Flow-Based Feasibility Test of Linear Interference Alignment with Arbitrary Interference Topology

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Outline

- Generic Feasibility of LIA
- Variants of Properness
- Reduced Interference Topology
- Flow-Based Tests
- Summary

Network Model

- $[\ell] = \{1, 2, \cdots, \ell\}$: users (or node-disjoint links, communication requests)

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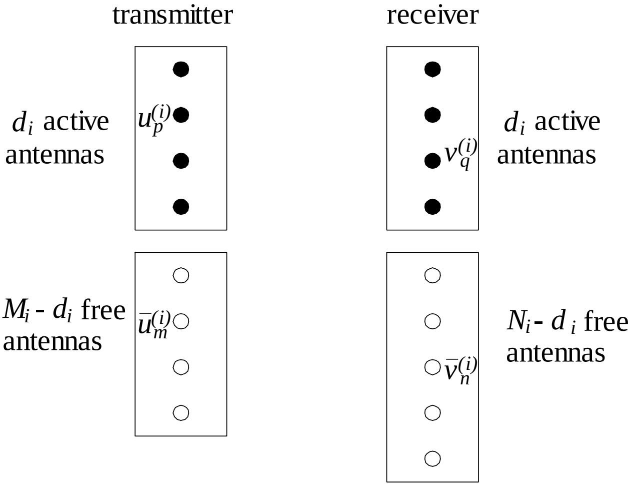
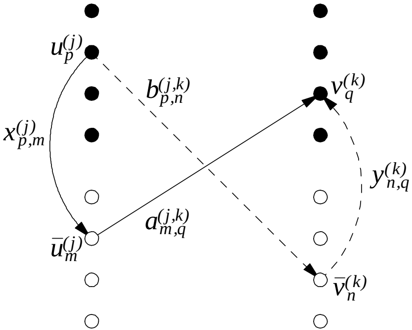


A Conceptual View of LIA



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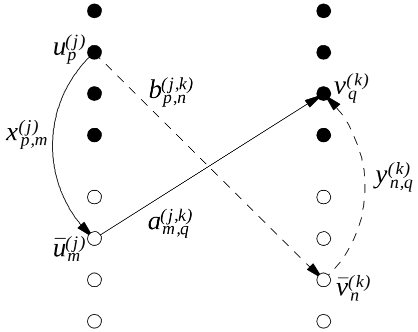

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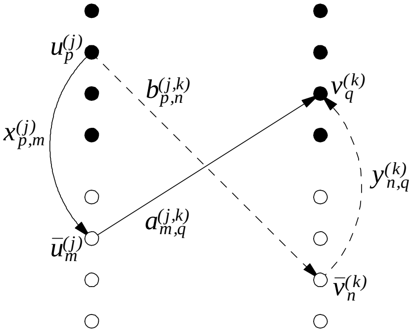

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- $\sum_{(j,k)\in\mathcal{E}} d_j d_k$ LIA mapping equations:

$$f_{pq}^{(j,k)} = \sum_{m\in[M_j-d_j]} x_{p,m}^{(j)} a_{m,q}^{(j,k)} + \sum_{n\in[N_k-d_k]} b_{p,n}^{(j,k)} y_{n,q}^{(k)}.$$

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- $\mathcal{L}$ is feasible $\Leftrightarrow$ LIA mapping is surjective a.s.. [González, Beltrán, Santamaría 2014]

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  - computationally intensive
  - prone to round-off errors in floating-point computations

Contributions of This Paper

- Tests via network flow

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- Tests via network flow
- Arbitrary interference topology
- Interesting implications

Roadmap

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  - Still inscalable

Strong Properness

- *Tx-strongly proper*: $\exists$ complete matching $\mathcal{M}$ of equations with variables s.t. for any $(j,k) \in \mathcal{E}$ and $q \in [d_k]$, the equations associated with
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- *strongly proper:* either Tx-strongly proper or Rx-strongly proper.

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- Subsequently, $\mathcal{E}$ is assumed to irreducible

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Test for Properness

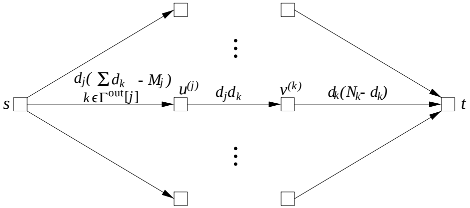

Figure: Flow network $\mathcal{N}_1$.

