Multiflows under Protocol Interference Model

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Outline

- Introduction
- Adaptive Zero-Sum Game
- Weak Dualities
- Flow Augmentation Methods
- Extensions to Budgeted Variants

Multiflow

- Multihop wireless network: $(V, A, \mathcal{I})$

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- k unicast requests.
- $\mathcal{F}_j$: the set of flows of the j-th request
- $\mathcal{P}_j$: the set of paths of the j-th request
- A k-flow is a sequence $f = \langle f_1, f_2, \cdots, f_k \rangle$ with $f_j \in \mathcal{F}_j$ $\forall j \in [k]$

Approximation-Preserving Reduction to MWIS

Theorem

If there is a polynomial (respectively, a polynomial μ-approximation) algorithm for **MWISL**, *then there exist polynomial time μ-approximate algorithms for all four variants of multiflow problems.*

PTAS

Theorem

All four variants of multiflow problems under protocol interference model have a PTAS.

Approximation-Preserving Reduction to Polynomial Approximate Capacity Subregions

Theorem

If there exists a polynomial μ-approximate capacity subregion, then there exist polynomial time μ-approximate algorithms for each of the four variants of multiflow problems.

Maximum Multiflow

Ω: capacity region

MWMF

$$\begin{array}{ll} \max & \sum_{j=1}^{k} w_j val(f_j) \\ s.t. & f_j \in \mathcal{F}_j, \forall 1 \leq j \leq k \\ & \sum_{j=1}^{k} f_j \in \Omega \end{array}$$

MCMF

$$\begin{array}{ll} \max & \phi \\ s.t. & f_j \in \mathcal{F}_j, \forall 1 \leq j \leq k \\ & val(f_j) \geq \phi w_j, \forall 1 \leq j \leq k \\ & \sum_{j=1}^{k} f_j \in \Omega \end{array}$$

Maximum Restricted Multiflow

Φ: a poly. μ-approx. capacity subregion

Φ-restricted **MWMF**

$$\begin{array}{ll} \max & \sum_{j=1}^{k} w_j val(f_j) \\ s.t. & f_j \in \mathcal{F}_j, \forall 1 \leq j \leq k \\ & \sum_{j=1}^{k} f_j \in \Phi \end{array}$$

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Maximum Restricted Multiflow

Step 1: solve the Φ-restricted LP to obtain a k-flow $f = \langle f_1, f_2, \cdots, f_k \rangle$,
Step 2: compute a fractional link schedule $\mathcal{S}$ of $\sum_{j=1}^{k} f_j$
Step 3: scale f and $\mathcal{S}$ by a factor $1/\|\mathcal{S}\|$
approximation bound: μ

Approximation Bounds

Restrictions to the polynomial capacity subregions constructed in the previous chapter yield the following approximation bounds:

- $2\left(\left\lceil \pi / \arcsin \frac{c-1}{2c} \right\rceil - 1\right)$ in general,
- $\left\lceil \frac{r+1}{h(r)} \right\rceil + 1$ with uniform interference radii.

Flow Augmentation Method for MWMF

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- **Flow Augmentation Stage**: Compute a k-flow f s.t.

$$\frac{\sum_{j\in[k]} w_j val\left(f_j\right)}{\Delta_D^{in}\left(\sum_{j\in[k]} f_j\right)} \geq \frac{opt}{(1+\varepsilon)\,\mu}.$$

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- **Scaling Stage**: Scale both Π and $\mathcal{S}$ by a factor $1/\left\|\mathcal{S}\right\|$ and then return them.

Approximation Bound

- $2(1+\varepsilon)\mu$ in general:

$$\frac{\sum_{j\in[k]} w_j val(f_j)}{\|\mathcal{S}\|} \geq \frac{\sum_{j\in[k]} w_j val(f_j)}{\Delta^*\left(\sum_{j\in[k]} f_j\right)}$$
$$\geq \frac{\sum_{j\in[k]} w_j val(f_j)}{2\Delta_D^{in}\left(\sum_{j\in[k]} f_j\right)} \geq \frac{opt}{2(1+\varepsilon)\mu}.$$

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- $(1+\varepsilon)\mu$ if D is acyclic:

$$\frac{\sum_{j\in[k]} w_j val(f_j)}{\|\mathcal{S}\|} \geq \frac{opt}{(1+\varepsilon)\mu}.$$

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Technical Issues Yet to Be addressed

- How to define an appropriate interference costs of links and paths?
- How much flow should be routed along those cheapest paths?

