How good is sink insertion?

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Abstract. Generating high quality meshes is one of the most important steps in many applications such as scientific computing. Sink insertion method is one of the mesh quality improvement methods that had been proposed recently. However, it is unknown whether this method is competitive to generate meshes with small number of elements. In this paper, we show that, given a two-dimensional polygonal domain with no small angles, the sink insertion method generates a well-shaped mesh with $O(n)$ triangles, where n is the minimum number of triangles generated by any method with the same quality guarantee.
We also show that the sink insertion method more likely can not guarantee the same result for a three-dimensional domain, while the other methods such as Delaunay refinement can achieve.

Keywords: Mesh generation, Delaunay triangulations, sink insertion, computational geometry, algorithms.

1 Introduction

Mesh generation is the process of breaking a geometric domain into a collection of primitive elements such as triangles in 2D and tetrahedra in 3D. It has plenty of applications in scientific computing, computer graphics, computer vision, geometric information system, and medical imaging. Some applications have a strict quality requirement on the underlying meshes used. For example, most of the numerical simulations require that the mesh is well-shaped. In addition, some numerical simulations methods, for example, the control volume method, prefer the mesh to be a Delaunay triangulation.

Recently, Edelsbrunner *et. al* [2, 3] proposed a new mesh improvement method based on sink insertion. This new method guarantees to generate a Delaunay mesh with a small radius-edge ratio. Edelsbrunner and Guoy (private communication) found that the sink insertion method tends to be more economical when we want to add as many points as possible at the same time to refine the mesh while maintaining the Delaunay property. It will also be useful in the software environment with off-line Delaunay triangulation or parallel meshing. In stead of dealing with all the circumcenters as many as the number of bad elements, they deal with small number of sinks of these bad elements. From experiments, they observe as many as 100 bad tetrahedra sharing the same sink. However, unlike Delaunay refinement, it is an open problem whether the sink insertion method generates an almost-good Delaunay mesh with $O(n)$ simplex elements, where n is the minimum number of d-dimensional simplex elements generated by any other methods with the same radius-edge ratio quality guarantee.

In this paper, we show that the sink insertion method guarantees to generate a well-shaped mesh with size $O(n)$ in $2D$. For a three-dimensional domain, unlike the Delaunay refinement methods, the size optimality is not guaranteed because of the existence of slivers in an almost-good Delaunay mesh. We give an example that suggests that the sink insertion method may not guarantee to generate a mesh with size $O(n)$ for a three-dimensional domain.

The rest of the paper is organized as follows. In Section 2, we review the sink insertion algorithm proposed by Edelsbrunner *et. al.* In Section 3, we prove that the sink insertion method guarantees to generate a well-shaped mesh with size $O(n)$ for a two-dimensional PLC domain with no small angles. In section 4, we discuss why the sink insertion method may not guarantee that the generated mesh has size $O(n)$ for a three-dimensional PLC domain. Section 5 concludes the paper with further discussions.

2 Preliminary

A simplicial mesh is called *almost good* if each of its simplex elements has a small radius-edge ratio, which is the circumradius divided by the shortest edge length of the simplex. Hereafter we use $\rho(\tau)$ to denote the radius-edge ratio of a simplex τ. Several theoretical and practical approaches have been proposed for generating almost-good Delaunay meshes. Assume the spatial domain that does not have small angles is given in terms of its piecewise linear complex boundary (PLC) [7]. It has been shown that the Delaunay refinement methods [1, 5, 6] generate an almost-good Delaunay mesh with size $O(n)$, where n is the minimum number of elements for any mesh with the same radius-edge ratio for the same geometric domain.

2.1 Sink

Edelsbrunner *et al.* [2, 3] defined the sink of a simplex σ in a Delaunay complex by the following recursive approach.

