Scaling Laws of Capacity for Large Scale Wireless Networks

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***Abstract*—In this paper, we focus on the *networking-theoretic* multicast capacity bounds for both *random extended networks* (REN) and *random dense networks* (RDN) under *Gaussian Channel* model, when all wireless nodes have the same constant transmission power P. During the transmission, the power decays along path with attenuation exponent $\alpha > 2$. In REN and RDN, n nodes are randomly distributed in the square region with side-length $\sqrt{n}$ and 1, respectively. We randomly choose n_s nodes as the sources of multicast sessions, and for each source v, we pick uniformly at random n_d nodes as the destination nodes of v. Based on percolation theory, we propose multicast schemes and analyze the achievable throughput taking account of all n_s and n_d. As the specific case of our results, we show that for $n_s = \Theta(n)$, the per-session multicast capacity of RDN is $\Theta(\frac{1}{\sqrt{n_d n}})$ when $n_d = O(\frac{n}{(\log n)^3})$ and is $\Theta(\frac{1}{n})$ when $n_d = \Omega(\frac{n}{\log n})$; the per-session multicast capacity of REN is $\Theta(\frac{1}{\sqrt{n_d n}})$ when $n_d = O(\frac{n}{(\log n)^{\alpha+1}})$ and is $\Theta(\frac{1}{n_d} \cdot (\log n)^{\frac{\alpha}{2}})$ when $n_d = \Omega(\frac{n}{\log n})$.**

***Index Terms*—Multicast Capacity, Percolation, Wireless ad hoc networks, Random networks, Achievable throughput**

I. Introduction

The problem of asymptotic scalability of capacity for wireless networks has received much attention from the researchers, especially after the pioneer work done by Gupta and Kumar [1]. According to the method of nodes placement and traffic pattern, Gupta and Kumar defined two types of networks: *arbitrary networks* and *random networks*. The randomness contributes to the fact that the throughput of random networks is not greater than that of arbitrary networks in general. According to the ways of letting the number of nodes n tend to infinity, there are two typical network models: *extended networks* and *dense networks*. Most of the succeeding researches follow those network models, and the derived capacity results differ from each other because of the diversity of analytical models and assumptions to be used.

There are generally two levels of capacity bounds. The first level is *information-theoretic* bounds that are obtained by allowing arbitrary (physical layer) cooperative relay strategies, [2]. The issue was first addressed by Xie and Kumar [3]. The second is *networking-theoretic* bounds that assumes that the signals received from nodes other than one particular transmitter are interference to be regarded as noise degrading the communication link. The pioneer work [1] is just done for this issue. It is intuitive that the optimal strategy for *networking-theoretic* bounds is to confine to nearest neighbor communication and maximize spatial reuse, because the interference generated by long communication would preclude most of the other nodes from communicating, [1], [4].

In the research of *networking-theoretic* capacity bounds, there are three types of channel models in general. The first is the *threshold-based channel* model under which if the value of a given conditional expression is beyond the threshold, the transmitter can send successfully to the receiver at a specific constant data rate, otherwise, it can not send any, *i.e.*, the transmission rate is assumed to be a binary function. The *protocol interference* model (PrIM) and *physical interference* model (PhIM) defined in [1] both belong to the *threshold-based channel* model. The former's conditional expression is the fraction of the distances from the particular transmitter and other transmitters to the particular receiver; the latter's conditional expression is SINR (signal to Interference plus noise ratio). This model is simple thus analytically attractive, and many works therein are based on this model, such as [5]–[13]. The second is the *probability-based channel* model [14] under which the receiver can receive packets at a specific rate successfully if the probability that SINR is below the threshold is less than a certain value. The third is the *rate-based channel model* that determines the transmission rate at which the transmitter can send its data to the receiver reliably, based on a continuous function of the receiver's SINR. Generally, any communication pair v_i and v_j can establish a direct communication link, over a channel of bandwidth B, of rate $R(v_i, v_j) = B\log(1 + (1/\eta)\text{SINR}(v_j))$. When $\eta > 1$, the receiver can achieve the maximum rate that meets a given BER requirement under a specific modulation and coding scheme; When $\eta = 1$, the receiver achieves the Shannon's capacity for a wireless channel with additive Gaussian white noise, see [15], [16]. Then, in the case of $\eta = 1$, *rate-based channel model* can be called the *Gaussian channel* model.

