Multiflows under Protocol Interference Model

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Outline

- Introduction
- Adaptive Zero-Sum Game
- Weak Dualities
- Flow Augmentation Methods
- Extensions to Budgeted Variants
• Multihop wireless network: \((V, A, \mathcal{I})\)
Multihop wireless network: $(V, A, I)$

- $k$ unicast requests.
- Multihop wireless network: \((V, A, I)\)
- \(k\) unicast requests.
- \(\mathcal{F}_j\): the set of flows of the \(j\)-th request
Multihop wireless network: \((V, A, I)\)

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- \(P_j\): the set of paths of the \(j\)-th request
Multihop wireless network: \((V, A, \mathcal{I})\)

- \(k\) unicast requests.
- \(\mathcal{F}_j\): the set of flows of the \(j\)-th request
- \(\mathcal{P}_j\): the set of paths of the \(j\)-th request

A \(k\)-flow is a sequence \(f = (f_1, f_2, \ldots, f_k)\) with \(f_j \in \mathcal{F}_j \ \forall j \in [k]\)
Theorem

If there is a polynomial (respectively, a polynomial $\mu$-approximation) algorithm for MWISL, then there exist polynomial time $\mu$-approximate algorithms for all four variants of multiflow problems.
Theorem

All four variants of multiflow problems under protocol interference model have a PTAS.
Approximation-Preserving Reduction to Polynomial Approximate Capacity Subregions

Theorem

If there exists a polynomial $\mu$-approximate capacity subregion, then there exist polynomial time $\mu$-approximate algorithms for each of the four variants of multiflow problems.
Maximum Multiflow

Ω: capacity region

\[ \text{MWMF} \]

\[
\begin{align*}
\text{max} \quad & \sum_{j=1}^{k} w_j \text{val}(f_j) \\
\text{s.t.} \quad & f_j \in \mathcal{F}_j, \forall 1 \leq j \leq k \\
& \sum_{j=1}^{k} f_j \in \Omega
\end{align*}
\]

\[ \text{MCMF} \]

\[
\begin{align*}
\text{max} \quad & \phi \\
\text{s.t.} \quad & f_j \in \mathcal{F}_j, \forall 1 \leq j \leq k \\
& \text{val}(f_j) \geq \phi w_j, \forall 1 \leq j \leq k \\
& \sum_{j=1}^{k} f_j \in \Omega
\end{align*}
\]
Maximum Restricted Multiflow

Φ: a poly. $\mu$-approx. capacity subregion

**Φ-restricted MWMF**

$$\max \sum_{j=1}^{k} w_j \text{val}(f_j)$$

s.t. $f_j \in \mathcal{F}_j$, $\forall 1 \leq j \leq k$

$$\sum_{j=1}^{k} f_j \in \Phi$$

**Φ-restricted MCMF**

$$\max \phi$$

s.t. $f_j \in \mathcal{F}_j$, $\forall 1 \leq j \leq k$

$$\text{val}(f_j) \geq \phi w_j$, $\forall 1 \leq j \leq k$

$$\sum_{j=1}^{k} f_j \in \Phi$$
Maximum Restricted Multiflow

Step 1: solve the Φ-restricted LP to obtain a $k$-flow $f = \langle f_1, f_2, \cdots, f_k \rangle$.
Step 2: compute a fractional link schedule $S$ of $\sum_{j=1}^{k} f_j$
Step 3: scale $f$ and $S$ by a factor $1/ \|S\|$
approximation bound: $\mu$
Restrictions to the polynomial capacity subregions constructed in the previous chapter yield the following approximation bounds:

- $2 \left( \left\lceil \frac{\pi}{\arcsin \frac{c-1}{2c}} \right\rceil - 1 \right)$ in general,
- $\left\lceil \frac{r+1}{h(r)} \right\rceil + 1$ with uniform interference radii.
Flow Augmentation Method for MWMF

- Parameters:

  \[ D \text{: an orientation of } D \text{ with } ILIN \mu \in \mathbb{R}^2 \text{ } (0, 1] : \text{ the trade-off parameter} \]

Flow Augmentation Stage: Compute a \( k \)-optimal flow \( f \) such that

\[
\sum_{j \in \left[ k \right]} w_{j \text{val}}(f_j) \Delta \text{in}_D \sum_{j \in \left[ k \right]} f_j \text{opt}(1 + \epsilon) \mu.
\]

Link-Scheduling Stage: Compute a link schedule \( S \) with \( GLS \).

Scaling Stage: Scale both \( \Pi \) and \( S \) by a factor \( \frac{1}{k_S} \) and then return them.
Flow Augmentation Method for MWMF

- **Parameters:**

  - $D$: an orientation of $D$ with ILIN $\mu$
Flow Augmentation Method for MWMF

- **Parameters:**
  - $D$: an orientation of $D$ with ILIN $\mu$
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Flow Augmentation Method for MWMF

- Parameters:
  - \( D \): an orientation of \( D \) with ILIN \( \mu \)
  - \( \epsilon \in (0, 1] \): the trade-off parameter

- Flow Augmentation Stage: Compute a \( k \)-flow \( f \) s.t.

\[
\frac{\sum_{j \in [k]} w_j \text{val}(f_j)}{\Delta^\text{in}_D \left( \sum_{j \in [k]} f_j \right)} \geq \frac{\text{opt}}{(1 + \epsilon) \mu}.
\]
Flow Augmentation Method for MWMF

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- **Link-Scheduling Stage:** Compute a link schedule \( S \) of \( x \) with GLS.
Flow Augmentation Method for MWMF

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  - $D$: an orientation of $D$ with ILIN $\mu$
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  \[
  \frac{\sum_{j \in [k]} w_j \text{val}(f_j)}{\Delta^\text{in}_D \left( \sum_{j \in [k]} f_j \right)} \geq \frac{\text{opt}}{(1 + \varepsilon) \mu}.
  \]

- Link-Scheduling Stage: Compute a link schedule $\mathcal{S}$ of $x$ with GLS.