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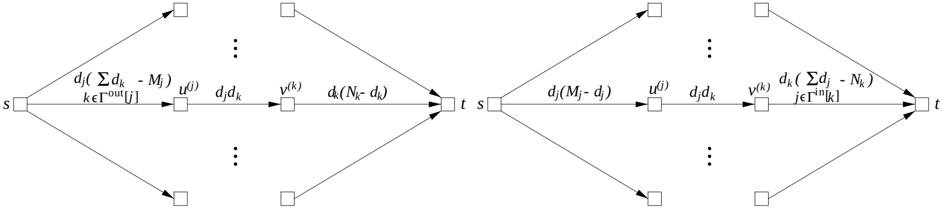

Figure: Flow network $\mathcal{N}_1$.

Figure: Flow network $\mathcal{N}_2$

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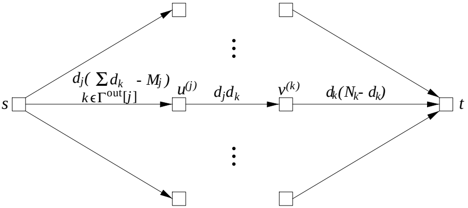

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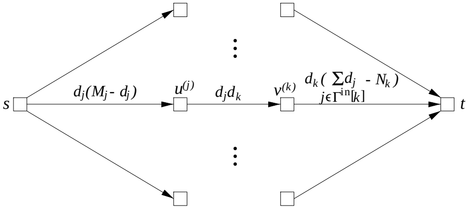

Figure: Flow network $\mathcal{N}_2$

$\mathcal{L}$ is proper
$\Leftrightarrow$ $\mathcal{N}_1$ has a flow saturating all arcs leaving s
$\Leftrightarrow$ $\mathcal{N}_2$ has a flow saturates all arcs entering t

Test for Properness: Uniform Stream

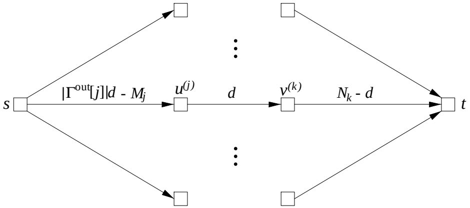

Figure: Flow network $\mathcal{N}_3$.

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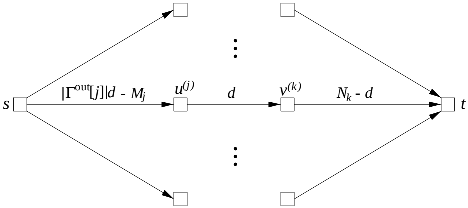

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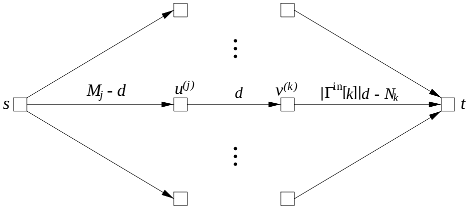
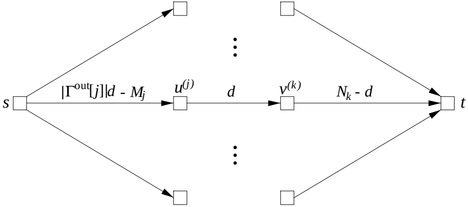


Figure: Flow network $\mathcal{N}_4$.

Test for Properness: Uniform Stream



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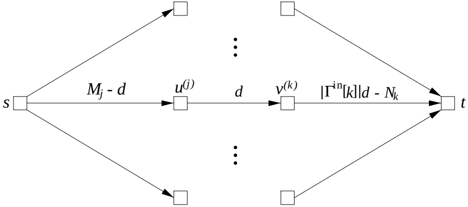

Figure: Flow network $\mathcal{N}_4$.

$\mathcal{L}$ is proper
$\Leftrightarrow$ $\mathcal{N}_3$ has a flow saturating all arcs leaving s
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Test for Properness: Symmetric User Instance

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- $\mathcal{L}$ is proper $\Leftrightarrow M + N \geq (\ell + 1)\, d$.

Test for Strong Properness: Uniform Stream

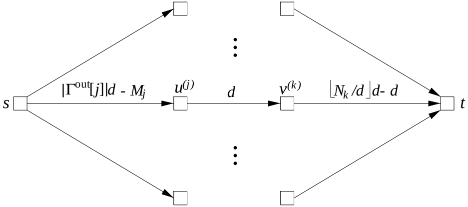

Figure: Flow network $\mathcal{N}_5$.

