Fractional Wireless Link Scheduling and Polynomial Approximate Capacity Regions of Wireless Networks

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***Abstract*—Fractional Link scheduling is one of the most fundamental problems in wireless networks. The prevailing approach for shortest fractional link scheduling is based on a reduction to the maximum-weighted independent set problem, which itself may not admit efficient approximation algorithms. In addition, except for the wireless networks under the protocol interference model, none of the existing scheduling algorithms can produce a link schedule with explicit upper bounds on its length in terms of the link demands. As the result, the polynomial approximate capacity regions in these networks remain blank. This paper develops a *purely* combinatorial paradigm for fractional link scheduling in wireless networks. In addition to the superior efficiency, it is able to provide explicit upper bounds on the lengths of the produced link schedule. By exploiting these upper bounds, polynomial approximate capacity regions are derived. The effectiveness of this new paradigm is demonstrated by its applications in wireless networks under the physical interference model and wireless MIMO networks under the protocol interference model.**

I. Introduction

Link scheduling is one of the most fundamental problems in wireless networks. Consider a general wireless networks represented by a *multigraph* $G_c = (V, A)$, where V is the set of nodes, and A is a set of communication links. A subset I of A is said to be *independent* if all the links in I can transmit successfully at the same time; and let $\mathcal{I}$ denote the collection of all independent subsets of A. In general, $\mathcal{I}$ is specified implicitly by an interference model possibly together with the signalling processing technologies at the physical layer. Suppose that $d \in \mathbb{R}_+^A$ is a traffic demand function on the A in terms the transmission time. A *link schedule* of d is a set

$$\mathcal{S} = \left\{ (I_j, x_j) \in \mathcal{I} \times \mathbb{R}^+ : 1 \leq j \leq k \right\}$$

satisfying that for each $a \in A$,

$$d(a) \leq \sum_{j=1}^{k} x_j \left| \{a\} \cap I_j \right|;$$

the value $\sum_{j=1}^{k} x_j$ are referred to as the *length* (or *latency*) of $\mathcal{S}$, and is denoted by $\|\mathcal{S}\|$. The minimum length of all fractional schedules of d is denoted by $\chi^*(d)$. Then, the problem finding a link schedule of d with minimum length is referred to as shortest link scheduling (**SLS**). The capacity region Ω of the network consists of all demands $d \in \mathbb{R}_+^A$ with $\chi^*(d) \leq 1$. In general, **SLS** is NP-hard, and the membership test of the capacity region is NP-complete. A set $\Phi \subseteq \mathbb{R}_+^A$ is said to be a polynomial (α, β)-approximate capacity region for some $\alpha, \beta > 0$ if the membership test of Φ can be done in polynomial time, and

$$\frac{1}{\alpha}\Omega \subseteq \Phi \subseteq \beta\Omega.$$

Polynomial approximate capacity regions of wireless networks play vital roles in the cross-layer communication scheduling.

SLS in wireless networks under protocol interference model has been well studied in [12], [14], [15], [16], [21], [23]. Very efficient greedy constant-approximation algorithm was proposed for the **SLS** in the single-channel single-radio (SCSR) setting [21] as well as the multi-channel multi-radio (MCMR) setting [23]. The prevailing paradigm for **SLS** in general wireless networks is a reduction to the problem **Maximum Weighted Independent Set** (**MWIS**): Given a non-negative weight function w on A, find an $I \in \mathcal{I}$ with maximum total weight $w(I) := \sum_{a \in I} w(a)$. In [22], a polynomial-time approximation-preserving reduction from **SLS** to **MWIS** was developed, and can be extended to arbitrary wireless networks. The power of this reduction is its generality and transparency to the underlying interference model and the physical-layer communication technologies. However, such reduction utilizes the ellipsoid method for linear programming, which is quite inefficient in practice [18]. In [2], an almost approximation-preserving reduction from **SLS** to the problem **MWIS** with improved running time was designed. Specifically, let $\mathcal{A}$ be a μ-approximation algorithm for **MWIS**, and $\varepsilon \in (0, 1/2]$ be an accuracy-efficiency trade-off parameter. The approximation algorithm for **SLS** developed in this paper produces a $(1+\varepsilon)\,\mu$-approximate solution by making only $O\left(\varepsilon^{-2}\,|A|\ln|A|\right)$ calls to $\mathcal{A}$. In certain wireless networks such as wireless SCSR networks under physical interference model [7], [22], [24] wireless MIMO networks under protocol interference model [11], [17], [25], this improved reduction is still less efficient and non-purely combinatorial method: Each iteration calls an approximation algorithm for the problem **MWIS**, which itself may be non-combinatorial. For examples, in wireless networks under the physical interference model with linear transmission power, or in MIMO wireless networks under protocol interference model with uniform number of antennas, the only known deterministic contant-approximation algorithms for **MWIS** were proposed in [24] and [25] respectively. Both algorithms solves a fractional linear program and then rounds the fractional solution and integer solution. Thus, the overall running-times of the algorithms are still very high.

Polynomial approximate capacity regions were only known for wireless networks under protocol interference model. The fractional link schedules produced by the approximation algorithms [21], [23] admit explicit upper bounds on the schedule lengths in terms of the link demands. Such explicit upper bounds lead to approximate capacity regions [21], [23] defined explicitly by a polynomial number of linear inequalities. In contrast, all existing studies on **SLS** in wireless networks under the physical interference model [2], [7], [22], [24] or in wireless MIMO networks under the protocol interference model [2], [11], [17], [25] have so far derived only the approximation bounds of the proposed algorithms. In these networks, there are no known explicit upper bounds on the fractional link schedule lengths, in terms of the link demands, which are achievable by some polynomial-time algorithms. As the result, the polynomial approximate capacity regions in these networks remain blank.