Definition 1. [SINK] *In a d-dimensional Delaunay complex, let c_σ be the circumcenter of a d-simplex σ; let $\mathcal{N}(\sigma)$ be the set of d-simplices that share a $d-1$ dimensional face with σ. For each simplex $\tau \in \mathcal{N}(\sigma)$, let H_τ be the half space containing τ bounded by the $d-1$ dimensional face shared by τ and σ. A point z is a* sink *of σ when*

- *z is c_σ and it is contained in σ; or*
- *z is a sink of $\tau \in \mathcal{N}(\sigma)$ and c_σ is contained in H_τ.*

A simplex containing its own circumcenter is called a *sink simplex*. Edelsbrunner *et al.* [2, 3] showed that there is no loop in the definition of sink among all d-dimensional simplices by proving that the circumradius of the simplex τ containing the sink of a simplex σ is not less than the circumradius of σ. Notice that if the circumcenter of a boundary simplex is not inside the domain, then its sink is not defined by the above definition. For this case, we just define its circumcenter as its sink. Given a boundary k-simplex σ ($k < d$) contained in a k-dimensional boundary polyhedron, its sink is defined by considering only that boundary polyhedron.

The *min-circumsphere* of a k-simplex τ in d-dimensions is the smallest d-dimensional sphere that contains all vertices of τ on its surface. When $k = d$, the min-circumsphere is also called the *circumsphere.* A point is said to *encroach* the domain boundary if it is contained inside the min-circumsphere of a boundary k-simplex, where $k < d$.

Let T be the set of tetrahedron in a 3-dimensional Delaunay mesh and $\mathcal{T} = T \cup \{\tau_\infty\}$, where τ_∞ represents the outside of the domain, called *dummy element.* Edelsbrunner *et al.* [2] defined the *flow relation* $\prec \subseteq \mathcal{T} \times \mathcal{T}$ with $\tau_1 \prec \tau_2$ if

1. τ_1 and τ_2 share a common triangle ν, and
2. the interior of τ_1 and the circumcenter c of τ_1 lie on the different side of the plane containing ν.

If $\tau_1 \prec \tau_2$, then τ_1 is called the *predecessor* of τ_2; and τ_2 is called the *successor* of τ_1. Here predecessor and successor are only meaningful for a Delaunay tetrahedron. The set of descendants of tetrahedron τ is defined as

$$desc(\tau) = \{\tau\} \cup desc_{\tau \prec \mu}(\mu), \quad \text{where } desc_{\tau \prec \mu}(\tau) = \bigcup_{\tau \prec \mu} desc(\mu).$$

Notice that for a triangle, there is only one successor defined, while there are only at most two successors defined for a tetrahedron. A sequence of tetrahedra with $\tau_1 \prec \tau_2 \ldots \prec \tau_n$ is called a flow path from τ_1 to τ_n, denoted by $\pi(\tau_1, \tau_n)$. See the left figure in Figure 1 for an illustration in 2D.

2.2 Sink Insertion Algorithms

Sink insertion method, proposed by Edelsbrunner *et al.* [2, 3], inserts the sinks of bad d-dimensional simplex elements instead of inserting their circumcenters directly. A simplex element is *bad* if its radius-edge ratio is larger than a constant ϱ. For the completeness of the presentation, we review the sink insertion algorithm for a three-dimensional domain.

Algorithm: SINK-INSERTION(ϱ_0)

Empty Encroachment: For any encroached boundary segment, add its midpoint and update the triangulation. For any encroached boundary triangle, add its sink and update the triangulation. If the sink to-be-added encroaches any boundary segment, we split that segment instead of adding that sink.

Bad Elements: For any tetrahedron σ with $\rho(\sigma) > \varrho_0$, find its sink s_σ. Assume that s_σ is the circumcenter of a tetrahedron τ. Insert the sink s_σ to split τ and update the Delaunay triangulation. However, if s_σ encroaches a boundary segment or triangle, we apply the following rules instead of adding s_σ.

Equatorial Sphere: For any boundary triangle μ encroached by the sink s_σ, add the sink s_μ of μ. Update the triangulation accordingly. However, if s_μ encroaches any boundary segment, we apply the following rule instead.

Diametral Sphere: For any boundary segment ν encroached by the sink s_σ or the sink s_μ, add the midpoint of ν. Update the triangulation accordingly.