In this paper, we study the *networking-theoretic* multicast capacity for both *random extended networks* (REN) and *random dense networks* (RDN) under *Gaussian Channel* model. We present both improved lower bound and improved upper bound on multicast capacity, compared with previous literatures.

For studying the lower bound of multicast capacity, we design two types of multicast schemes for REN and RDN.

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In one type of scheme, we construct the routing based on percolation theory and schedule respectively short-hops and long-hops. In the other type, we construct the routing without using the percolation theory in order to avoid the bottleneck on the accessing path into *highways* ([17]). Combining with the two types of schemes, we obtain the achievable throughput as the lower bounds of multicast capacity. A characteristics of this paper is that we take account of all cases of n_s and n_d, without any assumption of n_s and n_d as in most other literatures, which contributes to the generality of the paper. Our lower bounds on multicast capacity improve the previously best known results. We design our routing and schedule schemes based on several innovative techniques: both backbone highway system and second highway systems based on percolation theory, and parallel scheduling of nearby links. Using second highway systems and parallel scheduling of nearby links, to the best of our knowledge, are not used in previous studies.

On the other hand, using new analyzing techniques, we derive upper bounds on multicast capacity. Our upper bounds also rely on several new techniques. Two different approaches are proposed in this paper to study an upper bound. The first approach partitions the region into cells and we focus on cells with at most a constant number of nodes, in contrast to focusing on cells with at least a constant number of nodes for building highway systems [17]. We then show that some of these cells will have large load, regardless of the routing and scheduling schemes. The second approach is to study the bottleneck on some links. We show that there exist some special links that will be used by many multicast sessions (thus high load) and its own data rate is small, thus, implying an upper bound on per-session multicast capacity. Furthermore, we obtain the multicast capacity for both REN and RDN. As we know, unicast and broadcast can be regarded as two specific cases of multicast corresponding to the case $n_d = 1$ and $n_d = n - 1$, respectively, where n_d is the number of destinations for each multicast session. Thus, the capacity of multicast session can usually unify that of unicast and broadcast under the same channel models. Indeed, some existing results can be derived by our result as the specific cases, such as [2], [17], [23], [25].

The rest of the paper is structured as follows. In Section II, we propose the network model and related definitions. The main results is presented in Section III. In Section IV, we design the multicast schemes for *random extended networks*, and analyze the achievable throughput. In Section V, we extend the schemes to the *random extended networks* and derive the lower bounds of multicast capacity for *random extended networks*. In Section VI, we discuss the upper bound of the multicast capacity. In Section VII, we present a review of existing results on the capacity scaling of multihop wireless networks. In Section VIII, we conclude the paper.

II. Network Model

The asymptotic capacity is the capacity of the network as the number of nodes n in the network goes to infinity. According to the ways of letting n goes to infinity, there are two typical network models. The first is the *extended network model* that is to fix the node density to a given constant and increase the area of the deployment region to infinity. The second is the *dense network model* that is to fix the area of the deployment region and increase the node density to infinity. Under the *protocol interference model* (PrIM), Li *et al.* [11] proposed another model called *constant-range model* that is to fix the transmission range of all nodes to some constant, then increase the node density and the deployment area to increase the number of nodes in the networks.

In this paper, we focus on the former two typical network models. We construct a *random extended network* by placing nodes according to a Poisson point process (p.p.p.) of unit intensity on the square $\mathcal{A}_n = [0, \sqrt{n}] \times [0, \sqrt{n}]$. Similarly, we construct a dense network by placing nodes according to a Poisson point process of intensity n over a square of unit area, *i.e.*, $\mathcal{A}_1 = [0, 1] \times [0, 1]$. According to Chebyshev's inequality, we can easily obtain the number of nodes in $\mathcal{A}_n$ (or $\mathcal{A}_1$) is within $((1-\varepsilon_0)n, (1+\varepsilon_0)n)$. To simplify the description, we assume that the number of nodes are n, without changing our results in order sense, [2], [17]. We are mainly concerned with the events that occur inside these squares with high probability (*w.h.p.*); that is, with probability tending to one as $n \to \infty$.