- Scaling Stage: Scale both $\Pi$ and $\mathcal{S}$ by a factor $1/\|\mathcal{S}\|$ and then return them.
Approximation Bound

- $2(1 + \varepsilon)\mu$ in general:

\[
\frac{\sum_{j \in [k]} w_j \text{val}(f_j)}{\|S\|} \geq \frac{\sum_{j \in [k]} w_j \text{val}(f_j)}{\Delta^* \left(\sum_{j \in [k]} f_j\right)} \geq \frac{\sum_{j \in [k]} w_j \text{val}(f_j)}{2\Delta^{in}_D \left(\sum_{j \in [k]} f_j\right)} \geq \frac{\text{opt}}{2(1 + \varepsilon)\mu}.
\]
Approximation Bound

- $2 \left( 1 + \varepsilon \right) \mu$ in general:

$$\frac{\sum_{j \in [k]} w_j \text{val} \left( f_j \right)}{\|S\|} \geq \frac{\sum_{j \in [k]} w_j \text{val} \left( f_j \right)}{\Delta^* \left( \sum_{j \in [k]} f_j \right)} \geq \frac{\sum_{j \in [k]} w_j \text{val} \left( f_j \right)}{2 \Delta_D^{\text{in}} \left( \sum_{j \in [k]} f_j \right)} \geq \frac{\text{opt}}{2 \left( 1 + \varepsilon \right) \mu}.$$

- $(1 + \varepsilon) \mu$ if $D$ is acyclic:

$$\frac{\sum_{j \in [k]} w_j \text{val} \left( f_j \right)}{\|S\|} \geq \frac{\text{opt}}{(1 + \varepsilon) \mu}.$$
Flow Augmentation Method for MCMF

- Parameters:

- Flow Augmentation Stage: Compute a $k \in \mathbb{N}$ such that
  \[ \min_j \sum_{\mathcal{D}} \left( f_j \right) \Delta_{\mathcal{D}} \text{val}(f_j) w_j \Delta_{\mathcal{D}} \sum_j 2 \left[ k \right] f_j \text{opt} \left( 1 + \epsilon \right) \mu. \]

- Link-Scheduling Stage: Compute a link schedule $S$ with GLS.

- Scaling Stage: Scale both $\Pi$ and $S$ by a factor $1/k$ and then return them.
Flow Augmentation Method for MCMF

- **Parameters:**
  - $D$: an orientation of $D$ with ILIN $\mu$
Flow Augmentation Method for MCMF

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- $D$: an orientation of $D$ with ILIN $\mu$
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- **Parameters:**
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\[
\min_{j \in [k]} \frac{\text{val}(f_j)}{w_j} \geq \frac{\text{opt}}{(1 + \varepsilon) \mu}.
\]
Flow Augmentation Method for MCMF

- **Parameters:**
  - $D$: an orientation of $D$ with ILIN $\mu$
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\]

- **Link-Scheduling Stage:** Compute a link schedule $S$ of $x$ with GLS.

- **Scaling Stage:** Scale both $\Pi$ and $S$ by a factor $1 / \|S\|$ and then return them.
- How to define an appropriate interference costs of links and paths?
- How much flow should be routed along those cheapest paths?
Roadmap

- Introduction
- **Adaptive Zero-Sum Game**
- Weak Dualities
- Flow Augmentation Methods
- Extensions to Budgeted Variants
Adaptive Zero-Sum Game

- Players: an agent and a set $E$ of $m$ loss-incurring adversaries.
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- Rules in each sequential game round:
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- Rules in each sequential game round:
  - The agent declares an \textit{adaptive binding strategy} in terms of probabilistic distribution on adversaries.
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- Players: an agent and a set $E$ of $m$ loss-incurring adversaries.
- Rules in each sequential game round:
  - The agent declares an adaptive binding strategy in terms of probabilistic distribution on adversaries.
  - The adversaries picks their losses subjected to the Normalization Rule: the maximum value of the individual losses $= 1$. 

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Adaptive Zero-Sum Game

- Players: an agent and a set $E$ of $m$ loss-incurring adversaries.
- Rules in each sequential game round:
  - The agent declares an adaptive binding strategy in terms of probabilistic distribution on adversaries.
  - The adversaries picks their losses subjected to the **Normalization Rule**: the maximum value of the individual losses $= 1$.
  - The agent collects the profit according to the **Zero-Sum Rule**: the profit $= \text{expected loss of the adversaries w.r.t the agent's binding strategy.}$
Adaptive Zero-Sum Game

- Players: an agent and a set \( E \) of \( m \) loss-incurring adversaries.

Rules in each sequential game round:

- The agent declares an *adaptive binding strategy* in terms of probabilistic distribution on adversaries.
- The adversaries picks their losses subjected to the **Normalization Rule**: the maximum value of the individual losses is 1.
- The agent collects the profit according to the **Zero-Sum Rule**: the profit is the expected loss of the adversaries w.r.t the agent's binding strategy.

