcal{F}_j, \forall 1 \leq j \leq k \\
& \quad \sum_{j=1}^{k} f_j \in P
\end{align*}

\begin{align*}
\text{max. concurrent multflow} & \quad \max \phi \\
\text{s.t.} & \quad f_j \in \mathcal{F}_j, \forall 1 \leq j \leq k \\
& \quad \text{val} (f_j) \geq \phi d (j), \forall 1 \leq j \leq k \\
& \quad \sum_{j=1}^{k} f_j \in P
\end{align*}
**Restricted Multiflow**

\[
\begin{align*}
\text{max. } & \quad \text{Q-restricted multiflow} \\
& \quad \text{max } \sum_{j=1}^{k} \text{val}(f_j) \\
\text{s.t. } & \quad f_j \in F_j, \forall 1 \leq j \leq k \\
& \quad \sum_{j=1}^{k} f_j \in Q
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Restricted Multiflow

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\text{max } \sum_{j=1}^{k} \text{val}(f_j) & \quad \text{max } \phi \\
\text{s.t. } f_j \in F_j, \forall 1 \leq j \leq k & \quad \text{s.t. } f_j \in F_j, \forall 1 \leq j \leq k \\
\sum_{j=1}^{k} f_j \in Q & \quad \text{val}(f_j) \geq \phi d(j), \forall 1 \leq j \leq k \\
\end{align*}
\]

Step 1: solve the Q-restricted LP to obtain a k-flow \( \langle f_1, f_2, \ldots, f_k \rangle \).
Step 2: compute a fractional link schedule of length at most one for \( \sum_{j=1}^{k} f_j \).
Roadmap

- Problem Description
- Practical Approximation Algorithms
  - Episode: Independence Polytope And Fractional Coloring
    - Approximate Capacity Subregion
  
- Polynomial-Time Approximation Scheme
- Summary
Independence Polytope

- \( G = (V, E) \): an undirected graph
Independence Polytope

- $G = (V, E)$: an undirected graph
- $\mathcal{I}$: collection of independent sets of $G$
Independence Polytope

- $G = (V, E)$: an undirected graph
- $\mathcal{I}$: collection of independent sets of $G$
- Independence polytope $P \subset \mathbb{R}^V_+$: convex hull of $\{1^l : l \in \mathcal{I}\}$. 
Fractional (Weighted) Coloring

- For any $d \in \mathbb{R}_+^V$, a fractional coloring of $(G, d)$ is a set of $k$ pairs $(l_j, \lambda_j) \in \mathcal{I} \times \mathbb{R}_+$ s.t.

$$
\sum_{1 \leq j \leq k, v \in l_j} \lambda_j = d(v), \ \forall v \in V.
$$
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$k$: coloring number
For any $d \in \mathbb{R}_+^V$, a fractional coloring of $(G, d)$ is a set of $k$ pairs $(l_j, \lambda_j) \in \mathcal{I} \times \mathbb{R}_+^V$ s.t.

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Fractional (Weighted) Coloring

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\]

- \( k \): coloring number
- \( \sum_{j=1}^k \lambda_j \): coloring weight

Fractional chromatic number \( \chi_f(G, d) \): minimum weight of all fractional colorings of \((G, d)\).

\[
\chi_f(G, d) \geq \frac{d(V)}{\alpha(G)}.
\]
Fractional (Weighted) Coloring

For any $d \in \mathbb{R}_+^V$, a fractional coloring of $(G, d)$ is a set of $k$ pairs $(l_j, \lambda_j) \in I \times \mathbb{R}_+$ s.t.

$$\sum_{1 \leq j \leq k, \forall v \in l_j} \lambda_j = d(v), \forall v \in V.$$

- $k$: coloring number
- $\sum_{j=1}^k \lambda_j$: coloring weight

fractional chromatic number $\chi_f(G, d)$: minimum weight of all fractional colorings of $(G, d)$.

$$\chi_f(G, d) \geq \frac{d(V)}{\alpha(G)}.$$

$P = \{ d \in \mathbb{R}_+^V : \chi_f(G, d) \leq 1 \}$
Fractional (Weighted) Coloring: Example