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  - The agent collects the profit according to the **Zero-Sum Rule**: the profit $=$ expected loss of the adversaries w.r.t the agent's binding strategy.

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- Objective: to realize asap maximum cumulative loss of adversaries $\leq (1+\varepsilon) \times$ cumulative profit of the agent.

Exponential Binding Strategies

- $L(e)$: the cumulative loss of $e \in E$ at the beginning of a round
- The probability of $e \in E$: $\propto (1+\varepsilon)^{L(e)}$.

Design of The Game

Repeat

$$t = \left\lceil \frac{m \ln m}{\ln(1+\varepsilon) - \frac{\varepsilon}{1+\varepsilon}} \right\rceil$$

rounds:

1. **Declaration of exponential binding strategies** by agent

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rounds:

1. **Declaration of exponential binding strategies** by agent
2. **Generation of loss** by adversary s.t. the **Normalization Rule**

Analysis of The Game

$t = O\left(\varepsilon^{-2} m \ln m\right)$ as

$$\ln(1+\varepsilon) - \frac{\varepsilon}{1+\varepsilon} = -\ln\left(1 - \frac{\varepsilon}{1+\varepsilon}\right) - \frac{\varepsilon}{1+\varepsilon}$$
$$\geq \frac{1}{2}\left(\frac{\varepsilon}{1+\varepsilon}\right)^2 = \frac{1}{2}(1+1/\varepsilon)^{-2}.$$

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Theorem

At the end of the game, the maximum cumulative loss of the adversaries is at most $1+\varepsilon$ times the cumulative profit of the agent.

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- $y_r(e) = (1+\varepsilon)^{L_r(e)}$: $\forall 0 \leq r \leq t$ and $\forall e \in E$
- **Zero-Sum Rule**

$$\ell_r - \ell_{r-1} = \sum_{e \in E} \frac{y_{r-1}(e)}{y_{r-1}(E)} \left(L_r(e) - L_{r-1}(e) \right).$$

Analysis of The Game

$$
\begin{aligned}
y_r(E) &= \sum_{e\in E} y_r(e) = \sum_{e\in E} y_{r-1}(e)(1+\varepsilon)^{L_r(e)-L_{r-1}(e)} \\
&\leq \sum_{e\in E} y_{r-1}(e)\left(1+\varepsilon\left(L_r(e)-L_{r-1}(e)\right)\right) \\
&= \sum_{e\in E} y_{r-1}(e) + \varepsilon \sum_{e\in E} y_{r-1}(e)\left(L_r(e)-L_{r-1}(e)\right) \\
&= y_{r-1}(E)\left(1+\varepsilon \sum_{e\in E} \frac{y_{r-1}(e)}{y_{r-1}(E)}\left(L_r(e)-L_{r-1}(e)\right)\right) \\
&= y_{r-1}(E)\left(1+\varepsilon\left(\ell_r-\ell_{r-1}\right)\right) \\
&\leq y_{r-1}(E)\exp\left(\varepsilon\left(\ell_r-\ell_{r-1}\right)\right).
\end{aligned}
$$

Analysis of The Game

$$y_t(E) \leq y_0(E) \prod_{r=1}^{t} \exp\left(\varepsilon\left(\ell_r - \ell_{r-1}\right)\right) = m \exp\left(\varepsilon \ell_t\right).$$

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$$\varepsilon \ell_t \geq \ln \frac{y_t(E)}{m} \geq \ln \frac{y_t(e)}{m} = L_t(e) \ln(1+\varepsilon) - \ln m, \forall e \in E$$

Analysis of The Game

e: an adversary with the largest cumulative loss

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$$\varepsilon \ell_t \geq L_t(e)\ln(1+\varepsilon) - \ln m \geq \frac{\varepsilon}{1+\varepsilon} L_t(e)$$

$$L_t(e) \leq (1+\varepsilon)\,\ell_t.$$

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- y: positive function on A
- $\widehat{y}$: length function on A induced by y

$$\widehat{y}(a) = y\left(N_D^{out}[a]\right)$$

- $dist_j(\widehat{y})$ $\forall j \in [k]$: length of a shortest j-path w.r.t. $\widehat{y}$