- **Objective**: to realize asap maximum cumulative loss of adversaries \( \leq (1 + \varepsilon) \times \) cumulative profit of the agent.
Exponential Binding Strategies

- $L(e)$: the cumulative loss of $e \in E$ at the beginning of a round
- The probability of $e \in E$: $\propto (1 + \varepsilon)^{L(e)}$. 
Design of The Game

Repeat

\[ t = \left\lfloor \frac{m \ln m}{\ln (1 + \varepsilon) - \frac{\varepsilon}{1 + \varepsilon}} \right\rfloor \]

rounds:

1. Declaration of exponential binding strategies by agent
Repeat

\[ t = \left\lceil \frac{m \ln m}{\ln (1 + \varepsilon) - \frac{\varepsilon}{1+\varepsilon}} \right\rceil \]

rounds:

1. **Declaration of exponential binding strategies** by agent
2. **Generation of loss** by adversary s.t. the **Normalization Rule**
Analysis of The Game

\[ t = O (\varepsilon^{-2} m \ln m) \text{ as} \]

\[
\ln (1 + \varepsilon) - \frac{\varepsilon}{1 + \varepsilon} = - \ln \left(1 - \frac{\varepsilon}{1 + \varepsilon}\right) - \frac{\varepsilon}{1 + \varepsilon} \\
\geq \frac{1}{2} \left(\frac{\varepsilon}{1 + \varepsilon}\right)^2 = \frac{1}{2} \left(1 + 1/\varepsilon\right)^{-2}.
\]
Analysis of The Game

\[ t = O \left( \varepsilon^{-2} m \ln m \right) \text{ as} \]

\[
\ln (1 + \varepsilon) - \frac{\varepsilon}{1 + \varepsilon} = -\ln \left( 1 - \frac{\varepsilon}{1 + \varepsilon} \right) - \frac{\varepsilon}{1 + \varepsilon}
\]

\[
\geq \frac{1}{2} \left( \frac{\varepsilon}{1 + \varepsilon} \right)^2 = \frac{1}{2} \left( 1 + \frac{1}{\varepsilon} \right)^{-2}.
\]

**Theorem**

At the end of the game, the maximum cumulative loss of the adversaries is at most \(1 + \varepsilon\) times the cumulative profit of the agent.
Analysis of The Game

- For each round $r \in [t]$
Analysis of The Game

- For each round $r \in [t]$
  - $L_r(e)$: cumulative loss of $e \in E$ at the end of round $r$
For each round $r \in [t]$

- $L_r(e)$: cumulative loss of $e \in E$ at the end of round $r$
- $\ell_r$: cumulative profit of the agent at the end of round $r$
For each round $r \in [t]$

- $L_r(e)$: cumulative loss of $e \in E$ at the end of round $r$
- $\ell_r$: cumulative profit of the agent at the end of round $r$

$L_0(e) = 0, \forall e \in E; \ell_0 = 0.$
Analysis of The Game

- For each round $r \in [t]$
  - $L_r(e)$: cumulative loss of $e \in E$ at the end of round $r$
  - $\ell_r$: cumulative profit of the agent at the end of round $r$

- $L_0(e) = 0, \forall e \in E; \ell_0 = 0.$

- $y_r(e) = (1 + \varepsilon)^{L_r(e)}: \forall 0 \leq r \leq t$ and $\forall e \in E$

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Analysis of The Game

- For each round $r \in [t]$
  - $L_r(e)$: cumulative loss of $e \in E$ at the end of round $r$
  - $\ell_r$: cumulative profit of the agent at the end of round $r$

- $L_0(e) = 0, \forall e \in E$; $\ell_0 = 0$.

- $y_r(e) = (1 + \varepsilon)^{L_r(e)}$: $\forall 0 \leq r \leq t$ and $\forall e \in E$

- Zero-Sum Rule

$$\ell_r - \ell_{r-1} = \sum_{e \in E} \frac{y_{r-1}(e)}{y_{r-1}(E)} (L_r(e) - L_{r-1}(e)).$$
Analysis of The Game

\[ y_r (E) = \sum_{e \in E} y_r (e) = \sum_{e \in E} y_{r-1} (e) (1 + \varepsilon)^{L_r(e) - L_{r-1}(e)} \]

\[ \leq \sum_{e \in E} y_{r-1} (e) (1 + \varepsilon (L_r (e) - L_{r-1} (e))) \]

\[ = \sum_{e \in E} y_{r-1} (e) + \varepsilon \sum_{e \in E} y_{r-1} (e) (L_r (e) - L_{r-1} (e)) \]

\[ = y_{r-1} (E) \left( 1 + \varepsilon \sum_{e \in E} \frac{y_{r-1} (e)}{E} (L_r (e) - L_{r-1} (e)) \right) \]

\[ = y_{r-1} (E) (1 + \varepsilon (l_r - l_{r-1})) \]

\[ \leq y_{r-1} (E) \exp (\varepsilon (l_r - l_{r-1})). \]
Analysis of The Game

\[ y_t (E) \leq y_0 (E) \prod_{r=1}^{t} \exp (\varepsilon (\ell_r - \ell_{r-1})) = m \exp (\varepsilon \ell_t). \]
Analysis of The Game

\[ y_t(E) \leq y_0(E) \prod_{r=1}^{t} \exp(\varepsilon (\ell_r - \ell_{r-1})) = m \exp(\varepsilon \ell_t). \]

\[ \varepsilon \ell_t \geq \ln \frac{y_t(E)}{m} \geq \ln \frac{y_t(e)}{m} = L_t(e) \ln (1 + \varepsilon) - \ln m, \forall e \in E \]
Analysis of The Game