A. Capacity Definition

In this subsection, we give the formal definition of capacity in our model. We assume that $V = \{v_1, v_2, \cdots, v_n\}$ is the set of nodes in the network, and $\mathcal{S} \subseteq V$ is the set of source nodes of multicast, denote the number of multicast sessions as $|\mathcal{S}| = n_s$. For each source nodes, we can randomly select n_d nodes as destinations from the other nodes, where $n_d \leq n-1$. Let $\Lambda_{\mathcal{S},n_d} = (\lambda_{\mathcal{S},1}, \lambda_{\mathcal{S},2}, \cdots, \lambda_{\mathcal{S},n_s})$ be the *rate vector* of the multicast data rate of all multicast sessions.

Definition 1 (Feasible Rate Vector): Multicast rate vector $\Lambda_{\mathcal{S},n_d} = (\lambda_{\mathcal{S},1}, \lambda_{\mathcal{S},2}, \cdots, \lambda_{\mathcal{S},n_s})$ is feasible if there is a spatial and temporal scheme for scheduling transmissions such that by operating the network in a multi-hop fashion and buffering at intermediate nodes when awaiting transmission, the ith source node denoted as $v_{\mathcal{S},i}$ can deliver data to all n_d destinations at rate of $\lambda_{\mathcal{S},i}$ bit/second, where $i = 1, 2, \cdots, n_s$. That is, there is a $T < \infty$ such that in every time interval (with unit seconds) $[(i-1)\cdot T, i \cdot T]$, every node $v_{\mathcal{S},i} \in \mathcal{S}$ can send $T \cdot \lambda_{\mathcal{S},i}$ bits to all its n_d destinations.

Considering a multicast rate vector, we define the *total multicast throughput capacity* of such feasible rate vector as $\Lambda^{\mathrm{T}}_{\mathcal{S},n_d}(n) = \Sigma_{i=1}^{n_s}\lambda_{\mathcal{S},i}$, define the average per-session multicast throughput capacity as $\Lambda^{\mathrm{P}}_{\mathcal{S},n_d}(n) = \frac{\Sigma_{i=1}^{n_s}\lambda_{\mathcal{S},i}}{n_s}$, and define the *minimum per-session multicast throughput capacity* as $\Lambda^{\mathrm{M}}_{\mathcal{S},n_d}(n) = \min_{v_{\mathcal{S},i}\in\mathcal{S}} \lambda_{\mathcal{S},i}$.

Definition 2 (Throughput Capacity): An aggregated multicast throughput $\Lambda^{\mathrm{T}}_{\mathcal{S},n_d}(n)$ bits/sec is feasible for n_s multicast sessions (each session with n_d destinations) if there is a rate vector $\Lambda_{\mathcal{S},n_d} = (\lambda_{\mathcal{S},1}, \lambda_{\mathcal{S},2}, \cdots, \lambda_{\mathcal{S},n_s})$ that is feasible and $\Lambda^{\mathrm{T}}_{\mathcal{S},n_d}(n) = \Sigma_{i=1}^{n_s}\lambda_{\mathcal{S},i}$. Similarly, we say $\Lambda^{\mathrm{M}}_{\mathcal{S},n_d}(n) = \min_{v_{\mathcal{S},i}\in\mathcal{S}} \lambda_{\mathcal{S},i}$ is a feasible per-session multicast throughput capacity.

Definition 3 (Capacity of Random Networks): The aggregated multicast capacity of a class of random networks is of order $\Theta(g(n))$ bits/sec if there are deterministic constants $c > 0$ and $c < c' < +\infty$ such that

$$\lim_{n\to+\infty} \Pr(\Lambda^{\mathrm{T}}_{\mathcal{S},n_d}(n) = cg(n) \text{ is feasible}) = 1,$$
$$\lim_{n\to+\infty} \inf \Pr(\Lambda^{\mathrm{T}}_{\mathcal{S},n_d}(n) = c'g(n) \text{ is feasible}) < 1.$$

We can similarly define the per-session and minimum capacity for random networks.