**Figure:** For the all-one demand vector $d$, $\chi_f(G, d) = 2.5$. On the other hand, $\chi(G) = 3$. 
A polytope $Q \subseteq P$ is a polynomial $\mu$-approximation of $P$ if

1. $Q$ is a $\mu$-approximation of $P$, i.e., $P \subseteq \mu Q$,
2. $Q$ has an explicit polynomial representation, and
3. $\forall d \in Q$, a fractional coloring of $(G, d)$ with weight at most one can be computed in poly. time.
First-Fit Fractional Weighted Coloring

- $(v_1, v_2, \cdots, v_n)$: a vertex ordering
\[ \langle v_1, v_2, \ldots, v_n \rangle : \text{a vertex ordering} \]

- **Initialization:** \( S \leftarrow \emptyset ; U \leftarrow \{ v \in V : d(v) > 0 \} \)
First-Fit Fractional Weighted Coloring

- \( \langle v_1, v_2, \cdots, v_n \rangle \): a vertex ordering
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First-Fit Fractional Weighted Coloring

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1. select an IS \( I \subseteq U \) in the first-fit manner
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First-Fit Fractional Weighted Coloring

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  1. select an IS $I \subseteq U$ in the first-fit manner
  2. $\lambda \leftarrow \min_{v \in I} d(v)$
  3. add $(I, \lambda)$ to $S$
\( \langle v_1, v_2, \cdots, v_n \rangle \): a vertex ordering

Initialization: \( S \leftarrow \emptyset; U \leftarrow \{ v \in V : d(v) > 0 \} \)

Iterations: while \( U \neq \emptyset \) do

1. select an IS \( I \subseteq U \) in the first-fit manner
2. \( \lambda \leftarrow \min_{v \in I} d(v) \)
3. add \((I, \lambda)\) to \( S \)
4. \( \forall v \in U : d(v) \leftarrow d(v) - \lambda; \) if \( d(v) = 0 \), remove \( v \) from \( U \)
First-Fit Fractional Weighted Coloring

- \( \langle v_1, v_2, \ldots, v_n \rangle \): a vertex ordering
- **Initialization:** \( S \leftarrow \emptyset; \ U \leftarrow \{ v \in V : d(v) > 0 \} \)
- **Iterations:** while \( U \neq \emptyset \) do
  1. select an IS \( I \subseteq U \) in the first-fit manner
  2. \( \lambda \leftarrow \min_{v \in I} d(v) \)
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  4. \( \forall v \in U: d(v) \leftarrow d(v) - \lambda; \) if \( d(v) = 0 \), remove \( v \) from \( U \)
- **output** \( S \)
First-Fit Fractional Weighted Coloring

- Coloring number of $S \leq n$
First-Fit Fractional Weighted Coloring

- Coloring number of $S \leq n$
- Coloring weight of $S \leq \max_{1 \leq i \leq n} d(N_{\preceq}(v_i))$, where

$$N_{\preceq}(v_i) = \{ v_j : 1 \leq j \leq i, v_j \in N(v_i) \}$$

$$N_{\prec}(v_i) = \{ v_j : 1 \leq j < i, v_j \in N(v_i) \}.$$
First-Fit Fractional Weighted Coloring

- Coloring number of $S \leq n$
- Coloring weight of $S \leq \max_{1 \leq i \leq n} d \left( N_{\leq} (v_i) \right)$, where
  
  $$N_{\geq} (v_i) = \{ v_j : 1 \leq j \leq i, v_j \in N (v_i) \}$$
  $$N_{\leq} (v_i) = \{ v_j : 1 \leq j < i, v_j \in N (v_i) \} .$$

- $\max_{1 \leq i \leq n} d \left( N_{\leq} (v_i) \right)$: (closed) $d$-inductivity of $\langle v_1, v_2, \ldots, v_n \rangle$
First-Fit Fractional Weighted Coloring

- \( k \): \# of iterations/colors
- \( \forall 1 \leq j \leq k, \)
  - \( U_j \): the subset of nodes with residue demands at the beginning of the \( j \)-th iteration
  - \( (l_j, \lambda_j) \): the pair selected in the \( j \)-th iteration.
First-Fit Fractional Weighted Coloring

- \( k \): # of iterations/color
- \( \forall 1 \leq j \leq k \),
  - \( U_j \): the subset of nodes with residue demands at the beginning of the \( j \)-th iteration
  - \( (l_j, \lambda_j) \): the pair selected in the \( j \)-th iteration.