\( e \): an adversary with the largest cumulative loss

\[
L_t(e) \geq \frac{t}{m} \geq \frac{\ln m}{\ln (1 + \varepsilon) - \frac{\varepsilon}{1 + \varepsilon}}.
\]
e: an adversary with the largest cumulative loss

\[ L_t(e) \geq \frac{t}{m} \geq \frac{\ln m}{\ln(1 + \varepsilon) - \frac{\varepsilon}{1 + \varepsilon}}. \]

\[ L_t(e) \ln(1 + \varepsilon) - \ln m \geq \frac{\varepsilon}{1 + \varepsilon} L_t(e). \]
e: an adversary with the largest cumulative loss

\[ L_t(e) \geq \frac{t}{m} \geq \frac{\ln m}{\ln (1 + \varepsilon)} - \frac{\varepsilon}{1 + \varepsilon}. \]

\[ L_t(e) \ln (1 + \varepsilon) - \ln m \geq \frac{\varepsilon}{1 + \varepsilon} L_t(e). \]

\[ \varepsilon \ell_t \geq L_t(e) \ln (1 + \varepsilon) - \ln m \geq \frac{\varepsilon}{1 + \varepsilon} L_t(e) \]
e: an adversary with the largest cumulative loss

\[ L_t(e) \geq \frac{t}{m} \geq \frac{\ln m}{\ln (1 + \varepsilon) - \frac{\varepsilon}{1 + \varepsilon}}. \]

\[ L_t(e) \ln (1 + \varepsilon) - \ln m \geq \frac{\varepsilon}{1 + \varepsilon} L_t(e). \]

\[ \varepsilon \ell_t \geq L_t(e) \ln (1 + \varepsilon) - \ln m \geq \frac{\varepsilon}{1 + \varepsilon} L_t(e) \]

\[ L_t(e) \leq (1 + \varepsilon) \ell_t. \]
Roadmap

- Introduction
- Adaptive Zero-Sum Game
- **Weak Dualities**
- Flow Augmentation Methods
- Extensions to Budgeted Variants
Weak Dualities

- $y$: positive function on $A$
Weak Dualities

- \( y \): positive function on \( A \)
- \( \hat{y} \): length function on \( A \) induced by \( y \)

\[
\hat{y}(a) = y(N_{D}^{\text{out}}[a])
\]
Weak Dualities

- \( y \): positive function on \( A \)
- \( \hat{y} \): length function on \( A \) induced by \( y \)
  \[ \hat{y}(a) = y(N_{D}^{out}[a]) \]
- \( \text{dist}_{j}(\hat{y}) \ \forall \ j \in [k] \): length of a shortest \( j \)-path w.r.t. \( \hat{y} \)
The weight of the maximum weighted multflow

\[ \leq \mu y(A) \max_{j \in [k]} w_j \frac{\text{dist}_j(\hat{y})}{\text{dist}_j(\hat{y})}. \]
Weak Dualities

- The weight of the maximum weighted multiflow

\[ \leq \mu y(A) \max_{j \in [k]} \frac{w_j}{\text{dist}_j(\hat{y})}. \]

- The concurrency of the maximum concurrent multiflow is at most

\[ \leq \mu \frac{y(A)}{\sum_{j \in [k]} w_j \text{dist}_j(\hat{y})}. \]
A non-negative function $x$ on $\bigcup_{j \in [k]} \mathcal{P}_j$ is said to be a path-flow decomposition of a multiflow $f = (f_1, \cdots, f_k)$ if

$$f_j(a) = \sum_{P \in \mathcal{P}_j} |\{a\} \cap P| x(P), \forall a \in A, j \in [k]$$
A non-negative function $x$ on $\bigcup_{j \in [k]} P_j$ is said to be a \textit{path-flow decomposition} of a multiflow $f = (f_1, \cdots, f_k)$ if

$$f_j(a) = \sum_{P \in P_j} |\{a\} \cap P| x(P), \forall a \in A, j \in [k]$$

\[\text{Lemma}\]

\textit{For any path-flow decomposition $x$ of a multiflow $f = (f_1, \cdots, f_k)$,}

$$\sum_{j \in [k]} \sum_{P \in P_j} x(P) \hat{y}(P) = \sum_{a \in A} y(a) \sum_{j \in [k]} f_j(N_D^{in}[a])$$
\[
\sum_{j \in [k]} \sum_{P \in \mathcal{P}_j} x(P) \hat{y}(P)
\]

\[
= \sum_{j \in [k]} \sum_{P \in \mathcal{P}_j} x(P) \sum_{b \in P} \hat{y}(b)
\]

\[
= \sum_{j \in [k]} \sum_{P \in \mathcal{P}_j} x(P) \sum_{b \in P} y\left( N^\text{out}_D [b] \right)
\]

\[
= \sum_{j \in [k]} \sum_{P \in \mathcal{P}_j} x(P) \sum_{b \in \mathcal{A}} |\{b\} \cap P| y\left( N^\text{out}_D [b] \right)
\]

\[
= \sum_{j \in [k]} \sum_{P \in \mathcal{P}_j} x(P) \sum_{b \in \mathcal{A}} |\{b\} \cap P| \sum_{a \in N^\text{out}_D [b]} y(a)
\]