Weak Dualities

- The weight of the maximum weighted multiflow

$$\leq \mu y(A) \max_{j\in[k]} \frac{w_j}{dist_j(\widehat{y})}.$$

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$$\leq \mu y(A) \max_{j\in[k]} \frac{w_j}{dist_j(\widehat{y})}.$$

- The concurrency of the maximum concurrent multiflow is at most

$$\leq \mu \frac{y(A)}{\sum_{j\in[k]} w_j dist_j(\widehat{y})}.$$

Path-Flow Decomposition

A non-negative function x on $\bigcup_{j\in[k]}\mathcal{P}_j$ is said to be a *path-flow decomposition* of a multiflow $f = (f_1, \cdots, f_k)$ if

$$f_j(a) = \sum_{P\in\mathcal{P}_j} |\{a\} \cap P| \, x(P), \forall a \in A, j \in [k]$$

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Lemma

For any path-flow decomposition x of a multiflow $f=(f_1,\cdots,f_k)$,

$$\sum_{j\in[k]}\sum_{P\in\mathcal{P}_j}x(P)\,\widehat{y}(P)=\sum_{a\in A}y(a)\sum_{j\in[k]}f_j\left(N_D^{in}[a]\right)$$

Path-Flow Decomposition

$$
\begin{aligned}
& \sum_{j\in[k]} \sum_{P\in\mathcal{P}_j} x(P)\, \widehat{y}(P) \\
&= \sum_{j\in[k]} \sum_{P\in\mathcal{P}_j} x(P) \sum_{b\in P} \widehat{y}(b) \\
&= \sum_{j\in[k]} \sum_{P\in\mathcal{P}_j} x(P) \sum_{b\in P} y\left(N_D^{out}[b]\right) \\
&= \sum_{j\in[k]} \sum_{P\in\mathcal{P}_j} x(P) \sum_{b\in A} \left|\{b\}\cap P\right| y\left(N_D^{out}[b]\right) \\
&= \sum_{j\in[k]} \sum_{P\in\mathcal{P}_j} x(P) \sum_{b\in A} \left|\{b\}\cap P\right| \sum_{a\in N_D^{out}[b]} y(a) \\
&= \sum_{j\in[k]} \sum_{P\in\mathcal{P}_j} x(P) \sum_{b\in A} \left|\{b\}\cap P\right| \sum_{a\in A} y(a) \left|\{b\}\cap N_D^{in}[a]\right|
\end{aligned}
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&= \sum_{a\in A} y(a) \sum_{j\in[k]} \sum_{b\in A} \left|\{b\}\cap N_D^{in}[a]\right| \sum_{P\in\mathcal{P}_j} |\{b\}\cap P| \, x(P) \\
&= \sum_{a\in A} y(a) \sum_{j\in[k]} \sum_{b\in A} \left|\{b\}\cap N_D^{in}[a]\right| f_j(b) \\
&= \sum_{a\in A} y(a) \sum_{j\in[k]} \sum_{b\in N_D^{in}[a]} f_j(b) \\
&= \sum_{a\in A} y(a) \sum_{j\in[k]} f_j\left(N_D^{in}[a]\right),
\end{aligned}
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Feasible Multiflow

Lemma

For any feasible multiflow $f = (f_1, \cdots, f_k)$,

$$\sum_{j \in [k]} dist_j(\widehat{y})\, val(f_j) \leq \mu y(A)$$

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$$= \sum_{j \in [k]} dist_j(\widehat{y}) \sum_{P \in \mathcal{P}_j} x(P) = \sum_{j \in [k]} dist_j(\widehat{y})\, val(f_j).$$

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$$\sum_{a \in A} y(a) \sum_{j \in [k]} f_j\left(N_D^{in}[a]\right) \leq \Delta_D^{in}\left(\sum_{j \in [k]} f_j\right) \sum_{b \in A} y(a) \leq \mu y(A).$$