This paper develops a *purely* combinatorial paradigm for **SLS** in wireless networks. Instead of a reduction to **MWIS** as in the prevailing paradigm, it is based on a reduction to a much simpler problem of selecting a cost-efficient independent subset, which can be computed efficiently. In addition to the superior efficiency, it is able to provide explicit upper bounds on the lengths of the produced link schedule. By exploiting these upper bounds, polynomial approximate capacity regions are derived. The effectiveness of this new paradigm is demonstrated by the constant approximation bounds for **SLS** in wireless networks under the physical interference model or in MIMO wireless networks under the protocol interference model.

The remainder of this paper is organized as follows. Section II presents the general paradigm for approximating **SLS**. Section III introduces a general procedure to be utilized later for extracting a cost-efficient independent set. Section IV, Section V, and Section VI presents the applications of general paradigm in SCSR wireless networks under the physical interference model, MCMR wireless networks under the physical interference model, and MIMO wireless networks under the protocol interference model respectively. In each of these three sections, both the explicit upper-bound and the approximation bound of the schedule length are derived; in addition, a polynomial approximate capacity region is constructed. Finally, we conclude this paper in Section VII. The following standard notations will be adopted in this paper. For a real-valued function f on a finite set A and any $B \subseteq A$, $f(B)$ represents $\sum_{a\in B} f(a)$.

II. A General Paradigm

Consider a link demand $d \in \mathbb{R}_+^A$ and let A^+ denote the set of links with positive demands. Given a subset $S \subseteq A^+$ and a positive weight function w on S, the *cost-efficiency* of each $a \in S$ is defined to be $\frac{w(a)}{d(a)}$; the *cost-efficiency* of each subset I of S is defined to be the total cost-efficiency of all links in I. This section presents a paradigm for computing a factional link schedule of d assuming only that there is an algorithm $\mathcal{A}$ for selecting a cost-efficient independent set. Specifically, suppose that given any subset $S \subseteq A^+$ and a positive weight function w on S, the algorithm $\mathcal{A}$ produces an independent subset I of S whose cost-efficiency is at least $\frac{1}{\beta}w(S)$. Then, for any accuracy-efficiency trade-off parameter $\varepsilon \in (0, 1/2]$, the paradigm to be presented in this section computes a fractional link schedule of d length $(1+\varepsilon)\,\beta$ in $O\left(\varepsilon^{-2}\,|A^+|\ln|A^+|\right)$ calls of $\mathcal{A}$.

For better understanding the design and analysis of the paradigm, the paradigm is interpreted as an adaptive zero-sum game with retirement, which was introduced in [2], and generalizes the problem considered by Auer et al. [4], Vovk [20], Cesa-Bianchi et al. [6], Freund and Schapire [8], [9], Khandekar [13], and Arora et al. [3] in the context of learning or game theory. This game is briefly described in Subsection II-A. The link scheduling algorithm is presented in Subsection II-B.

A. An Adaptive Zero-Sum Game with Retirement

In [2], an adaptive zero-sum game with retirement was introduced, which generalizes the problem considered by Auer et al. [4], Vovk [20], Cesa-Bianchi et al. [6], Freund and

Schapire [8], [9], Khandekar [13], and Arora et al. [3] in the context of learning or game theory. This game is played in rounds between a set E of m profit-making agents and a loss-incurring adversary. At the end of the round, some agents may retire themselves *permanently*, and the set of agents not yet ret retired are said to be *active* agents. Initially, all agents are active. At the beginning of each round, the agents declare an *adaptive* binding strategy in terms of probabilistic distributions on active agents. Then, the adversary generates the profits of active agents in this round subject to the **Normalization Rule**: The maximum value of the individual profits is exactly one. The loss incurred by the adversary is determined by the **Zero-Sum Rule**: The loss of the adversary is *equal* to the expected profit of *active* agents with respect to the binding strategy on active agents. At the end of the round, some agents may decide to retire themselves to prevent the adversary from keeping a single agent overly wealthy while keeping other agents in poverty. The objective of the agents is to make it happen as early as possible that the cumulative profit of *every* agent is at least $\frac{1}{1+\varepsilon}$ times the cumulative loss of the adversary for some pre-specified $\varepsilon \in (0, 1/2]$; the objective of the adversary is exactly the opposite. The game has to be terminated whenever all agents are retired.

Two strategies for the agents were specified in [2] while the strategy for the adversary is left to specific applications.

- **Threshold-based retirement policy**: An agent will be retired permanently after making a cumulative profit at least the threshold
$$\phi = \frac{\ln m + \varepsilon}{\varepsilon (1+\varepsilon) + \ln (1-\varepsilon)}.$$
Since for $\varepsilon \in (0, 1/2]$,
$$\frac{1}{5} < \frac{\varepsilon (1+\varepsilon) + \ln (1-\varepsilon)}{\varepsilon^2} < \frac{1}{2}.$$
we have $\phi = \Theta \left(\varepsilon^{-2} \ln m \right)$.
- **Exponential Binding Strategies** strategy: Suppose at the beginning of a round each active agent a has the cumulative profit $Profit\,(a)$. Then, the exponential binding strategy on the active agents sets the probability of each active agent a proportional to $(1-\varepsilon)^{Profit(a)}$.

An implementation of the game playing with these strategies is described as follows. Let S be the set of active agents, which is initially E; let $Profit\,(a)$ be cumulative profit of each agent a, which is initially 0. Repeat following rounds while S is non-empty:

1) **Declaration of binding strategies**: The agent declares the exponential binding strategy on active agents.
2) **Generation of profits**: The adversary determines a non-negative profit in the present round subject to the **Normalization Rule** and update $Profit\,(a)$ for each active agent.
3) **Retirement of agents**: For each active agent $a \in S$, if $Profit\,(a) \geq \phi$ then the agent a is retired (i.e., removed) from S.

The effectiveness of above implementation of the game is asserted in the theorem below.