\[
= \sum_{j \in [k]} \sum_{P \in \mathcal{P}_j} x(P) \sum_{b \in \mathcal{A}} |\{b\} \cap P| \sum_{a \in \mathcal{A}} y(a) |\{b\} \cap N^\text{in}_D [a]|
\]
Path-Flow Decomposition

\[
\sum_{j \in [k]} \sum_{P \in \mathcal{P}_j} x(P) \sum_{b \in A} |\{b\} \cap P| \sum_{a \in A} y(a) |\{b\} \cap N^\text{in}_D [a]|
\]

\[
= \sum_{a \in A} y(a) \sum_{j \in [k]} \sum_{b \in A} |\{b\} \cap N^\text{in}_D [a]| \sum_{P \in \mathcal{P}_j} |\{b\} \cap P| x(P)
\]

\[
= \sum_{a \in A} y(a) \sum_{j \in [k]} \sum_{b \in A} |\{b\} \cap N^\text{in}_D [a]| f_j (b)
\]

\[
= \sum_{a \in A} y(a) \sum_{j \in [k]} \sum_{b \in N^\text{in}_D [a]} f_j (b)
\]

\[
= \sum_{a \in A} y(a) \sum_{j \in [k]} f_j (N^\text{in}_D [a]),
\]
Lemma

For any feasible multflow $f = (f_1, \cdots, f_k)$,

$$\sum_{j \in [k]} \text{dist}_j (\hat{y}) \text{val} (f_j) \leq \mu_y (A)$$
Lemma

For any feasible multflow \( f = (f_1, \cdots, f_k) \),

\[
\sum_{j \in [k]} \text{dist}_j (\hat{y}) \text{val} (f_j) \leq \mu \gamma (A)
\]

\[
\sum_{j \in [k]} \sum_{P \in \mathcal{P}_j} x (P) \hat{y} (P) \geq \sum_{j \in [k]} \sum_{P \in \mathcal{P}_j} x (P) \text{dist}_j (\hat{y})
\]

\[
= \sum_{j \in [k]} \text{dist}_j (\hat{y}) \sum_{P \in \mathcal{P}_j} x (P) = \sum_{j \in [k]} \text{dist}_j (\hat{y}) \text{val} (f_j).
\]
Lemma

For any feasible multflow $f = (f_1, \cdots, f_k)$,

$$
\sum_{j \in [k]} dist_j (\hat{y}) \text{val} (f_j) \leq \mu y (A)
$$

$$
\sum_{j \in [k]} \sum_{P \in \mathcal{P}_j} x (P) \hat{y} (P) \geq \sum_{j \in [k]} \sum_{P \in \mathcal{P}_j} x (P) \text{dist}_j (\hat{y})
$$

$$
= \sum_{j \in [k]} \text{dist}_j (\hat{y}) \sum_{P \in \mathcal{P}_j} x (P) = \sum_{j \in [k]} \text{dist}_j (\hat{y}) \text{val} (f_j).
$$

$$
\sum_{a \in A} y (a) \sum_{j \in [k]} f_j (N^\text{in}_D [a]) \leq \Delta^\text{in}_D \left( \sum_{j \in [k]} f_j \right) \sum_{b \in A} y (a) \leq \mu y (A).
$$
Weak Duality for MWMF

\[ f = (f_1, \cdots, f_k): \text{a maximum weighted multiflow} \]

\[
\mu y (A) \geq \sum_{j \in [k]} \text{dist}_j (\hat{y}) \text{val} (f_j) = \sum_{j \in [k]} \frac{\text{dist}_j (\hat{y})}{w_j} w_j \text{val} (f_j)
\]

\[
\geq \left( \min_{j \in [k]} \frac{\text{dist}_j (\hat{y})}{w_j} \right) \sum_{j \in [k]} w_j \text{val} (f_j)
\]
$f = (f_1, \ldots, f_k)$: a maximum weighted multi-flow

$$\mu y(A) \geq \sum_{j \in [k]} \text{dist}_j(\hat{y}) \text{val}(f_j) = \sum_{j \in [k]} \frac{\text{dist}_j(\hat{y})}{w_j} w_j \text{val}(f_j)$$

$$\geq \left( \min_{j \in [k]} \frac{\text{dist}_j(\hat{y})}{w_j} \right) \sum_{j \in [k]} w_j \text{val}(f_j)$$

$$\sum_{j \in [k]} w_j \text{val}(f_j) \leq \mu \frac{y(A)}{\min_{j \in [k]} \frac{\text{dist}_j(\hat{y})}{w_j}}$$
Weak Duality for MCMF

\[ f = (f_1, \cdots, f_k): \text{a maximum concurrent multiflow} \]

\[
\mu_y(A) \geq \sum_{j \in [k]} \text{dist}_j(\hat{y}) \cdot \text{val}(f_j) = \sum_{j \in [k]} w_j \text{dist}_j(\hat{y}) \frac{\text{val}(f_j)}{w_j} \\
\geq \left( \min_{j \in [k]} \frac{\text{val}(f_j)}{w_j} \right) \sum_{j \in [k]} w_j \text{dist}_j(\hat{y})
\]
Weak Duality for MCMF

\[ f = (f_1, \cdots, f_k): \text{a maximum concurrent multiflow} \]

\[ \mu y (A) \geq \sum_{j \in [k]} \text{dist}_j (\hat{y}) \text{val} (f_j) = \sum_{j \in [k]} w_j \text{dist}_j (\hat{y}) \frac{\text{val} (f_j)}{w_j} \]

\[ \geq \left( \min_{j \in [k]} \frac{\text{val} (f_j)}{w_j} \right) \sum_{j \in [k]} w_j \text{dist}_j (\hat{y}) \]

\[ \min_{j \in [k]} \frac{\text{dist}_j (\hat{y})}{w_j} \leq \mu \frac{y (A)}{\sum_{j \in [k]} w_j \text{dist}_j (\hat{y})} \]
Roadmap