Recall that the insertion of the circumcenter of a bad d-simplex will immediately remove the simplex. Inserting the sink of a bad tetrahedron may seem counter-intuitive, because the sink of a d-simplex could be far away from it. Consequently, the insertion of the sink may not remove the bad d-simplex immediately. The termination of the algorithm may be in jeopardy. However, it is proved that the circumradius of a tetrahedron σ is no more than that of the tetrahedron τ containing the sink of σ[2, 4]. Then the proofs of the termination of Delaunay refinement method [6] can be applied directly to prove the termination of the sink insertion algorithm. If we select $\varrho_0 > 1$, then the minimum distance among mesh vertices after the sink insertion will not decrease, which implies the algorithm's termination. If there are boundary constraints, the constant ϱ_0 has to be increased to $\sqrt{2}$ for $2D$ domain and 2 for $3D$ domain.

3 Good Grading Guarantee in 2D

This section is devoted to study the number of elements in the generated two-dimensional mesh by analyzing the relation between the nearest neighbor function $N()$ defined by the final mesh and the local feature size function $lfs()$ defined by the input domain. Here $N(x)$ is the distance from x to its second nearest mesh vertex. A mesh vertex v is always the nearest mesh vertex of itself. Local feature size $lfs(x)$ is the radius of the smallest disk centered at x intersecting two non-incident input segments or input vertices. Both $N()$ and $lfs()$ are 1-Lipschitz function. A mesh is said to have a *good grading* if the nearest neighbor function $N()$ defined on the mesh is within a constant factor of $lfs()$.

We study the spacing relations among intermediate meshes by using similar idea as Ruppert and Shewchuk did. With each vertex v, we associate an *insertion edge length* e_v equal to the length of the shortest edge connected to v immediately after v is introduced into the Delaunay mesh. Here v may not have to be inserted into the mesh actually. For the sake of convenience of analyzing, we also define a *parent vertex* $p(v)$ for each vertex v, unless v is an input vertex. Intuitively, for any non-input vertex v, $p(v)$ is the vertex "responsible" for the insertion of v. We discuss in detail what means by responsible here. If v is inserted as the sink of a triangle σ with $\rho(\sigma) \geq \varrho_0$, then $p(v)$ is the most recently inserted end point of the shortest edge of σ. If all vertices of σ are original input vertices, then $p(v)$ is one of the end points of the shortest edge of σ. If v is the midpoint of an encroached segment, then $p(v)$ is the encroaching vertex. For the sake of simplicity, we always assume that the encroaching vertex is not an input vertex, because Ruppert [5] and Shewchuk [6] showed that the nearest neighbor function $N()$ defined on the Delaunay mesh after enforcing the domain boundary is within a constant factor of the local feature size function, i.e., $N(v) \sim lfs(v)$. The parent vertex $p(v)$ of v does not need to be inserted into the mesh actually.

We then show that e_v of any introduced mesh vertex v is related to that of its parent vertex $p(v)$. Here v may also not be inserted due to encroaching. For a vertex v, as [6, 4], we define the *density ratio* at point v as $D_v = \frac{lfs(v)}{e_v}$. Clearly, D_v is at most one for an input vertex v, and for newly inserted vertex v, D_v tends to become larger. Notice that D_v is defined just immediately after v is introduced to the mesh; it is not defined based on the final mesh.

Lemma 1. [Radius-variation] *Consider a triangle σ. Let u and e_u be the circumcenter and the circumradius of σ. Assume that there is no triangle with a large radius-edge ratio in $desc(\sigma)$ except possibly σ itself. Assume triangle $\tau \neq \sigma$ contains the sink of σ inside. Let μ be the predecessor of triangle τ. Let w and e_w be the circumcenter and circumradius of the triangle μ. Then*

$$e_w - e_u \geq \frac{1}{4\varrho_0}||w - u||.$$

Proof. Let's consider a triangle $\tau_1 = pqr$ and its successor $\tau_2 = pqs$ in $desc(\sigma)$, where $\tau_2 \neq \tau$. Triangle τ_2 has a small radius-edge ratio from the assumption of $desc(\sigma)$. Let x and y be the circumcenter of τ_1 and τ_2 respectively. Let e_x and e_y be the circumradius of τ_1 and τ_2 respectively. The right figure in Figure 1 illustrates the proof that follows. Let c be the midpoint of the edge pq. Let

