Multiflows under Physical Interference Model

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Outline

- Introduction
- Weak Dualities
- An Adaptive Coupled Game
- Maximum Concurrent Multiflow
- Maximum Multiflow

Multiflow

- Multihop wireless network: $(V, A, \mathcal{I})$

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  - $\mathcal{F}_j$: the set of flows of the j-th request
  - $\mathcal{P}_j$: the set of paths of the j-th request
  - A k-flow is a sequence $f = \langle f_1, f_2, \cdots, f_k \rangle$ with $f_j \in \mathcal{F}_j \ \forall j \in [k]$

Reductions from MCMF And MMF to MWIS

- $\mathcal{A}$: μ-approximation algorithm for **MWIS**
- $\varepsilon \in (0, 1/2]$: trade-off between accuracy and efficiency
- **CMF-LS**(ε): $(1 + 2\varepsilon)\,\mu$-approx. alg. for **MCMF** by making $O\left(\varepsilon^{-2}\alpha m \ln m\right)$ calls to $\mathcal{A}$
- **MF-LS**(ε): $(1 + 2\varepsilon)\,\mu$-approx. alg. for **MMF** by making $O\left(\varepsilon^{-2}\alpha m \ln m\right)$ calls to $\mathcal{A}$

Applications to MCMF And MMF under Physical IM

- Linear power assignment: constant-approximation algorithms
- Other monotone and sublinear power assignment: $O(\ln \alpha)$-approximation algorithms
- Power control: $O(\ln \alpha)$-approximation algorithms

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- The total value of the maximum multiflow is at most

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Lemma

For any path-flow decomposition x of f and any link schedule z of f,

$$\sum_{j=1}^{k} \sum_{P \in \mathcal{P}_j} x(P) \, y(P) = \sum_{I \in \mathcal{I}} y(I) \, z(I)$$

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$$\sum_{j=1}^{k} \sum_{P \in \mathcal{P}_j} x(P)\, y(P) = \sum_{j=1}^{k} \sum_{P \in \mathcal{P}_j} x(P) \sum_{a \in A} y(a) \left| \{a\} \cap P \right|$$

$$= \sum_{a \in A} y(a) \sum_{j=1}^{k} \sum_{P \in \mathcal{P}_j} x(P) \left| \{a\} \cap P \right|$$

$$= \sum_{a \in A} y(a) \sum_{j=1}^{k} f_j(a) = \sum_{a \in A} y(a) \sum_{I \in \mathcal{I}} \left| \{a\} \cap I \right| z(I)$$

$$= \sum_{I \in \mathcal{I}} z(I) \sum_{a \in A} y(a) \left| \{a\} \cap I \right|$$

$$= \sum_{I \in \mathcal{I}} y(I)\, z(I)$$

Feasible Multiflow

Lemma

For any feasible f,

$$\sum_{j \in [k]} dist_j\left(\widehat{y}\right) val\left(f_j\right) \leq \max_{I \in \mathcal{I}} y\left(I\right)$$

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$$\sum_{j=1}^{k}\sum_{P\in\mathcal{P}_j} x(P)\, y(P) \geq \sum_{j=1}^{k}\sum_{P\in\mathcal{P}_j} x(P)\, dist_j(y)$$

$$= \sum_{j=1}^{k} dist_j(y) \sum_{P\in\mathcal{P}_j} x(P) = \sum_{j=1}^{k} dist_j(y)\, val(f_j)$$

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$$\sum_{I\in\mathcal{I}} y(I)\, z(I) \le \left(\max_{I\in\mathcal{I}} y(I)\right) \sum_{I\in\mathcal{I}} z(I) \le \max_{I\in\mathcal{I}} y(I).$$

Weak Duality for MMF

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$$\max_{I \in \mathcal{I}} y(I) \geq \sum_{j \in [k]} dist_j(y)\, val(f_j) \geq \left(\min_{j \in [k]} dist_j(y) \right) \sum_{j \in [k]} val(f_j)$$

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    - The agent may then update the weight $y(e)$ of each expert $e \in E$

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- Objective of the adversary: opposite

Multiplicative Weights Update (MWU)

- $\varepsilon \in (0, 1)$
- Initial weight: $y(e) = 1$, $\forall e \in E$
- MWU at the end of each round: $\forall e \in E$,

$$y(e) \leftarrow y(e)(1 - \varepsilon(p(e) - l(e))).$$

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$$L(e) \leftarrow L(e) + l(e);$$
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  - **Multiplicative Weights Update**: The agent updates $y(e)$ for each $e \in E$ by setting

$$y(e) \leftarrow y(e)(1 - \varepsilon(p(e) - l(e))).$$

Theorem

At the end of each round, for any $e \in E$,

$$\frac{\ln(1+\varepsilon)}{\ln(1-\varepsilon)^{-1}} L(e) \leq P(e) + \frac{\ln m}{\ln(1-\varepsilon)^{-1}}$$