- Introduction
- Adaptive Zero-Sum Game
- Weak Dualities
- Flow Augmentation Methods
- Extensions to Budgeted Variants
Interference-Aware Congestion, Costs, and Lengths

- \( f = (f_1, \ldots, f_k) \): a multiflow
Interference-Aware Congestion, Costs, and Lengths

- $f = (f_1, \cdots, f_k)$: a multiflow
- *congestion* of a link $a$ due to $f$: $\sum_{j \in [k]} f_j (N_{in}^D [a])$
- $f = (f_1, \cdots, f_k)$: a multiflow
- \textit{congestion} of a link $a$ due to $f$:  $\sum_{j \in [k]} f_j \left( N^\text{in}_D[a] \right)$
- \textit{congestion cost} of a link $a$ due to $f$: $y_f(a) = (1 + \epsilon) \sum_{j \in [k]} f_j \left( N^\text{in}_D[a] \right)$
- \( f = (f_1, \cdots, f_k) \): a multiflow

- \textit{congestion} of a link \( a \) due to \( f \): \( \sum_{j \in [k]} f_j \left( N_{D}^{in}[a] \right) \)

- \textit{congestion cost} of a link \( a \) due to \( f \): \( y_f(a) = (1 + \varepsilon) \sum_{j \in [k]} f_j(N_{D}^{in}[a]) \)

- \textit{length} of a link \( a \) due to \( f \): \( \hat{y}_f(a) = y_f(N_{D}^{out}[a]) \).
∀j ∈ [k], f_j ← 0;

repeat \[ \left\lfloor \frac{m \ln m}{\ln(1+ε) - \frac{ε}{1+ε}} \right\rfloor \] times

∀j ∈ [k], P_j ← a shortest j-path w.r.t. \( \hat{y}_f \);

j ← \arg \max_{j \in [k]} \frac{w_j}{\hat{y}_f(P_j)};

\( \delta \leftarrow \frac{1}{\max_{a \in A} |P_j \cap N_{in_D}[a]|} \);

∀a ∈ P_j, f_j(a) ← f_j(a) + δ;
Each link $a \in A$ corresponds to an adversary
Each link $a \in A$ corresponds to an adversary

- The cumulative loss of $a \equiv$ its interference-aware congestion due to the current $f$
Interpretation As An Adaptive Zero-Sum Game

- Each link \( a \in A \) corresponds to an adversary.
- The cumulative loss of \( a = \text{its interference-aware congestion} \) due to the current \( f \).
- \( \sigma \) is determined by the **Normalization Rule**: the loss of each link \( a \) is

\[
\delta \left| P_j \cap N^\text{in}_D [a] \right |.
\]
By **Zero-Sum Rule**, the agent earns a profit in a round

\[
\frac{1}{y_f (A)} \sum_{a \in A} y_f (a) \delta \left| P_j \cap N^i_D [a] \right| = \frac{\delta}{y_f (A)} \hat{y}_f (P_j) \\
= \frac{\delta w_j}{y_f (A)} \frac{\text{dist}_j (\hat{y}_f)}{w_j} \leq \mu \frac{\delta w_j}{\text{opt}}.
\]


By **Zero-Sum Rule**, the agent earns a profit in a round

\[
\frac{1}{y_f(A)} \sum_{a \in A} y_f(a) \delta |P_j \cap N_D^i[a]| = \frac{\delta}{y_f(A)} \hat{y}_f(P_j)
\]

\[
= \frac{\delta w_j}{y_f(A)} \frac{\text{dist}_j(\hat{y}_f)}{w_j} \leq \mu \frac{\delta w_j}{\text{opt}}.
\]

At the end of the last round, the cumulative profit of the agent is at most

\[
\mu \sum_{j \in [k]} w_j \text{val}(f_j) \frac{\text{opt}}{\text{opt}}.
\]
\[ \Delta_D^{in} \left( \sum_{j \in [k]} f_j \right) \leq (1 + \varepsilon) \mu \frac{\sum_{j \in [k]} w_j \text{val}(f_j)}{\text{opt}} \]
$$
\Delta_D^{in} \left( \sum_{j \in [k]} f_j \right) \leq (1 + \varepsilon) \mu \frac{\sum_{j \in [k]} w_j \text{val}(f_j)}{\text{opt}}
$$

$$
\frac{\sum_{j \in [k]} w_j \text{val}(f_j)}{\Delta_D^{in} \left( \sum_{j \in [k]} f_j \right)} \geq \frac{\text{opt}}{(1 + \varepsilon) \mu}.
$$
∀j ∈ [k], \( f_j \leftarrow 0 \);
repeat \( \left\lceil \frac{m \ln m}{\ln(1+\varepsilon) - \frac{\varepsilon}{1+\varepsilon}} \right\rceil \) times
∀j ∈ [k], \( P_j \leftarrow \) a shortest j-path w.r.t. \( \hat{y}_f \);
\( \sigma \leftarrow \frac{1}{\max_{a \in A} \sum_{j \in [k]} w_j |N_D^a[a] \cap P_j|} \);
∀j ∈ [k] and a ∈ \( P_j \), \( f_j(a) \leftarrow f_j(a) + \sigma w_j \);
Each link $a \in A$ corresponds to an adversary.
Interpretation As An Adaptive Zero-Sum Game

- Each link $a \in A$ corresponds to an adversary.
- The cumulative loss of $a = $ its interference-aware congestion due to the current $f$. 