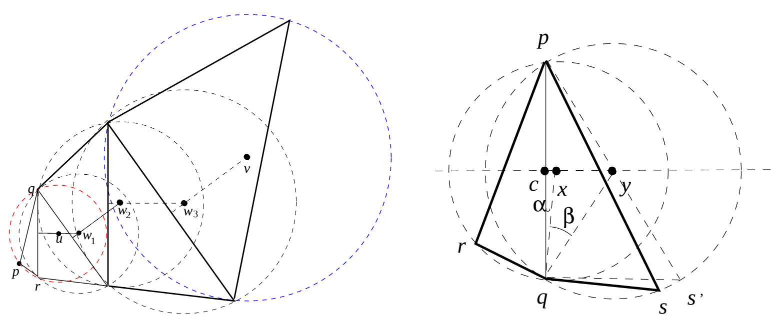

Fig. 1. Left: the descendants of a triangle pqr; Right: the relationship between the circumradius e_x of $\tau_1 = pqr$, the circumradius e_y of $\tau_2 = pqs$ and $||x - y||$.

$\alpha = \angle cqx$ and $\beta = \angle xqy$. Assume that cq has a unit length. Then $e_x = 1/\cos(\alpha)$, $e_y = 1/\cos(\alpha + \beta)$, and $||x - y|| = \tan(\alpha + \beta) - \tan(\alpha)$. It is easy to show that

$$\frac{e_y - e_x}{||x - y||} = \frac{\sin(\alpha + \frac{\beta}{2})}{\cos(\frac{\beta}{2})} = \sin(\alpha + \beta) - \cos(\alpha + \beta)\tan(\frac{\beta}{2}).$$

Assume we fix the value of $\alpha + \beta = \theta_0$, then it is easy to show that

$$\frac{\sin(\alpha + \frac{\beta}{2})}{\cos(\frac{\beta}{2})} \geq \sin(\theta_0) - \cos(\theta_0)\tan(\frac{\theta_0}{2}) = \tan(\frac{\theta_0}{2}).$$

Let ps' be a diameter of the circumcircle of the triangle pqs. It is easy to show that $||q-s|| < ||q-s'|| = 2||c-y|| = 2e_y \sin(\alpha+\beta)$. The triangle $\tau_2 = pqs$ has a small radius-edge ratio implies that $||q-s|| \geq \frac{e_y}{\varrho_0}$. Thus we have $\sin(\alpha+\beta) \geq \frac{1}{2\varrho_0}$. It follows that $\frac{e_y - e_x}{||x-y||} \geq \tan(\frac{\theta_0}{2}) = \frac{\sin\theta_0}{1+\cos\theta_0} > \frac{1}{4\varrho_0}$. Using the triangle inequality, it is easy to show that $e_w - e_u > \frac{1}{4\varrho_0}||w-u||$ by summing up the inequalities for all triangle and successor pairs τ_1 and τ_2 in $desc(\sigma)$ with $\tau_2 \neq \tau$. ◻

Theorem 1. [BOUNDED DENSITY RATIO] *There are fixed constants $D_1 \geq 1$ and $D_2 \geq 1$ such that for any vertex v inserted or rejected as the sink of a bad triangle, $D_v \leq D_2$; for any vertex v inserted or rejected as the midpoint of an encroached boundary segment, $D_v \leq D_1$. Hence, there is a constant $D = \max\{D_1, D_2\}$ such that $D_v \leq D$ for all mesh vertex v.*

PROOF. We prove this by induction. If v is an original input vertex, then the length e_v of the shortest edge connected to v is at least $lfs(v)$ from the definition of $lfs(v)$. Thus $D_v = \frac{lfs(v)}{e_v} \leq 1$ and the theorem holds.