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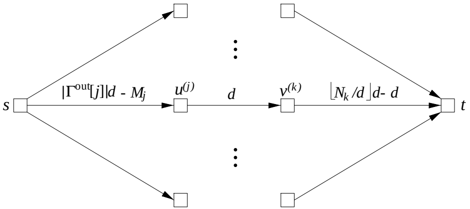

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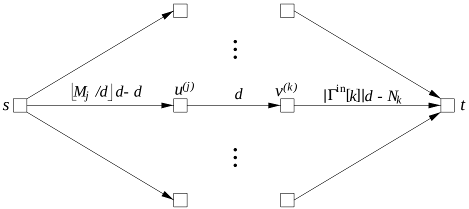
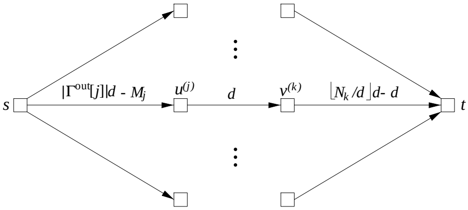


Figure: Flow network $\mathcal{N}_6$.

Test for Strong Properness: Uniform Stream



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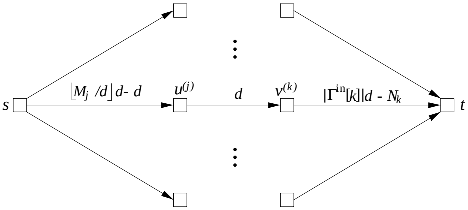

Figure: Flow network $\mathcal{N}_6$.

$\mathcal{L}$ is Tx-strongly proper $\Leftrightarrow \mathcal{N}_5$ has a flow saturating all arcs leaving s
$\mathcal{L}$ is Rx-strongly proper $\Leftrightarrow \mathcal{N}_6$ has a flow saturating all arcs leaving s

Equivalence: Divisible & Uniform Stream

Theorem

Suppose that either $d \mid M_j$ for each $j \in \mathcal{E}_T$, or $d \mid N_k$ for each $k \in \mathcal{E}_R$. Then feasibility, properness, and strong properness of $\mathcal{L}$ are all equivalent.

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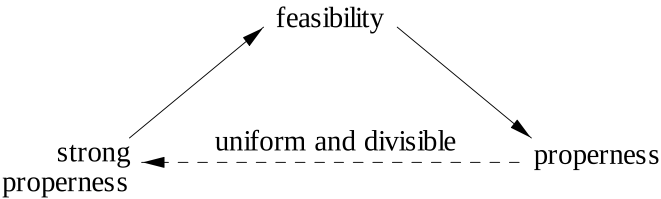

feasibility
strong
properness
uniform and divisible
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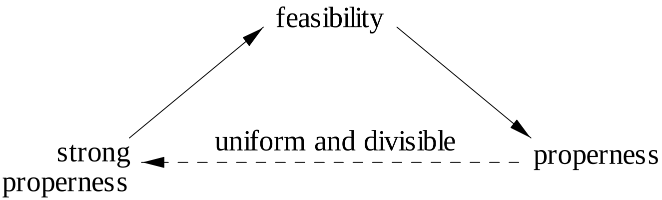

- Open Problem: computational complexity of the feasibility test.

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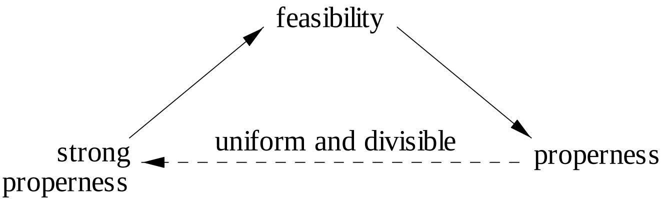

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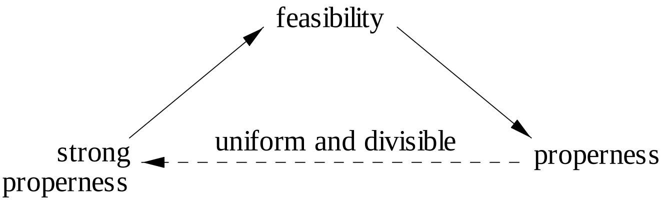

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- Conjecture. For $\mathcal{L} = (M, N, d)^{\ell}$ with $2d \leq \min\{M, N\}$, feasibility $\Leftrightarrow$ properness
  - holds with a *complete* interference topology [Ruan, Lau, Win 2013].