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  - $l_r = L_r - L_{r-1}$

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Claim 1: $y_t(E)$ decreases with t
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$$P_t(e)\ln(1-\varepsilon) + L_t(e)\ln(1+\varepsilon) \leq \ln m,$$

$$\frac{\ln(1+\varepsilon)}{\ln(1-\varepsilon)^{-1}} L_t(e) \leq P_t(e) + \frac{\ln m}{\ln(1-\varepsilon)^{-1}}.$$

Proof of Claim 1

$$
\begin{aligned}
y_t(E) &= \sum_{e\in E} y_t(e) \\
&= \sum_{e\in E} y_{t-1}(e)\left(1-\varepsilon\left(\left(p_t(e)-l_t(e)\right)\right)\right) \\
&= \sum_{e\in E} y_{t-1}(e) - \varepsilon \sum_{e\in E} y_{t-1}(e)\left(\left(p_t(e)-l_t(e)\right)\right) \\
&\leq \sum_{e\in E} y_{t-1}(e) \\
&\leq y_{t-1}(E)
\end{aligned}
$$

Proof of Claim 2

$$[t]^{+} = \{i \in [t] : p_i(e) \geq I_i(e)\},$$
$$[t]^{-} = \{i \in [t] : p_i(e) < I_i(e)\}.$$

Proof of Claim 2

$$[t]^+ = \{i \in [t] : p_i(e) \geq l_i(e)\},$$
$$[t]^- = \{i \in [t] : p_i(e) < l_i(e)\}.$$

$$\begin{aligned}
& y_t(e) \\
&= \prod_{i \in [t]} (1 - \varepsilon (p_i(e) - l_i(e))) \\
&= \prod_{i \in [t]^+} (1 - \varepsilon (p_i(e) - l_i(e))) \cdot \prod_{i \in [t]^-} (1 + \varepsilon (l_i(e) - p_i(e))) \\
&\geq \prod_{i \in [t]^+} (1 - \varepsilon)^{p_i(e) - l_i(e)} \cdot \prod_{i \in [t]^-} (1 + \varepsilon)^{l_i(e) - p_i(e)} \\
&= (1 - \varepsilon)^{\sum_{i \in [t]^+} (p_i(e) - l_i(e))} (1 + \varepsilon)^{\sum_{i \in [t]^-} (l_i(e) - p_i(e))}
\end{aligned}$$

Proof of Claim 2

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\begin{aligned}
&(1-\varepsilon)^{\sum_{i\in[t]^+}(p_i(e)-l_i(e))}(1+\varepsilon)^{\sum_{i\in[t]^-}(l_i(e)-p_i(e))} \\
&= (1-\varepsilon)^{\sum_{i\in[t]^+} p_i(e)}\left(\frac{1}{1-\varepsilon}\right)^{\sum_{i\in[t]^+}\ell_i(e)}\left(\frac{1}{1+\varepsilon}\right)^{\sum_{i\in[t]^-} p_i(e)}(1+\varepsilon)^{\sum_{i\in[t]^-} l_i(e)} \\
&\geq (1-\varepsilon)^{\sum_{i\in[t]^+} p_i(e)}(1+\varepsilon)^{\sum_{i\in[t]^+}\ell_i(e)}(1-\varepsilon)^{\sum_{i\in[t]^-} p_i(e)}(1+\varepsilon)^{\sum_{i\in[t]^-} l_i(e)} \\
&= (1-\varepsilon)^{\sum_{i\in[t]} p_i(e)}(1+\varepsilon)^{\sum_{i\in[t]} l_i(e)} \\
&= (1-\varepsilon)^{P_t(e)}(1+\varepsilon)^{L_t(e)}.
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- d': *deficit* demand function of links given by

$$d'(a) = \max\left\{0, \frac{\ln(1+\varepsilon)}{\ln(1-\varepsilon)^{-1}} \lambda \sum_{j \in [k]} f_j(a) - g(a)\right\}, \forall a \in A$$

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$$d'(a) \leq \frac{\ln m}{\ln(1-\varepsilon)^{-1}}, \forall a \in A$$