Then consider non-input vertex v. We first consider inserting v as the sink of a triangle σ. Let u be the circumcenter of the triangle σ. Notice that v is also the sink of any triangle from $desc(\sigma)$. Without loss of generality, assume that no triangle except σ from $desc(\sigma)$ has a large radius-edge ratio. Assume v is the circumcenter of a triangle τ. Notice that e_v is equal to the circumradius of τ.

Case 1: the triangles σ and τ are the same. Notice that σ has a radius-edge ratio at least ϱ_0, then parent p of vertex v is one of the end points of the shortest edge pq of σ. Here p could be the most recently inserted vertex or an original vertex of σ. Then q is an original vertex or is inserted before p. In both cases, we have $e_p \leq ||p-q||$. Then $e_p \leq ||p-q|| \leq \frac{R_\sigma}{\varrho_0}$. And $e_u = R_\sigma \geq \varrho_0 \cdot e_p$. Notice that $lfs(p) \leq D_p e_p$, where D_p is the density ratio bound of vertex p derived from induction. Thus $lfs(u) \leq lfs(p) + ||p-u|| \leq D_p e_p + e_u \leq (\frac{D_p}{\varrho_0} + 1)e_u$. It implies that $D_u = \frac{lfs(u)}{e_u} \leq \frac{D_p}{\varrho_0} + 1$. So a sufficient condition for bounding the density ratio of vertex u is

$$\frac{\max(D_1, D_2)}{\varrho_0} + 1 \leq D_2 \tag{1}$$

Case 2: the triangles σ and τ are not the same. Let w be the circumcenter of the predecessor triangle of τ. Similarly we define e_w as the circumradius of that triangle. Then by previous lemma 1, we know that there is a constant $\delta = \frac{1}{4\varrho_0}$ such that $e_w - e_u \geq \delta||w-u||$. Here w and u could be the same.

Subcase 2.1: vertices w and v are not close, i.e., $||v-w|| \geq \epsilon e_v$, where $\epsilon = \frac{1}{2\varrho_0}$. Then similar to the previous lemma 1, $e_v - e_u \geq \delta||v-u||$. For point v, we have

$lfs(v) \le lfs(u) + ||u - v|| \le lfs(u) + \frac{1}{\delta}(e_v - e_u) \le (1 + \frac{D_p}{\varrho_0} - \frac{1}{\delta})e_u + \frac{1}{\delta}e_v$. Thus a sufficient condition that $D_v = \frac{lfs(v)}{e_v} \le D_2$ is $(1 + \frac{D_p}{\varrho_0} - \frac{1}{\delta})e_u \le (D_2 - \frac{1}{\delta})e_v$. From $e_u \le e_v$, this inequality is satisfied if $1 + \frac{D_p}{\varrho_0} - \frac{1}{\delta} \le D_2 - \frac{1}{\delta}$, and $D_2 - \frac{1}{\delta} \ge 0$. From $D_p \le \max(D_1, D_2)$, a sufficient condition that $D_v \le D_2$ is

$$D_2 \ge \max(\frac{1}{\delta}, 1 + \frac{\max(D_1, D_2)}{\varrho_0}). \tag{2}$$

Subcase 2.2: vertices w and v are close, i.e., $||v - w|| < \epsilon e_v$, where $\epsilon = \frac{1}{2\varrho_0}$. For vertex w, similar to subcase 2.1, we have $lfs(w) \le (1 + \frac{D_p}{\varrho_0} - \frac{1}{\delta})e_u + \frac{1}{\delta}e_w$. Then $lfs(v) \le lfs(w) + ||v - w||$ implies that $lfs(v) \le (1 + \frac{D_p}{\varrho_0} - \frac{1}{\delta})e_u + \frac{1}{\delta}e_w + \epsilon e_v \le (1 + \frac{D_p}{\varrho_0} - \frac{1}{\delta})e_u + (\frac{1}{\delta} + \epsilon)e_v$. Thus, from $e_u \le e_v$, a sufficient condition that $D_v \le D_2$ is $1 + \frac{D_p}{\varrho_0} - \frac{1}{\delta} \le D_2 - \frac{1}{\delta} - \epsilon$, and $D_2 - \frac{1}{\delta} - \epsilon \ge 0$. Consequently, we need

$$D_2 \ge \max(1 + \epsilon + \frac{\max(D_1, D_2)}{\varrho_0}, \frac{1}{\delta} + \epsilon). \tag{3}$$