Theorem 2.1: The total number of rounds is at most $m \lceil \phi \rceil$; and at the end of the last round the cumulative profit of each agent is at least ϕ and the cumulative loss of the adversary is at most $(1+\varepsilon)\,\phi$.

B. Link Scheduling

Let
$$\phi = \frac{\ln |A^+| + \varepsilon}{\varepsilon (1+\varepsilon) + \ln (1-\varepsilon)}.$$
The algorithm **LS**(ε) outlined in Table I first builds up a link schedule $\mathcal{S}$ of ϕd from scratch with successive augmentations by a pair (I, x) in each iteration and then returns $\frac{1}{\phi}\mathcal{S}$ as the output link schedule of d. The design of **LS**(ε) is based on the general framework of an adaptive zero-sum game with retirement introduced in the previous subsection. Each link $a \in A^+$ corresponds to an agent, and and each augmenting iteration of the **LS**(ε) corresponds to a game round. The agents plays exactly with the strategies described in the previous subsection. For each agent $a \in A^+$, $Profit\,(a)$ is its cumulative profit, which is initially 0; ϕ is the retirement threshold of the agents; and S is the set of active agents, which is initially A^+. In addition, each agent $a \in A^+$ *implicitly* maintains a weight
$$w(a) = (1-\varepsilon)^{Profit(a)}.$$
The profit generation strategy of the adversary is coupled with the link schedule augmentation: In each round of the game, the profit of each agent is the *proportion* of its demand served by the augmentation pair. Consequently, at the end of each round the cumulative profit of each agent is the *proportion* of its demand that has been served by the present $\mathcal{S}$. Specifically, at the beginning of each round the adversary computes an IS I of S by the algorithm $\mathcal{A}$ with respect to the weight w. The length l of I is determined by the **Normalization Rule** as follows. Due to the augmentation (I, l), each $a \in I$ earns a profit $\frac{l}{d(a)}$. The **Normalization Rule** dictates that
$$l = \min_{l \in I} d(a).$$
This completes the specification of the adversary's strategy on generating losses in each round. After augmenting $\mathcal{S}$ with the pair (I, λ), $Profit\,(a)$ for all $a \in I$ are explicitly updated

accordingly (and $w(a)$ for all $a \in I$ are implicitly updated accordingly); and if $Profit(a) \geq \phi$ then a is retired from S. By Theorem 2.1, the number of rounds is at most

$$\left|A^+\right| \lceil \phi \rceil = O\left(\varepsilon^{-2}\left|A^+\right| \ln\left|A^+\right|\right).$$

After the last round, the proportion of the demand by each $a \in A^+$ served by $\mathcal{S}$ is at least ϕ. Thus, $\frac{1}{\phi}\mathcal{S}$ is a link schedule of d and is returned as the output.

Algorithm **LS**(ε):

$\mathcal{S} \leftarrow \emptyset, Profit \leftarrow \mathbf{0}, S \leftarrow A^+$, $\phi \leftarrow \frac{\ln\left|A^+\right|+\varepsilon}{\varepsilon(1+\varepsilon)+\ln(1-\varepsilon)}$;
while $S \neq \emptyset$ do
  $I \leftarrow$ output of $\mathcal{A}$ on (S, w);
  $l \leftarrow \min_{a \in I} d(a)$;
  $\mathcal{S} \leftarrow \mathcal{S} \cup \{(I, l)\}$;
  for each $a \in I$ do
    $Profit(a) \leftarrow Profit(a) + \frac{l}{d(a)}$;
    if $P(a) \geq \phi$ then $S \leftarrow S \setminus \{a\}$;
return $\frac{1}{\phi}\mathcal{S}$.

TABLE I
OUTLINE OF THE ALGORITHM **LS**(ε).

The theorem below analyzes the performance of the algorithm **LS**(ε).

Theorem 2.2: The schedule output by algorithm **LS**(ε) has length at most $(1+\varepsilon)\beta$.

Proof. Consider a specific round in which $\mathcal{S}$ is augmented by a pair (I, λ). Then,

$$\textstyle\sum_{a \in I} \frac{w(a)}{d(a)} \geq \frac{w(S)}{\beta}.$$

By the **Zero-Sum Rule**, the loss of the adversary in this round is

$$\frac{1}{w(S)} \textstyle\sum_{a \in I} w(a) \frac{l}{d(a)} = \frac{l}{w(S)} \sum_{a \in I} \frac{w(a)}{d(a)} \geq \frac{l}{\beta}.$$

So, the cumulative loss of the adversary at the end of last round is at least $\frac{\|\mathcal{S}\|}{4}$. On the other hand, by Theorem 2.1 the cumulative loss of the adversary at the end of last round is at most $(1+\varepsilon)\phi$. Thus,

$$\frac{\|\mathcal{S}\|}{\beta} \leq (1+\varepsilon)\phi.$$

Hence, the output link schedule has length

$$\frac{\|\mathcal{S}\|}{\phi} \leq (1+\varepsilon)\beta.$$

So, the theorem holds. ■

III. An Extraction Procedure

Consider a non-empty finite set S, a non-negative function ρ on S^2, a positive demand function d on S, and a positive weight function w on S. In this section, we present a simple procedure **WIS** to compute a subset I of satisfying that

$$\max_{a \in I} \sum_{b \in I \setminus \{a\}} \rho(b, a) < 1,$$

and

$$\sum_{a \in I} \frac{w(a)}{d(a)} \geq \frac{w(S)}{4 \max_{a \in S}\left[d(a) + \sum_{b \in S \setminus \{a\}} \rho(b, a) d(b)\right]}.$$

This procedure is very useful in the selection of cost-effective independent set in specific wireless networks.