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- If

$$\frac{\ell'}{\ell} \leq \varepsilon' := \frac{(1+2\varepsilon)\ln(1+\varepsilon)+\ln(1-\varepsilon)}{\ln(1-\varepsilon)^{-1}},$$

then return

$$\left(\frac{1}{\ell+\ell'}\frac{\ln(1+\varepsilon)}{\ln(1-\varepsilon)^{-1}}\lambda f, \frac{1}{\ell+\ell'}\left(\Gamma\cup\Gamma'\right)\right)$$

Overview: Approximation Bound

The concurrency of the returned multiflow is

$$\begin{aligned}
&\frac{\ell}{\ell+\ell'}\frac{\ln(1+\varepsilon)}{\ln(1-\varepsilon)^{-1}}\lambda \\
&= \frac{1}{1+\ell'/\ell}\frac{\ln(1+\varepsilon)}{\ln(1-\varepsilon)^{-1}}\lambda \\
&\geq \frac{1}{1+\varepsilon'}\frac{\ln(1+\varepsilon)}{\ln(1-\varepsilon)^{-1}}\lambda \\
&\geq \frac{\lambda}{1+2\varepsilon} \\
&\geq \frac{opt}{(1+2\varepsilon)\mu}.
\end{aligned}$$

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for each $J \in \mathcal{J}$ do
$\quad \delta' \leftarrow \max_{a \in J} d'(a)$;
$\quad$ if $\delta' > 0$ then $\Gamma' \leftarrow \Gamma' \cup \{(J, \delta')\}, \ell' \leftarrow \ell' + \delta'$;

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$$\ell' = \sum_{J \in \mathcal{J}} \max_{a \in J} d'(a) \leq \frac{l \ln m}{\ln (1 - \varepsilon)^{-1}}.$$

Outline of Algorithm Design

Algorithm **CMF-LS**(ε):
// initialization $\vdots$

repeat forever
    // primary schedule and flow augmentation
    $\vdots$

// complementary schedule computation $\vdots$
// test of termination   if $\ell' \leq \varepsilon' \ell$ then return $\frac{1}{\ell+\ell'} \frac{\ln(1+\varepsilon)}{\ln(1-\varepsilon)^{-1}} f$ and $\frac{1}{\ell+\ell'} (\Gamma \cup \Gamma')$.
//MWU $\vdots$

Initialization

$\forall j \in [k]$, $f_j \leftarrow \mathbf{0}$; $\Gamma \leftarrow \emptyset$, $\ell \leftarrow 0$, $g \leftarrow \mathbf{0}$; $\lambda \leftarrow \infty$;
$y \leftarrow \mathbf{1}$, $\varepsilon' \leftarrow \frac{(1+2\varepsilon)\ln(1+\varepsilon)+\ln(1-\varepsilon)}{\ln(1-\varepsilon)^{-1}}$;
$\mathcal{J} \leftarrow$ a link partition output by **GreedyFC** with $\mathcal{A}$;

Augmentation And MWU

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\boxed{
\begin{array}{l}
I \leftarrow \text{a } y\text{-weighted IS in } \mathcal{I} \text{ output by } \mathcal{A};\\
\forall j \in [k],\ P_j \leftarrow \text{a shortest } j\text{-path w.r.t. } y;\\
\lambda \leftarrow \min\left\{\lambda, \frac{y(I)}{\sum_{j\in[k]} d_j y(P_j)}\right\};\\
\delta \leftarrow \frac{1}{\max_{a\in A}\left| |\{a\}\cap I| - \lambda \sum_{j=1}^{k} d_j |\{a\}\cap P_j| \right|};\\
\Gamma \leftarrow \Gamma \cup \{(I,\delta)\},\ \ell \leftarrow \ell + \delta;\\
\forall a \in I,\ g(a) \leftarrow g(a) + \delta;\\
\forall j \in [k] \text{ and } \forall a \in P_j,\ f_j(a) \leftarrow f_j(a) + \delta d_j;
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}
$$

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Interpretation As An Adaptive Coupled Game

- Each link $a \in A \leftrightarrow$ an adversary

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- Each link $a \in A \leftrightarrow$ an adversary
- Each iteration $\leftrightarrow$ a game round
- Agent: WMU strategy
- Adversary: At the end of each round,

$$\begin{aligned} \text{cumulative profit of } a &= g(a) \\ \text{cumulative loss of } a &\geq \lambda \sum_{j \in [k]} f_j(a). \end{aligned}$$

Profits/Losses Generation In Each Round

$$\boxed{\begin{array}{l} I \leftarrow \text{a } y\text{-weighted IS in } \mathcal{I} \text{ output by } \mathcal{A}; \\ \forall j \in [k],\ P_j \leftarrow \text{a shortest } j\text{-path w.r.t. } y; \\ \lambda \leftarrow \min\left\{\lambda, \frac{y(I)}{\sum_{j\in[k]} d_j y(P_j)}\right\}; \\ \delta \leftarrow \frac{1}{\max_{a\in A}\left| |\{a\}\cap I| - \lambda \sum_{j=1}^{k} d_j |\{a\}\cap P_j| \right|}; \end{array}}$$