Normalization Rule: the loss of each link $a$ is

$$\sigma \sum_j 2^{-|k|} w_j P_j \in D[a].$$
Each link \( a \in A \) corresponds to an adversary.

The cumulative loss of \( a \) is its interference-aware congestion due to the current \( f \).

\( \sigma \) is determined by the **Normalization Rule**: the loss of each link \( a \) is

\[
\sigma \sum_{j \in [k]} w_j \left| P_j \cap N^\text{in}_D[a] \right|.
\]
By **Zero-Sum Rule**, the agent earns a profit in a round

\[
\frac{1}{y_f(A)} \sum_{a \in A} y_f(a) \sigma \sum_{j \in [k]} w_j |P_j \cap N^\text{in}_D[a]|
\]

\[
= \frac{\sigma}{y_f(A)} \sum_{j \in [k]} w_j \hat{y}_f(P_j) = \frac{\sigma}{y_f(A)} \sum_{j \in [k]} w_j \text{dist}_j(\hat{y}_f) \leq \mu \frac{\sigma}{\text{opt}}.
\]
Profit of The Agent

- By **Zero-Sum Rule**, the agent earns a profit in a round

\[
\frac{1}{y_f(A)} \sum_{a \in A} y_f(a) \sigma \sum_{j \in [k]} w_j |P_j \cap N_D^{in}[a]|
\]

\[
= \frac{\sigma}{y_f(A)} \sum_{j \in [k]} w_j \hat{y}_f(P_j) = \frac{\sigma}{y_f(A)} \sum_{j \in [k]} w_j \text{dist}_j(\hat{y}_f) \leq \mu \frac{\sigma}{\sigma_{\text{opt}}}.
\]

- At the end of the last round, the cumulative profit of the agent is at most

\[
\mu \frac{\min_{j \in [k]} \text{val}(f_j) / w_j}{\text{opt}},
\]
\[ \Delta^{in}_D \left( \sum_{j \in [k]} f_j \right) \leq (1 + \varepsilon) \mu \frac{\min_{j \in [k]} \text{val}(f_j)}{w_{j_{opt}}} \]
\[ \Delta_{in}^D \left( \sum_{j \in [k]} f_j \right) \leq (1 + \varepsilon) \mu \frac{\min_{j \in [k]} \text{val}(f_j) / w_j}{\text{opt}} \]

\[ \frac{\min_{j \in [k]} \text{val}(f_j) / w_j}{\Delta_{in}^D \left( \sum_{j \in [k]} f_j \right)} \geq \frac{\text{opt}}{(1 + \varepsilon) \mu}. \]
Roadmap

- Introduction
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Flow Augmentation Method for Budgeted Multiflows

// Flow Augmentation Stage

// Link-Scheduling Stage
S ← the greedy link schedule of $\sum_{j \in [k]} f_j$;

// Scaling Stage

\[
\begin{align*}
    \mathbf{f} & \leftarrow \frac{1}{\max \left\{ \|S\|, \sum_{a \in A} g(a) \sum_{j \in [k]} f_j(a) \right\}} \mathbf{f}; \\
    S & \leftarrow \frac{1}{\max \left\{ \|S\|, \sum_{a \in A} g(a) \sum_{j \in [k]} f_j(a) \right\}} S;
\end{align*}
\]

Output $\mathbf{f}$ and $S$. 
• $g$: flow expense function of $A$
A Virtual Link Representing Budget Constraint

- $g$: flow expense function of $A$
- "virtual" link $a^+$: disjoint from all links in $A$
A Virtual Link Representing Budget Constraint

- $g$: flow expense function of $A$
- "virtual" link $a^+$: disjoint from all links in $A$
- $A^+ = A \cup \{a^+\}$
Weak Dualities

- $y$: positive function on $A^+$
Weak Dualities

- $y$: positive function on $A^+$
- $\hat{y}$: length function on $A$ induced by $y$:

$$\hat{y}(a) = y(N_{D}^{out}[a]) + y(a^+)g(a)$$
Weak Dualities

- $y$: positive function on $A^+$
- $\hat{y}$: length function on $A$ induced by $y$:
  \[
  \hat{y}(a) = y(N_D^{\text{out}}[a]) + y(a^+)g(a)
  \]
- $dist_j(\hat{y}) \forall j \in [k]$: length of a shortest $j$-path w.r.t. $\hat{y}$
The weight of the budgeted maximum weighted multiflow

\[ \leq \mu \frac{y(A^+)}{\min_{j \in [k]} \text{dist}_j(\hat{y}) / w_j}. \]
The weight of the budgeted maximum weighted multiflow

\[ \leq \mu \frac{y(A^+)}{\min_{j \in [k]} \text{dist}_j(\hat{y}) / w_j}. \]