Case 3: vertex v is the midpoint of a segment that is encroached by a vertex w. Here w is the sink of a triangle σ with large radius-edge ratio. Assume that the sink w is contained in triangle τ. Let u be the circumcenter of triangle σ.

Subcase 3.1: vertices w and u are not same. From the analysis of case 2, we have $lfs(w) \le (1 + \frac{D_p}{\varrho_0} - \frac{1}{\delta})e_u + (\frac{1}{\delta} + \epsilon)e_w$ by substituting v by w in the results. The vertex w is inside the circumcircle centered at v with radius e_v. Therefore $e_w \le \sqrt{2}e_v$. From $lfs(v) \le lfs(w) + ||w - u|| \le (1 + \frac{D_p}{\varrho_0} - \frac{1}{\delta})e_u + (\frac{\sqrt{2}}{\delta} + \sqrt{2}\epsilon + 1)e_v$, inequality $D_v \le D_1$ holds if $1 + \frac{D_p}{\varrho_0} - \frac{1}{\delta} \le D_1 - \sqrt{2}(\frac{1}{\delta} + \epsilon) - 1$ and $D_1 \ge \sqrt{2}(\frac{1}{\delta} + \epsilon) + 1$. Consequently, a sufficient condition is that

$$D_1 \ge \max\{2 + \sqrt{2}(\frac{1}{\delta} + \epsilon) - \frac{1}{\delta} + \frac{\max(D_1, D_2)}{\varrho_0}, \sqrt{2}(\frac{1}{\delta} + \epsilon) + 1\}. \tag{4}$$

Subcase 3.2: vertices u and w are the same. We have $lfs(v) \le lfs(u) + ||u - v|| \le (1 + \frac{D_p}{\varrho_0})e_u + e_v \le (1 + \frac{D_p}{\varrho_0})\sqrt{2}e_v + e_v$. To prove $D_v \le D_1$, we need

$$\sqrt{2}(1 + \frac{D_p}{\varrho_0}) + 1 \le D_1. \tag{5}$$

As conclusion, if we choose D_1 and D_2 as the follows,

$$D_1 = \max \begin{cases} 1 + \sqrt{2}(\frac{1}{\delta} + \epsilon), \\ \frac{(\sqrt{2}+1)\varrho_0}{\varrho_0 - \sqrt{2}}, \\ \frac{\varrho_0(2\delta + \sqrt{2} - 1 + \sqrt{2}\epsilon\delta)}{\delta(\varrho_0 - 1)}, \end{cases} \quad \text{and} \quad D_2 = \max(\frac{1}{\delta} + \epsilon, 1 + \epsilon + \frac{D_1}{\varrho_0}),$$

then all inequalities are satisfied. Notice here $\delta = \frac{1}{4\varrho_0}$, and $\epsilon = \frac{1}{2\varrho_0}$. $\square$

The following theorem concludes that the generated mesh has a good grading, i.e., for any mesh vertex v, $N(v)$ is at least some constant factor of $lfs(v)$.

Theorem 2. [GOOD GRADING] *For any mesh vertex v generated by the sink insertion method, the edge incident on v has length at least $\frac{lfs(v)}{D+1}$.*