B. Rate between two nodes

We assume all nodes transmit at constant power P, and that node v_j receives the transmitted signal from node v_i with power $P \cdot l(v_i, v_j)$, where $l(v_i, v_j)$ indicates the path loss between v_i and v_j. We restrict ourselves to a model of communication where the interference at the receiver is simply regarded as noise, *i.e.*, we focus on the *networking-theoretic bounds* instead of the *information-theoretic bounds*, see [3], [18]–[22]. Hence, any two nodes can establish a direct communication link, over a channel of unit bandwidth B, of rate

$$R(v_i, v_j) = B\log(1 + \frac{P \cdot l(v_i, v_j)}{N_0 + \sum_{v_k\in A(i)} P\cdot l(v_k, v_j)}) \quad (1)$$

where $N_0 \geq 0$ is the ambient noise power at the receiver, and $A(i)$ is the set of nodes that transmit when v_i is scheduled.

C. Theory Preparations

In this section, we provide some related results to be used in the analysis. Firstly, we recall the following useful lemma from [17].

Lemma 1 (Upper Tail on Chernoff Bounds): For a Poisson random variable X of parameter λ, we have

$$\Pr(X \geq x) \leq \frac{e^{-\lambda}(e\lambda)^x}{x^x}, \text{ for } x > \lambda \quad (2)$$

Similarly, we give the lower tail version of Chernoff bounds.

Lemma 2 (Lower Tail on Chernoff Bounds): For a Poisson random variable X of parameter λ, we have

$$\Pr(X \leq x) \leq \frac{e^{-\lambda}(e\lambda)^x}{x^x}, \text{ for } 0 \leq x < \lambda \quad (3)$$

Proof: For any $t > 0$ and $0 \leq x < \lambda$, by Markov-Inequality we have

$$\Pr(X \leq x) = \Pr(-X \geq -x) \leq \frac{E(e^{-tX})}{e^{-tx}} \quad (4)$$

Because $E(e^{-tX}) = \sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!} e^{-tk} = e^{\lambda(e^{-t}-1)}$ and by equation (4), we have $\Pr(X \leq x) \leq e^{\lambda(e^{-t}-1)+tx}$. Let $t = \ln(\lambda/x) > 0$ we obtain Equation (3). ∎

Notations: To facilitate the expression, define a function as

$$\max_{order}\{\varphi(n),\ \phi(n)\} = \begin{cases} \Theta(\varphi(n)), & \text{if } \varphi(n) = \Omega(\phi(n)) \\ \Theta(\phi(n)), & \text{if } \phi(n) = \Omega(\varphi(n)) \end{cases}$$

Similarly, we define another function as

$$\min_{order}\{\varphi(n),\ \phi(n)\} = \begin{cases} \Theta(\varphi(n)), & \text{if } \varphi(n) = O(\phi(n)) \\ \Theta(\phi(n)), & \text{if } \phi(n) = O(\varphi(n)) \end{cases}$$

To simplify the description, let $\theta(n) : [\theta_1(n), \theta_2(n)]$ represents that $\theta(n) = \Omega(\theta_1(n))$ and $\theta(n) = O(\theta_2(n))$.

III. Main Results

Let d_{ij} be the Euclidean distance between two nodes v_i and v_j. Let the power attenuation function be $l(v_i, v_j)$. We study the throughput issue taking account of all n_s and n_d, the general results have more complex forms and they are shown in Theorem 6 and Theorem 7. In this section, we give the results under the assumption $n_s = \Theta(n)$ as the specific cases of our general results.

A. Random Extended Networks

For the *random extended network model*, we assume that $l(v_i, v_j) = \min\{1,\ d_{ij}^{-\alpha}\}$ with $\alpha > 2$ and $N_0 > 0$.