For simplicity, we denote

$$\Delta = \max_{a \in S}\left[d(a) + \sum_{b \in S \setminus \{a\}} \rho(b, a) d(b)\right];$$

for and $a, b \in S$, we denote

$$\rho_w(b, a) = \frac{w(b)/d(b)}{w(a)/d(a)} \rho(a, b) + \rho(b, a).$$

Note that $\rho_w(b, a) \geq \rho(b, a)$. The algorithm **WIS** is outlined in Table II. It initializes I to an empty set and another set F to S, and proceeds in two phases:

- **Growing Phase:** While F is non-empty, remove any a from F, and add it to I if
$$\sum_{b \in I} \rho_w(b, a) + \frac{1}{2\Delta} \sum_{b \in F} \rho_w(b, a) d(b) < 1.$$
- **Pruning Phase:** While
$$\max_{a \in I} \sum_{b \in I \setminus \{a\}} \rho(b, a) \geq 1,$$
remove from I any a satisfying that
$$\sum_{b \in I \setminus \{a\}} \rho(b, a) \geq 1.$$

The theorem below asserts the correctness of the algorithm.

Theorem 3.1: For the output I,

$$\sum_{a \in I} \frac{w(a)}{d(a)} \geq \frac{1}{4\Delta} w(S).$$

Proof. We define a real-valued function f on the set of non-negative vectors x indexed by the set of elements in S by

$$f(x) = \sum_{a \in S} \frac{w(a)}{d(a)} x(a) \left(1 - \sum_{b \in S \setminus \{a\}} \rho(b, a) x(b)\right).$$

Algorithm **WIS**:
//Initialization
$I \leftarrow \emptyset, F \leftarrow S$;
//Growing Phase
while $F \neq \emptyset$ do
remove an arbitrary a from F;
if $\sum_{b\in I}\rho_w(b,a)+\frac{1}{2\Delta}\sum_{b\in F}\rho_w(b,a)\,d(b)<1$ then
add a to I;
//Pruning Phase
while $\max_{a\in I}\sum_{b\in I\setminus\{a\}}\rho(b,a)\geq 1$ do
remove from I any a s.t. $\sum_{b\in I\setminus\{a\}}\rho(b,a)\geq 1$;
return I.

TABLE II
OUTLINE OF THE ALGORITHM **WIS**.

Then,

$$\begin{aligned}
& f\left(\frac{1}{2\Delta}d\right) \\
&= \sum_{a\in S}\frac{w(a)}{d(a)}\frac{d(a)}{2\Delta}\left(1-\sum_{b\in S\setminus\{a\}}\rho(b,a)\frac{d(b)}{2\Delta}\right) \\
&= \frac{1}{4\Delta}\sum_{a\in S}w(a)\left(2-\frac{1}{\Delta}\sum_{b\in S\setminus\{a\}}\rho(b,a)\,d(b)\right) \\
&\geq \frac{1}{4\Delta}\sum_{a\in S}w(a)\left(2-\frac{1}{\Delta}(\Delta-d(a))\right) \\
&= \frac{1}{4\Delta}\sum_{a\in S}w(a)(1+d(a)) \\
&\geq \frac{1}{4\Delta}\sum_{a\in S}w(a) \\
&= \frac{1}{4\Delta}w(S).
\end{aligned}$$

In addition, for each fixed $a\in S$, f is a linear function of $x(a)$ with slope

$$\begin{aligned}
& \frac{w(a)}{d(a)}\left(1-\sum_{b\in S\setminus\{a\}}\left(\frac{w(b)/d(b)}{w(a)/d(a)}\varrho(a,b)+\varrho(b,a)\right)x(b)\right) \\
&= \frac{w(a)}{d(a)}\left(1-\sum_{b\in S\setminus\{a\}}\rho_w(b,a)\,x(b)\right),
\end{aligned}$$

which is referred to as the *a-slope of f at x*.

We modify x starting from $\frac{1}{2\Delta}d$ through the executions of the algorithm as follows:

- For each iteration in **Growing Phase**, let a be the link removed from F, and reset $x(a)$ to 1 if a is added to I, and to 0 otherwise.
- For each iteration in **Pruning Phase**, let a be the link removed from I, and reset $x(a)$ to 0.

We remark that at the end of the **Growing Phase**, x is $\{0,1\}$-valued We shall show that $f(x)$ is non-decreasing after each update. As the result, the final output I satisfies that

$$\begin{aligned}
\sum_{a\in I}\frac{w(a)}{d(a)} &= \sum_{a\in S}\frac{w(a)}{d(a)}x(a) \\
&\geq f(x) \\
&\geq f\left(\frac{1}{2\Delta}d\right) \\
&\geq \frac{1}{4\Delta}w(S),
\end{aligned}$$

and hence the theorem holds.

Now, we show that $f(x)$ is non-decreasing after each iteration in **Growing Phase**. Consider a particular iteration of **Growing Phase** and let a be the link removed from F. Right after the removal of a, the a-slope of f at x is

$$\begin{aligned}
& \frac{w(a)}{d(a)}\left(1-\sum_{b\in S\setminus\{a\}}\rho_w(b,a)\,x(b)\right) \\
&= \frac{w(a)}{d(a)}\left(1-\sum_{b\in I}\rho_w(b,a)-\frac{1}{2\Delta}\sum_{b\in F}\rho_w(b,a)\,d(b)\right).
\end{aligned}$$

If

$$\sum_{b\in I}\rho_w(b,a)+\frac{1}{2\Delta}\sum_{b\in F}\rho_w(b,a)\,d(b)<1.$$

then the a-slope of f at x is positive and hence by increasing $x(a)$ to 1, $f(x)$ increases. Otherwise, the a-slope of f at x is non-negative and hence by decreasing $x(a)$ to 0, $f(x)$ either increases or remains unchanged. So, in either case, $f(x)$ is non-decreasing after each iteration in **Growing Phase**.