Weak Duality for MWMF

$f = (f_1, \cdots, f_k)$: a maximum weighted multiflow

$$\mu y(A) \geq \sum_{j\in[k]} dist_j(\widehat{y})\, val(f_j) = \sum_{j\in[k]} \frac{dist_j(\widehat{y})}{w_j} w_j val(f_j)$$
$$\geq \left(\min_{j\in[k]} \frac{dist_j(\widehat{y})}{w_j}\right) \sum_{j\in[k]} w_j val(f_j)$$

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$$\sum_{j \in [k]} w_j val(f_j) \leq \mu \frac{y(A)}{\min_{j \in [k]} \frac{dist_j(\widehat{y})}{w_j}}$$

Weak Duality for MCMF

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$$\min_{j \in [k]} \frac{dist_j(\widehat{y})}{w_j} \leq \mu \frac{y(A)}{\sum_{j \in [k]} w_j dist_j(\widehat{y})}$$

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Interference-Aware Congestion, Costs, and Lengths

- $f = (f_1, \cdots, f_k)$: a multiflow
- *congestion* of a link a due to f: $\sum_{j \in [k]} f_j \left(N_D^{in}[a]\right)$

Interference-Aware Congestion, Costs, and Lengths

- $f = (f_1, \cdots, f_k)$: a multiflow
- *congestion* of a link a due to f: $\sum_{j\in[k]} f_j \left(N_D^{in}[a]\right)$
- *congestion cost* of a link a due to f: $y_f(a) = (1+\varepsilon)^{\sum_{j\in[k]} f_j\left(N_D^{in}[a]\right)}$

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- *length* of a link a due to f: $\widehat{y}_f(a) = y_f\left(N_D^{out}[a]\right)$.

Flow Augmentation Stage for MWMF

$\forall j \in [k], f_j \leftarrow \mathbf{0}$;

repeat $\left\lceil \frac{m \ln m}{\ln(1+\varepsilon) - \frac{\varepsilon}{1+\varepsilon}} \right\rceil$ times

$\quad \forall j \in [k],\ P_j \leftarrow$ a shortest j-path w.r.t. $\widehat{y}_f$;

$\quad j \leftarrow \arg\max_{j \in [k]} \frac{w_j}{\widehat{y}_f(P_j)}$;

$\quad \delta \leftarrow \frac{1}{\max_{a \in A} \left| P_j \cap N_D^{in}[a] \right|}$;

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Profit of The Agent

- By **Zero-Sum Rule**, the agent earns a profit in a round

$$\frac{1}{y_f(A)}\sum_{a\in A} y_f(a)\,\delta\left|P_j\cap N_D^{in}[a]\right| = \frac{\delta}{y_f(A)}\widehat{y}_f(P_j)$$
$$= \frac{\delta w_j}{y_f(A)}\frac{dist_j(\widehat{y}_f)}{w_j} \leq \mu\frac{\delta w_j}{opt}.$$

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$$\mu\frac{\sum_{j\in[k]} w_j val(f_j)}{opt}.$$

Profit vs. Maximum Loss

$$\Delta_D^{in}\left(\sum_{j\in[k]} f_j\right) \leq (1+\varepsilon)\,\mu\frac{\sum_{j\in[k]} w_j val\left(f_j\right)}{opt}$$

Profit vs. Maximum Loss

$$\Delta_D^{in}\left(\sum_{j\in[k]} f_j\right) \leq (1+\varepsilon)\,\mu \frac{\sum_{j\in[k]} w_j val\left(f_j\right)}{opt}$$

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$$\begin{aligned}&\frac{1}{y_f(A)}\sum_{a\in A} y_f(a)\,\sigma\sum_{j\in[k]} w_j\left|P_j\cap N_D^{in}[a]\right|\\&=\frac{\sigma}{y_f(A)}\sum_{j\in[k]} w_j\widehat{y}_f(P_j)=\frac{\sigma}{y_f(A)}\sum_{j\in[k]} w_j dist_j(\widehat{y}_f)\leq\mu\frac{\sigma}{opt}.\end{aligned}$$

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$$= \frac{\sigma}{y_f(A)} \sum_{j \in [k]} w_j \widehat{y}_f(P_j) = \frac{\sigma}{y_f(A)} \sum_{j \in [k]} w_j dist_j(\widehat{y}_f) \leq \mu \frac{\sigma}{opt}.$$

