Maximum-Weighted λ-Colorable Subgraph: Revisiting And Applications

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Outline

- Introduction
- Local-Ratio Scheme
- Fractional Local-Ratio Scheme
- Comparability Graph
- Applications
- Summary

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- Applications: Maximum-weighted schedulable requests in multi-channel wireless networks

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- μ-approx. alg. for **MWIS** $\Longrightarrow f_\lambda(\mu)$-approx. alg. for λ-**MWCS**, where

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  - $f_\lambda(\mu)$ strictly increases with λ

$$\mu = f_1(\mu) < f_\lambda(\mu) < \mu + 1 - \frac{1}{\lambda}$$

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- $\Longrightarrow 2\gamma$-approx. alg. for **MWIS**
- $\Longrightarrow f_\lambda(2\gamma)$-approx. alg. for λ-**MWCS**

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- Whether and when a better approx. bound may be achieved than prior art

Contributions of This Paper

$$g_\lambda(\mu) := \min\left\{\mu, \mu\left(1 - \frac{1}{\lambda}\right) + 1\right\}$$

- an ordering with BLIN $\beta \Longrightarrow g_\lambda(\beta)$-approx. alg. for λ-**MWCS**

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- For $1 < \lambda < 2\mu$, $g_\lambda(\mu) < f_\lambda(\mu)$

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- *Cocomparable* vertex ordering $\prec$: For any $v \prec v' \prec v''$,

$$\left(vv' \notin E\right) \wedge \left(v'v'' \notin E\right) \Longrightarrow \left(vv'' \notin E\right).$$

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- *Cocomparability graph*: $\exists$ cocomparable vertex ordering
- Exact alg. for λ-**MWCS**: $1 < f_\lambda(1)$

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Local-Ratio Scheme

$\prec$: a vertex ordering with BLIN β

1. Compute the greedy candidate subset S of V in $\prec$.
2. Compute the maximal inductively feasible subset F of S in $\prec$.

Greedy Candidate Subset

greedy candidate subset S of V in $\prec$
$S \leftarrow \emptyset$; for each $v \in V$ in the *reverse* order of $\prec$ do $\quad \overline{w}(v) \leftarrow w(v) - \frac{1}{\lambda}\overline{w}(N(v) \cap S)$; $\quad$ if $\overline{w}(v) > 0$, $S \leftarrow S \cup \{v\}$;
return S.

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$$\overline{w}(S) \geq \frac{1}{g_\lambda(\beta)} opt.$$

Maximal Inductively Feasible Subset

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Fractional Local-Ratio Scheme

D: an orientation with ILIN γ

1. Solve an auxiliary LP
$$\begin{array}{ll} \max & \sum_{v \in V} w(v)\, x(v) \\ s.t. & x(v) + \frac{2}{\lambda} x\left(N_D^{in}(v)\right) \leq 1, \forall v \in V \\ & x(v) \geq 0, \forall v \in V \end{array}$$
2. Compute a surplus-preserving ordering $\prec$ of V w.r.t. x.
3. Compute the greedy candidate subset S of A in $\prec$.
4. Compute the maximal inductively feasible subset F of S in $\prec$.

Surplus-preserving Ordering

surplus-preserving ordering w.r.t. x

$U \leftarrow V$;
for $i = n$ **down to** 1 do
    $v_i \leftarrow$ an x-surplus vertex in U;
        $//x\left(\ddot{N}^{in}(v_i) \cap U\right) \geq x\left(\ddot{N}^{out}(v_i) \cap U\right)$.
    $U \leftarrow U \setminus \{v_i\}$;

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$$w(F) \geq \overline{w}(S) \geq \sum_{v \in V} w(v)\, x(v) \geq \frac{1}{g_\lambda(2\gamma)} opt.$$

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Exact Algorithm on Comparability Graphs

- $G = (V, E)$: a cocomparability graph with a cocomparability ordering $\langle v_1, v_2, \cdots, v_n \rangle$.
- Exact algorithm: reduction to min-cost flow

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$$\{(s, x_i) : 1 \leq i \leq n\} \cup \{(y_i, t) : 1 \leq i \leq n\} \cup$$
$$\{(x_i, y_i) : 1 \leq i \leq n\} \cup$$
$$\{(y_i, x_j) : 1 \leq i < j \leq n, v_i v_j \notin E\}$$

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- Capacity: unit capacity on each arc

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- Capacity: unit capacity on each arc
- Cost: each arc (x_i, y_i) has a cost $-w(v_i)$, and each other arc has cost 0.

Reduction

- For any $i_1 < i_2 < \cdots < i_l$,

$$I = \{v_{i_1}, v_{i_2}, \cdots, v_{i_l}\} \leftrightarrows$$
$$P = \langle s, x_{i_1}, y_{i_1}, x_{i_2}, y_{i_2}, \cdots, x_{i_l}, y_{i_l}, t \rangle$$

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- cost of $P = -w(I)$
- min. cost of s-t flows of value $\leq \lambda = -opt$

Algorithm

- Compute an integral min-cost flow s–t flow f of value $\leq \lambda$ in D,
- Decompose f into s–t paths flows: $\#$ of paths $\leq \lambda$
- For each path P, let I be the IS induced by P. The collection of all these IS's is returned as the output.

Roadmap

- Problem Description
- A Finer Treatment of Conflicts
- Light Requests
- Heavy Requests
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- First-fit transmission schedule