Next, we show that $f(x)$ is non-decreasing after each iteration in **Pruning Phase**. Consider a particular iteration of **Pruning Phase** and let a be the link removed from I. At the beginning of this iteration, the a-slope of f at x is

$$\begin{aligned}
& \frac{w(a)}{d(a)}\left(1-\sum_{b\in S\setminus\{a\}}\rho_w(b,a)\,x(b)\right) \\
&= \frac{w(a)}{d(a)}\left(1-\sum_{b\in I\setminus\{a\}}\rho_w(b,a)\right).
\end{aligned}$$

Since

$$\sum_{b\in I\setminus\{a\}}\rho_w(b,a)\geq\sum_{b\in I\setminus\{a\}}\rho(b,a)\geq 1,$$

the a-slope of f at x is non-negative and hence by decreasing $x(a)$ to 0, $f(x)$ either increases or remains unchanged. ■

IV. SCSR WIRELESS NETWORKS UNDER PHYSICAL INTERFERENCE MODEL

Consider a SCSR wireless network consisting of a set V of nodes in a plane. The signal strength attenuates with a path loss factor $\eta r^{-\kappa}$, where r is the distance from the transmitter, κ is *path-loss exponent* (a constant between 2 and 5 depending on the wireless environment), and η is the *reference loss factor*.

The signal quality perceived by a receiver is measured by the *signal to interference and noise ratio* (*SINR*), which is the quotient between the power of the wanted signal and the total power of unwanted signals and the ambient noise ξ. In order to correctly interpret the wanted signal under the physical interference model, the SINR must be greater than certain threshold $\sigma > 1$. Thus, for any pair a of nodes u and v to communicate with each other even without any interference, their transmission powers should exceed

$$p_0(a) = \tfrac{\sigma\xi}{\eta}\ell(a)^\kappa,$$

where $\ell(a)$ is the distance between u and v. In the setting of *linear* transmission power, the transmission power of a is $p(a) = \gamma p_0(a)$ for some constant $\gamma > 1$. Assume all nodes have maximum transmission power P. Then, the set A of communication links consists of all possible pairs a of nodes satisfying that $p(a) \leq P$. For any two links a and b in A, we denote by $\ell(a,b)$ the the distance between the sender of a and the receiver of b. The interference of a link $a \in A$ toward another link $b \in A$ is $p(a) \cdot \eta\ell(a,b)^{-\kappa}$ when they transmit at the same time. A set I of links in A is *independent* if when all links in I transmit at the same time, the SINR of each link in I is greater than σ. Assume the above network setting, we present an efficient link scheduling algorithm following the general paradigm developed Section II, and derive an polynomial approximate capacity region.

The independence can be conveniently characterized in terms of conflict factors [24]. The *conflict factor* of a link a toward another link $b \in A$, denoted by $\rho(a,b)$, is defined as follows:

- If a and b share a common node, then $\rho(a,b) = 1$;
- otherwise,

$$\rho(a,b) = \min\left\{\frac{\sigma\gamma}{\gamma-1}\left(\frac{\ell(a)}{\ell(a,b)}\right)^\kappa, 1\right\}.$$

Then, a subset I of A is independent if and only if

$$\max_{a\in I}\sum_{b\in I\setminus\{a\}}\rho(b,a) < 1.$$

Let $\mathcal{I}$ be the collection of independent subsets of A.

Consider a link demand $d \in \mathbb{R}_+^A$ and let A^+ denote the set of links with positive demands. Let

$$\Delta(d) := \max_{a\in A}\left[d(a) + \sum_{b\in A\setminus\{a\}}\rho(b,a)\,d(b)\right]$$

Given any subset S of A^+ and a positive weight function w on S, the algorithm $\mathcal{A}$ for selecting cost-effective independent sets simply applies the algorithm **WIS** to computes an independent subset I of S. The cost-efficiency of I satisfies that

$$\begin{aligned}&\sum_{a\in I}\frac{w(a)}{d(a)}\\&\geq \frac{w(S)}{4\max_{a\in S}\left[d(a) + \sum_{b\in S\setminus\{a\}}\rho(b,a)\,d(b)\right]}\\&\geq \frac{w(S)}{4\Delta(d)}.\end{aligned}$$

By Theorem 2.2, for any $\varepsilon \in (0,1/2]$ the algorithm **LS**(ε) computes a fractional link schedule $\mathcal{S}$ of d with

$$\|\mathcal{S}\| \leq 4(1+\varepsilon)\Delta(d)$$

in $O\left(\varepsilon^{-2}|A^+|\ln|A^+|\right)$ calls of $\mathcal{A}$.

Next, we derive an approximation bound of the above algorithm. The *local independence number* (LIN) of A is defined to be

$$\mu := \max_{I\in\mathcal{I}}\max_{a\in A}\left(|I\cap\{a\}| + \sum_{b\in I\setminus\{a\}}\rho(b,a)\right).$$

It was shown in [24] that

$$\mu \leq 2\left(\left\lceil \pi/\arcsin\frac{1-\left(\frac{\gamma-1}{\sigma\gamma}\right)^{1/\kappa}}{2}\right\rceil - 1\right).$$

The LIN μ relates $\Delta(d)$ and $\chi^*(d)$ as follows:

Lemma 4.1: $\Delta(d) \leq \mu\chi^*(d)$.