- At the end of the last round, the cumulative profit of the agent is at most

$$\mu \frac{\min_{j \in [k]} val(f_j) / w_j}{opt},$$

Profit vs. Maximum Loss

$$\Delta_D^{in}\left(\sum_{j\in[k]} f_j\right) \leq (1+\varepsilon)\,\mu \frac{\min_{j\in[k]} val\left(f_j\right)/w_j}{opt}$$

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$$\frac{\min_{j\in[k]} val\left(f_j\right)/w_j}{\Delta_D^{in}\left(\sum_{j\in[k]} f_j\right)} \geq \frac{opt}{(1+\varepsilon)\,\mu}.$$

Roadmap

- Introduction
- Adaptive Zero-Sum Game
- Weak Dualities
- Flow Augmentation Methods
- Extensions to Budgeted Variants

Flow Augmentation Method for Budgeted Multiflows

// **Flow Augmentation Stage**

⋮

// **Link-Scheduling Stage**

$\mathcal{S} \leftarrow$ the greedy link schedule of $\sum_{j\in[k]} f_j$;

// **Scaling Stage**

$f \leftarrow \frac{1}{\max\left\{\|\mathcal{S}\|, \sum_{a\in A} g(a) \sum_{j\in[k]} f_j(a)\right\}} f$;

$\mathcal{S} \leftarrow \frac{1}{\max\left\{\|\mathcal{S}\|, \sum_{a\in A} g(a) \sum_{j\in[k]} f_j(a)\right\}} \mathcal{S}$;

Output f and $\mathcal{S}$.

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$$\widehat{y}(a) = y\left(N_D^{out}[a]\right) + y\left(a^+\right) g(a)$$

- $dist_j(\widehat{y})$ $\forall j \in [k]$: length of a shortest j-path w.r.t. $\widehat{y}$

Weak Dualities

- The weight of the budgeted maximum weighted multiflow

$$\leq \mu \frac{y\left(A^{+}\right)}{\min_{j\in[k]} dist_j\left(\widehat{y}\right)/w_j}.$$

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$$\leq \mu \frac{y\left(A^{+}\right)}{\min_{j\in[k]} dist_j\left(\widehat{y}\right)/w_j}.$$

- The concurrency of the budgeted maximum concurrent multiflow is at most

$$\leq \mu \frac{y\left(A^{+}\right)}{\sum_{j\in[k]} w_j dist_j\left(\widehat{y}\right)}.$$

Path-Flow Decomposition

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Lemma

For any path-flow decomposition x of a multiflow $f = (f_1, \cdots, f_k)$,

$$\sum_{j \in [k]} \sum_{P \in \mathcal{P}_j} x(P) \widehat{y}(P) = \sum_{a \in A} y(a) \sum_{j \in [k]} f_j\left(N_D^{in}[a]\right) + y\left(a^+\right) \sum_{a \in A} g(a) \sum_{j \in [k]} f_j(a)$$

Feasible Multiflow

Lemma

For any feasible multiflow $f = (f_1, \cdots, f_k)$,

$$\sum_{j \in [k]} dist_j(\widehat{y})\, val(f_j) \leq \mu y(A^+)$$

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- *bottleneck congestion* of f

$$\max \left\{ \Delta_D^{in} \left(\sum_{j \in [k]} f_j \right), \sum_{a \in A} g(a) \sum_{j \in [k]} f_j(a) \right\}$$

- *congestion cost* of $a \in A$ due to f:

$$y_f(a) = (1+\varepsilon)^{\sum_{j\in[k]} f_j\left(N_D^{in}[a]\right)}$$

Interference/Budget-Aware Costs and Lengths

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FA Stage for Budgeted MWMF

$$
\begin{array}{|l|}
\hline
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\text{repeat } \left\lceil \frac{(m+1)\ln(m+1)}{\ln(1+\varepsilon)-\frac{\varepsilon}{1+\varepsilon}} \right\rceil \text{ times} \\
\qquad \forall j \in [k], P_j \leftarrow \text{a shortest } j\text{-path w.r.t. } \widehat{y}_f; \\
\qquad j \leftarrow \arg\max_{j\in[k]} \frac{w_j}{\widehat{y}_f(P_j)}; \\
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  $\forall j \in [k]$ and $a \in P_j$, $f_j(a) \leftarrow f_j(a) + \sigma w_j$;