- profit of $a \in A$: $\delta \left|\{a\} \cap I\right|$

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- profit of $a \in A$: $\delta \left|\{a\} \cap I\right|$
- loss of $a \in A$: $\delta\lambda \sum_{j=1}^{k} d_j \left|\{a\} \cap P_j\right|$
- σ is determined by the **Normalization Rule**

Generalized Zero-Sum Rule

$$
\begin{aligned}
&\sum_{a\in A} y(a)\left(\delta\left|\{a\}\cap I\right| - \delta\lambda\sum_{j\in[k]} d_j\left|\{a\}\cap P_j\right|\right)\\
&= \delta\left(\sum_{a\in A} y(a)\left|\{a\}\cap I\right| - \lambda\sum_{a\in A} y(a)\sum_{j=1}^{k} d_j\left|\{a\}\cap P_j\right|\right)\\
&= \delta\left(y(I) - \lambda\sum_{j=1}^{k} d_j\sum_{a\in A} y(a)\left|\{a\}\cap P_j\right|\right)\\
&= \delta\left(y(I) - \lambda\sum_{j=1}^{k} d_j y_{t-1}(P_j)\right)\\
&\geq 0
\end{aligned}
$$

Algorithm Analysis

- h_j: the hop number of a min-hop path of the j-th request for each $j \in [k]$
- α: the size of a maximum-sized independent set of A

Theorem

The algorithm **CMF-LS**(ε) *runs in*

$$O\left(\varepsilon^{-2} \max\left\{1, \frac{\alpha}{\min_{j\in[k]} h_j}\right\} l \ln m\right).$$

iterations and has an approximation bound $(1+2\varepsilon)\,\mu$.

Invariant Properties

Claim 1: In each iteration, the deficit of each link is at most $\frac{\ln m}{\ln(1-\varepsilon)^{-1}}$, and hence

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$$\delta \geq \frac{1}{\max\left\{1, \frac{\alpha}{\min_{j\in[k]} h_j}\right\}}.$$

Proof of Claim 3

For each a,

$$
\begin{aligned}
&\left| \left|\{a\} \cap I\right| - \lambda \sum_{j=1}^{k} d_j \left|\{a\} \cap P_j\right| \right| \\
&\leq \max \left\{ \left|\{a\} \cap I\right|, \lambda \sum_{j=1}^{k} d_j \left|\{a\} \cap P_j\right| \right\} \\
&\leq \max \left\{ 1, \lambda \sum_{j=1}^{k} d_j \right\} \\
&\leq \max \left\{ 1, \frac{\alpha}{\sum_{j \in [k]} d_j h_j} \sum_{j=1}^{k} d_j \right\} \\
&\leq \max \left\{ 1, \frac{\alpha}{\min_{j \in [k]} h_j} \right\}
\end{aligned}
$$

Number of Iterations

Claim 4: The number of iterations is at most

$$
\begin{aligned}
t &= \left\lceil \frac{\max\left\{1, \frac{\alpha}{\min_{j\in[k]} h_j}\right\} / \ln m}{(1+2\varepsilon)\ln(1+\varepsilon)+\ln(1-\varepsilon)} \right\rceil \\
&= O\left(\varepsilon^{-2} \max\left\{1, \frac{\alpha}{\min_{j\in[k]} h_j}\right\} / \ln m\right).
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$$= O\left(\varepsilon^{-2} \max\left\{1, \frac{\alpha}{\min_{j\in[k]} h_j}\right\} l \ln m\right).$$

The primary schedule Γ in the iteration t would have length

$$\ell \geq t \frac{1}{\max\left\{1, \frac{\alpha}{\min_{j\in[k]} h_j}\right\}} \geq \frac{l \ln m}{(1+2\varepsilon)\ln(1+\varepsilon) + \ln(1-\varepsilon)}$$

$$= \frac{l \ln m}{\varepsilon' \ln(1-\varepsilon)^{-1}} \geq \frac{\ell'}{\varepsilon'}.$$

Roadmap

- Introduction
- Weak Dualities
- An Adaptive Coupled Game
- Maximum Concurrent Multiflow
- **Maximum Multiflow**

Outline of Algorithm Design

Algorithm **MF-LS**(ε):
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& \leq \max \{1, \lambda\} \\
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$$

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