Proof. Let

$$\left\{(I_j,l_j) \in \mathcal{I}\times\mathbb{R}^+ : 1 \leq j \leq k\right\}$$

be a shortest fractional schedule of d. For any $a \in A$, we have

$$\begin{aligned}&d(a) + \sum_{b\in A\setminus\{a\}}\rho(b,a)\,d(b)\\&\leq \sum_{j=1}^k \ell_j|I_j\cap\{a\}| + \sum_{b\in A\setminus\{a\}}\rho(b,a)\sum_{j=1}^k \ell_j|I_j\cap\{b\}|\\&= \sum_{j=1}^k \ell_j\left(|I_j\cap\{a\}| + \sum_{b\in A\setminus\{a\}}\rho(b,a)|I_j\cap\{b\}|\right)\\&= \sum_{j=1}^k \ell_j\left(|I_j\cap\{a\}| + \sum_{b\in I_j\setminus\{a\}}\rho(b,a)\right)\\&\leq \sum_{j=1}^k \ell_j\mu\\&= \mu\sum_{j=1}^k \ell_j\\&= \mu\chi^*(d).\end{aligned}$$

So, the lemma holds. ■

The above lemma immediately implies an approximation bound $4(1+\varepsilon)\mu$ of the algorithm $\mathbf{LS}(\varepsilon)$. In summary, we have the following theorem.

Theorem 4.2: For any $\varepsilon \in (0, 1/2]$, the algorithm $\mathbf{LS}(\varepsilon)$ computes a fractional link schedule $\mathcal{S}$ of d with

$$\|\mathcal{S}\| \leq 4(1+\varepsilon)\Delta(d) \leq 4(1+\varepsilon)\mu\chi^*(d).$$

in $O\left(\varepsilon^{-2}|A^+|\ln|A^+|\right)$ calls of $\mathcal{A}$.

Finally, we present a polynomial approximate capacity subregion.

Theorem 4.3: Let

$$\Phi = \left\{d \in \mathbb{R}_+^A : \Delta(d) \leq 1\right\}.$$

Then, Φ is a polynomial $(\mu, 4)$-approximate capacity region.

Proof. We first show that $\Phi \subseteq 4\Omega$. Consider any $d \in \Phi$. Then, for any $\varepsilon \in (0, 1/2]$,

$$\chi^*(d) \leq 4(1+\varepsilon)\Delta(d) \leq 4(1+\varepsilon).$$

Since ε can be arbitrarily small, we have

$$\chi^*(d) \leq 4.$$

Hence, $d \in 4\Omega$. Thus, $\Phi \subseteq 4\Omega$.

Next, we show that $\frac{1}{\mu}\Omega \subseteq \Phi$. Consider any $d \in \Omega$. Since

$$\Delta(d) \leq \mu\chi^*(d) \leq \mu,$$

we have $d \in \mu\Phi$. So, $\Omega \subseteq \mu\Phi$ or equivalently, $\frac{1}{\mu}\Omega \subseteq \Phi$. ■

V. MCMR Wireless Networks under Physical Interference Model

We consisder an instance of MCMR wireless network under phyiscal interference model, which is a generalization of the instance in the previous section by having λ channels and $\tau(v)$ radios at each node $v \in V$. For any two neighboring nodes u and v, there are $\lambda\tau(u)\tau(v)$ radio-level links from u to v, each of which is specified by a tuple (u, v, i, j, k) where i is the radio index at u, j is the radio index at v, and k is the channel index. The *conflict factor* $\rho(a, b)$ of a link a toward another link $b \in A$, is modified as follows:

- If a and b are node-disjoint and have different channels, then $\rho(a, b) = 0$;
- If a and b share a common radio, then $\rho(a, b) = 1$;
- otherwise,

$$\rho(a, b) = \min\left\{\frac{\sigma\gamma}{\gamma - 1}\left(\frac{\ell(a)}{\ell(a, b)}\right)^\kappa, 1\right\}.$$

Then, a subset I of A is independent if and only if

$$\max_{a \in I} \sum_{b \in I \setminus \{a\}} \rho(b, a) < 1.$$

Let $\mathcal{I}$ be the collection of independent subsets of A. The *local independence number* (LIN) of A is defined to be

$$\mu := \max_{I \in \mathcal{I}} \max_{a \in A}\left(|I \cap \{a\}| + \sum_{b \in I \setminus \{a\}} \rho(b, a)\right).$$

Since each independent set contains at most two links sharing a common radio with a given link, the following modified upper bound on the LIN holds:

$$\mu \leq 2\left\lceil \pi / \arcsin \frac{1 - \left(\frac{\gamma - 1}{\sigma\gamma}\right)^{1/\kappa}}{2} \right\rceil.$$

Consider a link demand $d \in \mathbb{R}_+^A$ and let A^+ denote the set of links with positive demands. Let

$$\Delta(d) := \max_{a \in A}\left[d(a) + \sum_{b \in A \setminus \{a\}} \rho(b, a)\, d(b)\right]$$

Similar to Lemma 4.1 we have

$$\Delta(d) \leq \mu\chi^*(d).$$

Also similar to the SCSR setting, the algorithm $\mathcal{A}$ for selecting cost-effective independent sets adopts the algorithm **WIS**. By Theorem 2.2, we have the following theorem.

Theorem 5.1: For any $\varepsilon \in (0, 1/2]$, the algorithm $\mathbf{LS}(\varepsilon)$ computes a fractional link schedule $\mathcal{S}$ of d with

$$\|\mathcal{S}\| \leq 4(1+\varepsilon)\Delta(d) \leq 4(1+\varepsilon)\mu\chi^*(d).$$

in $O\left(\varepsilon^{-2}|A^+|\ln|A^+|\right)$ calls of $\mathcal{A}$.

Finally, we present a polynomial approximate capacity subregion.

Theorem 5.2: Let

$$\Phi = \left\{d \in \mathbb{R}_+^A : \Delta(d) \leq 1\right\}.$$

Then, Φ is a polynomial $(\mu, 4)$-approximate capacity region.

The proof of the above theorem is similar to the proof of Theorem 4.3 and is omitted.