Consider an arbitrary node \( v_i \in U_k \). \( \forall 1 \leq j \leq k \), let \( V_{i,j} = N(v_i) \cap U_j \).

Then, \( \forall 1 \leq j \leq k \), \( l_j \cap V_{i,j} \neq \emptyset \). Hence,

\[
d(N(v_i)) = \sum_{j=1}^{k} \lambda_j |l_j \cap V_{i,j}| \geq \sum_{j=1}^{k} \lambda_j.
\]
Smallest-Last Ordering

**Question**: how to compute a vertex ordering with least closed $d$-inductivity?
**Question**: how to compute a vertex ordering with least closed $d$-inductivity?

- $H \leftarrow G$.
- For $i = n$ down to 1,
  - $v_i \leftarrow$ a vertex of smallest closed weighted degree in $H$
  - $H \leftarrow H - \{v_i\}$
Smallest-Last Ordering

closed $d$-inductivity of $G$:

$$
\bar{\delta}^* (G, d) = \max_{U \subseteq V} \min_{u \in U} d \left( N_{G[u]} [u] \right)
$$

Theorem

The smallest-last ordering achieves the smallest $d$-inductivity $\bar{\delta}^* (G, d)$ among all vertex orderings.
For any ordering $\langle v_1, v_2, \cdots, v_n \rangle$, 

$$\max_{1 \leq i \leq n} d(N_{\leq i}(v_i)) \geq \delta^*(G, d).$$
Proof of Theorem

For any ordering $\langle v_1, v_2, \cdots, v_n \rangle$,

$$\max_{1 \leq i \leq n} \ d \left( N_{\leq} (v_i) \right) \geq \bar{\delta}^* (G, d).$$

Let $U \subseteq V$ be s.t.

$$\bar{\delta}^* (G, d) = \min_{u \in U} d \left( N_{G[U]} [u] \right)$$

and $j$ be the last index such that $v_j \in U$. Then,

$$\max_{1 \leq i \leq n} \ d \left( N_{\leq} (v_i) \right) \geq d \left( N_{\leq} (v_j) \right) \geq d \left( N_{G[U]} [v_j] \right) = \bar{\delta}^* (G, d).$$
Proof of The Theorem

For the smallest-last ordering \( \langle v_1, v_2, \cdots, v_n \rangle \),

\[
\max_{1 \leq i \leq n} d (N_{\preceq} (v_i)) \leq \overline{\delta}^* (G, d).
\]
Proof of Theorem

For the smallest-last ordering \( \langle v_1, v_2, \cdots, v_n \rangle \),

\[
\max_{1 \leq i \leq n} d(N \preceq (v_i)) \leq \delta^*(G, d).
\]

For any \( 1 \leq i \leq n \), let \( V_i = \{v_1, v_2, \cdots, v_i\} \). Then,

\[
d(N \preceq (v_i)) = \min_{u \in V_i} d\left(N_{G[V_i][u]}\right) \leq \delta^*(G, d).
\]

Hence,

\[
\max_{1 \leq i \leq n} d(N \preceq (v_i)) \leq \delta^*(G, d).
\]
Inductive Local Independence Number (LIN)

\[ \langle v_1, v_2, \cdots, v_n \rangle: \text{a vertex ordering} \]
Inductive Local Independence Number (LIN)

- $\langle v_1, v_2, \cdots, v_n \rangle$: a vertex ordering
- Its Inductive LIN:

$$\alpha^* = \max_{1 \leq i \leq n} \{ |I| : I \in \mathcal{I}, I \subseteq N_\prec(v_i) \}.$$
Inductive Local Independence Number (LIN)

1. \( \langle v_1, v_2, \ldots, v_n \rangle \): a vertex ordering
2. Its Inductive LIN:

\[
\alpha^* = \max_{1 \leq i \leq n} \{|I| : I \in \mathcal{I}, I \subseteq N_{<} (v_i)\}.
\]