The proof is omitted here. The values corresponding to D_1 and D_2 guaranteed by the Delaunay refinement method [5, 6] are small: $D_v = \frac{lfs(v)}{e_v}$ is at least $\frac{1}{\varrho_0 - \sqrt{2}}$ for a vertex v inserted as the circumcenter of a bad triangle and at least $\frac{\varrho_0}{\varrho_0 - \sqrt{2}}$ for a vertex v inserted as the midpoint of an encroached segment. For instance, Ruppert claims that if the smallest angle is 10 degrees, then no edge is smaller than $\frac{1}{6}$ of the local feature size of either of its endpoints. To guarantee the minimum angle 10 degrees, we need $\varrho_0 = \frac{1}{2\sin 10^\circ} \approx 2.88$. Then $\delta \approx 0.087$ and $\epsilon \approx 0.174$. So $D_1 \approx 17.54$ and $D_2 \approx 16.69$. It then implies that no edge is smaller than $\frac{1}{19}$ of the local feature size of either of its endpoints in any mesh generated by the sink insertion method. Therefore, the theoretical bound on the number of elements of the mesh generated by sink insertion method is more likely larger than that by the Delaunay refinement method.

Ruppert shows that the nearest neighbor value $N(v)$ of a mesh vertex v of any almost-good mesh is at most a constant factor of $lfs(v)$, where the constant depends on the radius-edge ratio. The above Theorem 2 shows that the nearest neighbor $N(v)$ for the $2D$ Delaunay mesh generated by sink insertion is at least some constant factor of $lfs(v)$. Then we have the following theorem.

Theorem 3. [LINEAR SIZE] *The number of triangles in the $2D$ mesh generated by the sink insertion method is within a constant factor of any Delaunay mesh for the same domain, where the constant depends on the radius-edge ratios of the meshes.*

4 Discussions for 3D Domain

Shewchuk [6] showed that the Delaunay refinement method generates almost-good meshes with a good grading guarantee in two and three dimensions. We had showed that the sink insertion method also generates a almost-good mesh with a good grading guarantee in two dimensions. Unfortunately, the proofs presented here can not be directly applied to three dimensions. The reason is as follows. To guarantee the size optimality of the sink insertion method, the nearest neighbor function $N()$ defined on the generated mesh must be within a constant factor of the local feature size function $lfs()$. Notice that $lfs(v) \geq e_v \geq N(v)$, where e_v is the length of the shortest edge connected to v when vertex v is inserted. Therefore, when a vertex v is introduced to the mesh, e_v should be within a constant factor of $lfs(v)$ to guarantee the good grading of the generated mesh. In other words, we have to prove the existence of a constant D such that for each vertex v inserted into the mesh, $lfs(v) \leq De_v$. Or more specifically, there should exist three constants D_1, D_2, and D_3 such that

- for each vertex v inserted as the circumcenter of a tetrahedron, $lfs(v) \leq D_3 e_v$;
- for v inserted as the circumcenter of a boundary triangle, $lfs(v) \leq D_2 e_v$;
- for v inserted as the midpoint of a boundary segment, $lfs(v) \leq D_1 e_v$.

Assume that we insert a vertex v as a sink of a tetrahedron σ, and v is the circumcenter of a tetrahedron τ. Notice that the structure of $desc(\sigma)$ is a DAG whose node out-degree is at most 2. Consider a tetrahedron τ from $desc(\sigma)$. Without loss of generality, we assume that there is no tetrahedron in $\pi(\sigma, \tau)$ with a large radius-edge ratio except tetrahedron σ itself. In three dimensions, the fact that a tetrahedron has a small radius-edge ratio does not guarantee that the tetrahedron has no small angles. Slivers are the only tetrahedra that have a small radius-edge ratio but their aspect ratio could be arbitrarily large. Let u and e_u be the circumcenter and the circumradius of σ. Let e_v be the circumradius of the tetrahedron τ. It is possible that e_v is almost the same as e_u even vertex v is far away from u. Consequently, $\frac{e_v - e_u}{||v-u||}$ could not be bounded from below by any constant. Figure 2 gives an example of a configuration such that $\frac{e_v - e_u}{||v-u||}$ could be arbitrarily small. This, together with the fact that $lfs(v)$ could be as large as $lfs(u) + ||v-u||$ implies that $D_v = \frac{lfs(v)}{e_v}$ could be much larger than $D_u = \frac{lfs(u)}{e_u}$. Therefore we can not bound the density D_v using the relation between D_u and D_v even assuming that the density D_u of vertex u is bounded by a constant D_3.