VI. MIMO Wireless Networks under Protocol Interference Model

Consider an instance of wireless MIMO network on a set V of networking nodes. Each node v has τ antennas and operates in the half-duplex mode, i.e. it cannot transmit and receive at the same time. Then, for each pair of neighboring nodes u and v, there are τ links. Under a protocol interference model, each link is associated with an interference range. With the receiver-side interference suppression, a set I of links is independent if and only if the following two constraints are satisfied:

1) **Half-Duplex Constraint**: A node cannot be both a sender of some link in I and a receiver of some other link in I.
2) **Receiver Constraint**: Each node v which is a receiver of some link in I lies in the interference range of at most τ links in I.

Let $\mathcal{I}$ be the collection of independent subsets of A. We remark that if I independent, each node is a sender of at most τ links in I. Indeed, consider any node u which is a sender of some link in I. Let a be a shortest link with u as the sender. Then the receiver of a lies in the interference range of every link in I with u as the sender (including a itself). Thus, the number of links in I with u as the sender is at most τ.

Consider a link demand $d \in \mathbb{R}_+^A$ and let A^+ denote the set of links with positive demands. Given any subset S of A^+ and a positive weight function w on S, the algorithm $\mathcal{A}$ for selecting cost-effective independent sets is less straightforward than in the previous two sections. The selection proceeds in two steps:

- **Phase 1**: Select a cost-effective subset J of S which satisfies the **Receiver Constraint**.
- **Phase 2**: Select a cost-effective subset I of J which satisfies the **Half-Duplex Constraint**.

We begin with the description of **Phase 1**. For any pair of links a and b in A, define $\rho(a,b)$ to be $1/\tau$ if the receiver of b lies within the interference range of a, and to be 0 otherwise. Then, a set J of links satisfies the **Receiver Constraint** if and only if

$$\max_{a\in J} \sum_{b\in J\setminus\{a\}} \rho(b,a) < 1.$$

Thus, the algorithm **WIS** can be applied to select a set J of links in S satisfying the **Receiver Constraint**. Let

$$\Delta(d) := \max_{a\in A}\left[d(a) + \sum_{b\in A\setminus\{a\}} \rho(b,a)\, d(b)\right]$$

Then, the cost-efficiency of J is at least

$$\frac{w(S)}{4\max_{a\in S}\left[d(a) + \sum_{b\in S\setminus\{a\}} \rho(b,a)\, d(b)\right]} \geq \frac{w(S)}{4\Delta(d)}.$$

We move on to the description of **Phase 2**. Let U be the set of end nodes of the streams in J, and $\langle u_1, u_2, \cdots, u_k\rangle$ be an arbitrary ordering of U. Conceptually, this phase proceeds in two steps

- **Node Partition Step**: U is greedily partitioned into U' and U'' as follows. Initially, $U' = \{u_1\}$ and $U'' = \emptyset$. For $j = 2$ to k, if the set of links in J between u_j and U' is more cost-effective than the set of links in J between u_j and U'', u_j is added to U''; otherwise, u_j is added to U'.
- **Link Partition Step**: The set of links between U' and U'' is partitioned into two subsets: I_1 consists of the links from U' to U'', and I_2 consists of the links from U'' to U'. The more cost-effective one between I_1 and I_2 is output as I, with ties broken arbitrarily.

It is obvious that I satisfies the **Half-Duplex Constraint**. Using the same argument as in Theorem 2.1 in [25], we can show that the cost-efficiency of I is at least $1/4$ of the cost-efficiency of J. Hence the cost-efficiency of the independent I is at least

$$\frac{1}{4}\cdot\frac{w(S)}{4\Delta(d)} = \frac{w(S)}{16\Delta(d)}.$$

By Theorem 2.2, for any $\varepsilon \in (0, 1/2]$ the algorithm **LS**(ε) computes a fractional link schedule $\mathcal{S}$ of d with

$$\|\mathcal{S}\| \leq 16(1+\varepsilon)\,\Delta(d)$$

in $O\left(\varepsilon^{-2}\left|A^+\right|\ln\left|A^+\right|\right)$ calls of $\mathcal{A}$. Next, we derive an approximation bound of this algorithm. The *local independence number* (LIN) of A is defined to be

$$\mu := \max_{I\in\mathcal{I}}\max_{a\in A}\left(|I\cap\{a\}| + \sum_{b\in I\setminus\{a\}}\rho(b,a)\right).$$

In the widely adopted setting in which the interference range of a link a is the disk of radius at least $\frac{1}{c}$ times the length of a for some constant $c \in (0,1)$, it was shown in [24] that

$$\mu \leq \left\lceil \pi/\arcsin\frac{1-c}{2}\right\rceil - 1.$$

In general, similar to Lemma 4.1, we have

$$\Delta(d) \leq \mu\chi^*(d).$$

In summary, we have the following theorem.

Theorem 6.1: For any $\varepsilon \in (0, 1/2]$, the algorithm **LS**(ε) computes a fractional link schedule $\mathcal{S}$ of d with

$$\|\mathcal{S}\| \leq 16(1+\varepsilon)\,\Delta(d) \leq 16(1+\varepsilon)\,\mu\chi^*(d).$$

in $O\left(\varepsilon^{-2}\left|A^+\right|\ln\left|A^+\right|\right)$ calls of $\mathcal{A}$.

Finally, we present a polynomial approximate capacity subregion.

Theorem 6.2: Let

$$\Phi = \left\{ d \in \mathbb{R}_+^A : \Delta(d) \leq 1 \right\}.$$

Then, Φ is a polynomial $(\mu, 16)$-approximate capacity region.

The proof of the above theorem is similar to the proof of Theorem 4.3 and is omitted.