3. \( \forall d \in \mathbb{R}_+^V, \)

\[
\bar{\delta}^* (G, d) \leq \max_{1 \leq i \leq n} d (N_{\leq} (v_i)) \leq \alpha^* \chi_f (G, d).
\]
\(\forall 1 \leq i \leq n,\)

\[\chi_f (G, d) \geq d (N_{\preceq} (v_i)) / \alpha^* + d (v_i)\]

\[\Rightarrow d (N_{\preceq} (v_i)) \leq \alpha^* (\chi_f (G, d) - d (v_i)) + d (v_i)\]

\[= \alpha^* \chi_f (G, d) - (\alpha^* - 1) d (v_i)\]

\[\leq \alpha^* \chi_f (G, d) - (\alpha^* - 1) \min_{v \in V} d (v).\]
The polytope

\[ Q = \left\{ d \in \mathbb{R}_+^V : \max_{1 \leq i \leq n} d \left( N_{\leq} (v_i) \right) \leq 1 \right\}. \]

is a polynomial \( \alpha^* \)-approx. of \( P \).
$D = (V, A)$: an orientation of $G$
Inward Local Independence Number (LIN)

- \( D = (V, A) \): an orientation of \( G \)
- Its inward LIN:

\[
\beta^* = \max_{u \in V} \left\{ |I| : I \in \mathcal{I}, I \subseteq N^\text{in}_D(v) \right\}.
\]
Inward Local Independence Number (LIN)

- $D = (V, A)$: an orientation of $G$
- Its inward LIN:
  \[
  \beta^* = \max_{u \in V} \{|I| : I \in \mathcal{I}, I \subseteq N_D^{in}(v)\}.
  \]
- For all $d \in \mathbb{R}_+^V$, 
  \[
  \overline{\delta}^* (G, d) \leq \max_{v \in V} \left( d(v) + 2d(N_D^{in}(v)) \right) \leq 2\beta^* \chi_f (G, d).
  \]
Inward LIN: The First Inequality

For any $U \subseteq V$, $\exists u \in U$ s.t.

$$d \left( N_{D[U]}^{in} (u) \right) \geq d \left( N_{D[U]}^{out} (u) \right).$$

Thus,

$$d \left( N_{G[U]} [u] \right) = d (u) + d \left( N_{D[U]}^{in} (u) \right) + d \left( N_{D[U]}^{out} (u) \right)
\leq d (u) + 2d \left( N_{D[U]}^{in} (u) \right)
\leq d (u) + 2d \left( N_{D}^{in} (u) \right)
\leq \max_{v \in V} (d (v) + 2d \left( N_{D}^{in} (v) \right)).$$

So,

$$\overline{\delta}^* (G, d) \leq \max_{v \in V} (d (v) + 2d \left( N_{D}^{in} (v) \right)).$$
$\forall v \in V,$

$$\chi_f (G, d) \geq d (N_D^{in} (v)) / \beta^* + d (v)$$

$$\Rightarrow d (N_D^{in} (v)) \leq \beta^* (\chi_f (G, d) - d (v))$$

$$\Rightarrow d (v) + 2d (N_D^{in} (v)) \leq 2\beta^* \chi_f (G, d) - (2\beta^* - 1) d (v)$$
The polytope

\[ Q = \left\{ d \in \mathbb{R}_+^V : \max_{v \in V} \left( d(v) + 2d(N_D^{in}(v)) \right) \leq 1 \right\}. \]

is a polynomial $2\beta^*$-approx. of $P$. 
Roadmap

- Problem Description
- Practical Approximation Algorithms
  - Episode: Independence Polytope And Fractional Coloring
  - Approximate Capacity Subregion
- Polynomial-Time Approximation Scheme
- Summary
Orientation of the conflict graph: for any a pair of conflicting links \( a_1 = (u_1, v_1) \) and \( a_2 = (u_2, v_2) \)
Protocol IM

- Orientation of the conflict graph: for any a pair of conflicting links $a_1 = (u_1, v_1)$ and $a_2 = (u_2, v_2)$
  - if $v_1$ is in the interference range of $u_2$, take $(a_2, a_1)$;
Orientation of the conflict graph: for any a pair of conflicting links $a_1 = (u_1, v_1)$ and $a_2 = (u_2, v_2)$

- if $v_1$ is in the interference range of $u_2$, take $(a_2, a_1)$;
- otherwise, take $(a_1, a_2)$.
Orientation of the conflict graph: for any a pair of conflicting links $a_1 = (u_1, v_1)$ and $a_2 = (u_2, v_2)$

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- otherwise, take $(a_1, a_2)$.