Figure 2 is constructed as follows. Let sliver $p_0q_0r_0s_0$ be a successor of the tetrahedron σ, which is not shown in the left figure of Figure 2. Assume that the circumcenter of the sliver $p_0q_0r_0s_0$ is on the different side of the plane $H_{p_0r_0s_0}$ passing p_0, r_0 and s_0 with sliver $p_0q_0r_0s_0$. One of the successor of sliver $p_0q_0r_0s_0$ is constructed by lifting vertex q_0 to a new position q_1 such that tetrahedron $p_0q_1r_0s_0$ is a sliver and its circumcenter is on the different side of the plane $H_{p_0q_1s_0}$ passing p_0, q_1 and s_0 with the tetrahedron $p_0q_1r_0s_0$. Then we lift the node r_0 to r_1 to construct a sliver successor $p_0q_1r_1s_0$ of tetrahedron $p_0q_1r_0s_0$ whose circumcenter is on the different side of the plane $H_{p_0q_1r_1}$ with $p_0q_1r_1s_0$. We then construct a sliver successor $p_0q_1r_1s_1$ of $p_0q_1r_1s_0$ by lifting s_0 to s_1. Sliver successor $p_1q_1r_1s_1$ is constructed by lifting p_0 to a new position p_1. The above procedure could be repeated many rounds if $p_0q_0r_0s_0$ is carefully configured and every lifting is carefully chosen. The middle figure in Figure 2 give the sliver pattern used in constructing this example. It is easy to show that using only the tetrahedron τ also can not bound the density ratio $D_v = \frac{lfs(v)}{e_v}$. Assume that p is a vertex of the shortest edge of tetrahedron τ; see the right figure in Figure 2. Then we have $D_v \leq 1 + \frac{4}{\sqrt{6}} D_p$. Here the situation $D_v = 1 + \frac{4}{\sqrt{6}} D_p$ could be

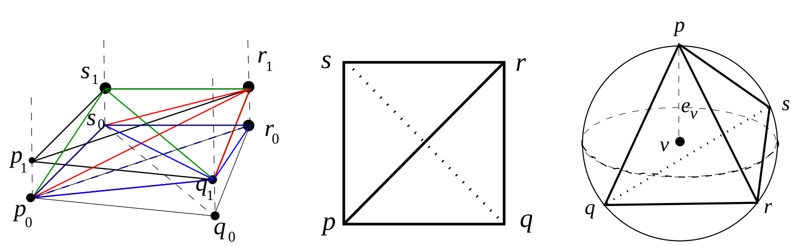

Fig. 2. Left: all sliver descendants; Right: the sink simplex does not help.

achieved if τ is a regular tetrahedron. It implies that the upper bound for D_v could be always larger than that for D_p. However, we doubt that these situations can really happen in practice.

5 Conclusion

In this paper, we show that the sink insertion method guarantees to generate a two-dimensional mesh with good grading. On the other hand, we also give an example of three-dimensional local mesh configuration to show that the sink insertion method may fail to generate a mesh with size $O(n)$, where n is the minimum number of the mesh elements with the same radius-edge ratio quality.

As reported by the experimental results (Guoy and Edelsbrunner, private communication), the sink insertion method usually generates meshes whose sizes are not much larger than that by Delaunay refinement method. However, it is interesting to see if we can theoretically prove the good grading guarantee of the sink insertion method or give an example of three-dimensional domain such that a sequence of sink insertions will generate a mesh whose size is larger than $O(n)$. Notice Li [4] recently gave a new refinement-based algorithm that generates well-shaped three-dimensional meshes with size $O(n)$. Li [4] also proposed a variation of the sink insertion method, which inserts a point near the sink and its insertion will not introduce small slivers compared to τ instead of inserting the sink of a tetrahedron τ with large radius-edge ratio or sliver. This variation guarantees to generate well-shaped 3D Delaunay meshes. However, it is open whether this variation will have a good grading guarantee. It is interesting to see what is the mesh size relation between two meshes generated by the sink insertion method and this variation proposed in [4].

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