VII. Conclusion

In this paper, we have developed a purely combinatorial paradigm for approximating the problem **SLS** in wireless networks, which is not only efficient but also provides an explicit upper bound on the length of the produced schedule. The power of this paradigm was demonstrated by its applications in SCSR wireless networks under the physical interference model, MCMR wireless networks under the physical interference model, and MIMO wireless networks under the protocol interference model respectively. In these three networks, constant-approximation algorithms for **SLS** based on the reduction to the problem **MWIS** have been developed in the literature. However, the constant-approximation algorithms for the problem **MWIS** in these networks have to solve a linear programming, which is not purely combinatorial and is less efficient. In addition, none of existing approximation algorithms for **SLS** in these networks can provide an explicit upper-bound on the length of the produced schedule. Furthermore, there is no known polynomial approximate capacity region in each of these three networks. In contrast, our new paradigm leads to very efficient link scheduling algorithms, with matchable approximations bounds. The produced link schedule by these algorithms have explicit upper-bounds on their lengths. Such attractive feature is exploited to construct polynomial approximate capacity regions in these three networks.

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References

[1] R. K. Ahuja, T. L. Magnanti, and J. B. Orlin. *Network flows: theory, algorithms, and applications*. Prentice-Hall, 1993.

[2] F. Al-Dhelaan, P.-J. Wan, and H.Q. Yuan, A New Paradigm for Shortest Link Scheduling in Wireless Networks: Theory And Applications, *WASA* 2016.

[3] S. Arora, E. Hazan, and S. Kale. The multiplicative weights update method: A meta-algorithm and application. *Theory of Computing* 8(1):121–164, 2012.

[4] P. Auer, N. Cesa-Bianchi, Y. Freund, and R. Schapire. Gambling in a rigged casino: the adversarial multi-armed bandit problem. In *Proceedings of the 36th IEEE Symposium on Foundations of Computer Science*, pages 322–331, 1995.

[5] D. Bienstock. *Potential function methods for approximately solving linear programs: theory and practice*. Kluwer Academic Publisher, 2002.

[6] N. Cesa-Bianchi, Y. Freund, D. Helmbold, D. Haussler, R. Schapire, and M. Warmuth. How to use expert advice. *Journal of the Association for Computing Machinery*, 44(3):427–485, 1997.

[7] D. Chafekar, V. Kumar, M. Marathe, S. Parthasarathy, and A. Srinivasan, Approximation algorithms for computing capacity of wireless networks with SINR constraints, *IEEE INFOCOM* 2008, pages 1166–1174.

[8] Y. Freund and R. Schapire. A decision-theoretic generalization of online learning and an application to boosting. *Journal of Computer and System Sciences*, 55(1):119–139, 1997.

[9] Y. Freund and R. Schapire. Adaptive game playing using multiplicative weights. *Games and Economic Behavior*, 29:79–103, 1999.

[10] A. Gkelias, F. Boccardi, C.H. Liu, and K.K. Leung, MIMO routing with QoS provisioning, *Proc. ISWPC* 2008, pages 46–50.

[11] B. Hamdaoui, K. Shin, D. Jiang, Characterization and Analysis of Multi-Hop Wireless MIMO Network Throughput, *Proc. ACM MobiHoc* 2007, pp. 120–129.

[12] B. Han, V. S. A. Kumar, M. V. Marathe, S. Parthasarathy, and A. Srinivasan. Distributed Strategies for Channel Allocation and Scheduling in Software-Defined Radio Networks. *Proc. IEEE INFOCOM* 2009, pp. 1521-1529.

[13] R. Khandekar, Lagrangian relaxation based algorithms for convex programming problems, *PhD thesis*, Indian Institute of Technology, Delhi, 2004.

[14] M. Kodialam and T. Nandagopal. Characterizing achievable rates in multi-hop wireless networks: the joint routing and scheduling problem. *Proc. of ACM MobiCom* 2003.

[15] M. Kodialam and T. Nandagopal. Characterizing the capacity region in multi-radio multi-channel wireless mesh networks. *Proc. of ACM MobiCom*, 2005.

[16] V.S.A. Kumar, M.V. Marathe, S. Parthasarathy, and A. Srinivasan. Algorithmic aspects of capacity in wireless networks. S*IGMETRICS Perform. Eval. Rev.* 33(1):133–144, 2005.

[17] J. Liu, Y.T. Hou, Y. Shi, and H. Sherali, Cross-Layer Optimization for MIMO-Based Wireless Ad Hoc Networks: Routing, Power Allocation, and Bandwidth Allocation, *IEEE Journal on Selected Areas in Communications* (6):913–926, August 2008.

[18] A. Schrijver. *Combinatorial Optimization*. Number 24 in Algorithms and Combinatorics. Springer, 2003.

[19] K. Sundaresan and R. Sivakumar, Routing in ad-hoc networks with MIMO links, *Proc. IEEE ICNP* 2005, pages 85–98.

[20] V. Vovk, A game of prediction with expert advice. In *Proceedings of the 8th Annual Conference on Computational Learning Theory*, pages 51–60, 1995.

[21] P.-J. Wan, Multiflows in Multihop Wireless Networks, *ACM MOBIHOC* 2009, pp. 85-94.

[22] P.-J. Wan, O. Frieder, X. Jia, F. Yao, X.-H. Xu, S.-J. Tang, Wireless Link Scheduling under Physical Interference Model, *IEEE INFOCOM* 2011.

[23] P.-J. Wan, X. Jia, G. Dai, H. Du, Z.G. Wan, and O. Frieder, Scalable Algorithms for Wireless Link Schedulings in Multi-Channel Multi-Radio Wireless Networks, *IEEE INFOCOM* 2013.

[24] P.-J. Wan, L. Wang, C. Ma, Z. Wang, B. Xu, and M. Li, Maximizing Wireless Network Capacity with Linear Power: Breaking The Logarithmic Barrier, *IEEE INFOCOM Mini-Conference* 2013.

[25] P.-J. Wan, B. Xu, O. Frieder, S. Ji, B. Wang, and X. Xu, Capacity Maximization in Wireless MIMO Networks with Receiver-Side Interference Suppression, *ACM MOBIHOC* 2014.