$Q$: inward independence polytope of this orientation
Protocol IM

- Orientation of the conflict graph: for any a pair of conflicting links $a_1 = (u_1, v_1)$ and $a_2 = (u_2, v_2)$
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- $\beta^*$: inward LIN of this orientation
Protocol IM

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- $\beta^*$: inward LIN of this orientation

**Lemma**

$$\beta^* \leq \left\lceil \frac{\pi}{\arcsin \frac{c-1}{2c}} \right\rceil - 1, \text{ and hence } Q \text{ is a poly.}$$

$$2 \left( \left\lceil \frac{\pi}{\arcsin \frac{c-1}{2c}} \right\rceil - 1 \right)$$-approx capacity subregion.
Lexicographic ordering: sort all edges in the lexicographic order of their right endpoints
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Q: inductive independence polytope of this ordering
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$Q$: inductive independence polytope of this ordering

$\alpha^*$: inductive LIN of this ordering
**Lemma**

\[ \alpha^* \leq 7 \text{ and hence } Q \text{ is a poly. } 7\text{-approx capacity subregion.} \]
Interference radius decreasing ordering: sort all edges in the decreasing order of their larger interference radii
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Lemma

$\alpha^* \leq 23$ hence $Q$ is a poly. 23-approx capacity subregion.
Roadmap

- Problem Description
- Practical Approximation Algorithms
- **Polynomial-Time Approximation Scheme**
- Summary
**Theorem** Restricted to a network class $\mathcal{N}$, if there is a poly. (resp., a poly. $\mu$-approx.) alg. for **MWISL**, then there is a poly. (resp., a poly. $\mu$-approx.) alg. for (1) **SWLS**, (2) **MMF**, and (3) **MCMF**.
Ellipsoid Method with (Approx.) Separation Oracle

\( \mathcal{P}_j \) for \( 1 \leq j \leq k \): the set of paths for commodity \( j \)
\( \mathcal{P} \): union of \( \mathcal{P}_1, \ldots, \mathcal{P}_k \)
\( \mathcal{P}_e \) for \( e \in A \): the set of paths in \( \mathcal{P} \) that use link \( e \)
\( \mathcal{I}_e \) for \( e \in A \): the set of \( I \) in \( \mathcal{P} \) containing \( e \)

<table>
<thead>
<tr>
<th>Primary (path-flow) LP for MMF</th>
<th>dual LP</th>
</tr>
</thead>
<tbody>
<tr>
<td>max ( x(\mathcal{P}) )</td>
<td>min ( \tau )</td>
</tr>
<tr>
<td>s.t. ( x(\mathcal{P}_e) \leq \lambda(\mathcal{I}_e) ), ( \forall e \in A )</td>
<td>s.t. ( y(p) \geq 1, \forall p \in \mathcal{P} )</td>
</tr>
<tr>
<td>( \lambda(\mathcal{I}) \leq 1 )</td>
<td>( y(I) \leq \tau, \forall I \in \mathcal{I} )</td>
</tr>
<tr>
<td>( x \in \mathbb{R}<em>+^\mathcal{P} ), ( \lambda \in \mathbb{R}</em>+^\mathcal{I} )</td>
<td>( y \in \mathbb{R}<em>+^A ), ( \tau \in \mathbb{R}</em>+ )</td>
</tr>
</tbody>
</table>
**Theorem** Restricted to any of the three network classes, **MWISL** has a PTAS:

1. 802.11 IM;
2. Protocol IM, and the interference radius of each node is at least $c$ times its communication radius for some constant $c > 1$;
3. Protocol IM, and every $k$-hop neighborhood in the conflict-graph contains at most $O(k^c)$ independent links for some constant $c > 0$.

**Approaches:**
- for the first two classes: shifting strategy + dynamic programming
- for the third class: polynomial growth
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\item 802.11 IM;\linebreak
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PTAS in broader ranges of networks