The concurrency of the budgeted maximum concurrent multiflow is at most

\[ \leq \mu \frac{y(A^+)}{\sum_{j \in [k]} w_j \text{dist}_j(\hat{y})}. \]
Path-Flow Decomposition

For any path-flow decomposition \( \mathbf{x} \) of a multi-flow \( \mathbf{f} = (f_1, \ldots, f_k) \),

\[
\sum_{j=1}^{k} \left\{ \sum_{P: x(P) = b} \sum_{a \in D(a)} (y(a) + \sum_{j=1}^{k} f_j(a)) \right\}
\]
Lemma

For any path-flow decomposition \( x \) of a multiflow \( f = (f_1, \cdots, f_k) \),

\[
\sum_{j \in [k]} \sum_{P \in \mathcal{P}_j} x(P) \hat{y}(P) = \sum_{a \in A} y(a) \sum_{j \in [k]} f_j(N_D^{in}[a])
\]

\[
+ y(a^+) \sum_{a \in A} g(a) \sum_{j \in [k]} f_j(a)
\]
Feasible Multiflow

**Lemma**

For any feasible multiflow $f = (f_1, \ldots, f_k)$,

$$\sum_{j \in [k]} \text{dist}_j (\hat{y}) \text{ val} (f_j) \leq \mu y (A^+)$$
Interference/Budget-Aware Congestions

- $f = (f_1, \ldots, f_k)$: a multiflow
Interference/Budget-Aware Congestions

- $f = (f_1, \ldots, f_k)$: a multiflow
- congestion of $a \in A$ due to $f$:

$$\sum_{j \in [k]} f_j \left( N_D^\text{in} [a] \right)$$
Interference/Budget-Aware Congestions

- \( f = (f_1, \ldots, f_k) \): a multiflow
- congestion of \( a \in A \) due to \( f \):
  \[
  \sum_{j \in [k]} f_j \left( N_D^{\text{in}}[a] \right)
  \]
- congestion of \( a^+ \) due to \( f \):
  \[
  \sum_{a \in A} g(a) \sum_{j \in [k]} f_j(a)
  \]
Interference/Budget-Aware Congestions

- $f = (f_1, \cdots, f_k)$: a multiflow

- Congestion of $a \in A$ due to $f$:
  \[
  \sum_{j \in [k]} f_j (N_D^\text{in} [a])
  \]

- Congestion of $a^+$ due to $f$:
  \[
  \sum_{a \in A} g(a) \sum_{j \in [k]} f_j (a)
  \]

- Bottleneck congestion of $f$
  \[
  \max \left\{ \Delta_D^\text{in} \left( \sum_{j \in [k]} f_j \right), \sum_{a \in A} \sum_{j \in [k]} g(a) f_j (a) \right\}
  \]
congestion cost of $a \in A$ due to $f$:

$$y_f(a) = (1 + \varepsilon) \sum_{j \in [k]} f_j\left(\mathcal{N}_{D[a]}^{in}\right)$$
- **congestion cost of** $a \in A$ due to $f$:

$$y_f (a) = (1 + \varepsilon) \sum_{j \in [k]} f_j (N^{in}_D [a])$$

- **congestion cost of** $a^+$ due to $f$:

$$y_f (a^+) = (1 + \varepsilon) \sum_{a \in A} g(a) \sum_{j \in [k]} f_j (a)$$
• *congestion cost* of \( a \in A \) due to \( f \):

\[
y_f (a) = (1 + \varepsilon) \sum_{j \in [k]} f_j (N_D^{in} [a])
\]

• *congestion cost* of \( a^+ \) due to \( f \):

\[
y_f (a^+) = (1 + \varepsilon) \sum_{a \in A} g(a) \sum_{j \in [k]} f_j (a)
\]

• *length* of \( a \in A \) due to \( f \):

\[
\hat{y}_f (a) = y_f (N_D^{out} [a]) + y_f (a^+) g (a)
\]
FA Stage for Budgeted MWMF

∀j ∈ [k], f_j ← 0;  
repeat \left\lceil \frac{(m+1) \ln(m+1)}{\ln(1+\varepsilon) - \frac{\varepsilon}{1+\varepsilon}} \right\rceil \text{ times}  
\forall j ∈ [k], P_j ← \text{a shortest } j\text{-path w.r.t. } \hat{y}_f;  
j ← \arg \max_{j ∈ [k]} \frac{w_j}{\hat{y}_f(P_j)};  
\delta ← \frac{1}{\max_{a ∈ A} \left( \max_{P_j ∈ \mathcal{P}} |P_j ∩ N_D^{in}[a]|, g(P_j) \right)};  
\forall a ∈ P_j, f_j(a) ← f_j(a) + \delta;
\begin{align*}
\forall j \in [k], & \quad f_j \leftarrow 0; \\
\text{repeat} & \quad \left\lceil \frac{(m+1) \ln(m+1)}{\ln(1+\varepsilon) - \frac{\varepsilon}{1+\varepsilon}} \right\rceil \text{ times} \\
\forall j \in [k], & \quad P_j \leftarrow \text{a shortest } j\text{-path w.r.t. } \hat{y}_f; \\
\sigma & \leftarrow \frac{1}{\max \left\{ \max_{a \in A} \sum_{j \in [k]} w_j | N_D^{in}[a] \cap P_j | , \sum_{j \in [k]} w_j g(P_j) \right\}}; \\
\forall j \in [k] & \quad \text{and } a \in P_j, \quad f_j(a) \leftarrow f_j(a) + \sigma w_j;
\end{align*}