Theorem 1: The achievable per-session multicast throughput for *random extended networks* is of order

$$\begin{cases} \Omega(\frac{1}{\sqrt{n_d n}}) & \text{when } n_d : [1, \frac{n}{(\log n)^{\alpha+1}}] \\ \Omega(\frac{1}{n_d(\log n)^{\frac{\alpha+1}{2}}}) & \text{when } n_d : [\frac{n}{(\log n)^{\alpha+1}}, \frac{n}{(\log n)^2}] \\ \Omega(\frac{1}{\sqrt{nn_d}\cdot(\log n)^{\frac{\alpha-1}{2}}}) & \text{when } n_d : [\frac{n}{(\log n)^2}, \frac{n}{\log n}] \\ \Omega(\frac{1}{n_d(\log n)^{\frac{\alpha}{2}}}) & \text{when } n_d : [\frac{n}{\log n}, n] \end{cases} \quad (5)$$

For the upper bound, we have

Theorem 2: The per-session multicast throughput for *random extended networks* is at most of order

$$\begin{cases} O(\frac{1}{\sqrt{n_d n}}) & \text{when } n_d : [1, \frac{n}{(\log n)^{\alpha}}] \\ O(\frac{1}{n_d(\log n)^{\frac{\alpha}{2}}}) & \text{when } n_d : [\frac{n}{(\log n)^{\alpha}}, n] \end{cases} \quad (6)$$

From Theorem 1 and Theorem 2, we obtain that

Theorem 3: The per-session multicast capacity for *random extended networks* is

$$\begin{cases} \Theta(\frac{1}{\sqrt{n_d n}}) & \text{when } n_d : [1, \frac{n}{(\log n)^{\alpha+1}}] \\ \Theta(\frac{1}{n_d(\log n)^{\frac{\alpha}{2}}}) & \text{when } n_d : [\frac{n}{\log n}, n] \end{cases} \quad (7)$$

Observe that there is a gap between the upper bound and lower bound when $n_d : [\frac{n}{(\log n)^{\alpha+1}}, \frac{n}{\log n}]$. The gap would be closed by presenting possibly new tight upper bound and designing algorithms to achieve it.

B. Random Dense Networks

For the *random dense network model*, we assume that $l(v_i, v_j) = d_{ij}^{-\alpha}$ with $\alpha > 2$ and ignore *near field effects of electromagnetic propagation*.

Theorem 4: The achievable per-session multicast throughput for *random extended networks* is of order

$$\begin{cases} \Omega(\frac{1}{\sqrt{n_d n}}) & \text{when } n_d : [1, \frac{n}{(\log n)^3}] \\ \Omega(\frac{1}{n_d(\log n)^{\frac{3}{2}}}) & \text{when } n_d : [\frac{n}{(\log n)^3}, \frac{n}{(\log n)^2}] \\ \Omega(\frac{1}{\sqrt{nn_d\log n}}) & \text{when } n_d : [\frac{n}{(\log n)^2}, \frac{n}{\log n}] \\ \Omega(\frac{1}{n}) & \text{when } n_d : [\frac{n}{\log n}, n] \end{cases} \quad (8)$$

Combining with the upper bound proposed in [23] (described in Lemma 22), we obtain

Theorem 5: The per-session multicast capacity for *random dense networks* is

$$\begin{cases} \Theta(\frac{1}{\sqrt{n_d n}}) & \text{when } n_d : [1, \frac{n}{(\log n)^3}] \\ \Theta(\frac{1}{n}) & \text{when } n_d : [\frac{n}{\log n}, n] \end{cases} \quad (9)$$

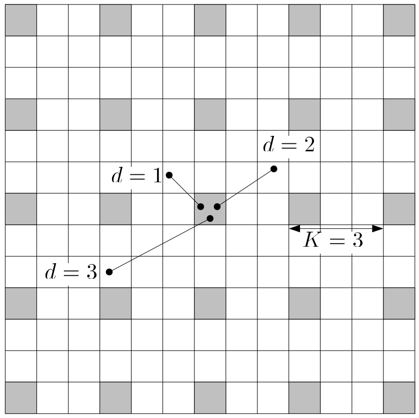


Fig. 1. Transmission scheduling for *first-class highway phase*. Gray squares can be scheduled simultaneously. Around each gray square there is a silence region of squares in which nodes are prohibited to transmit in a given time slot. The *layer-distance* d represents that the receiver locates on the cell of the $d\mathrm{th}$ layer from the transmitting cell.

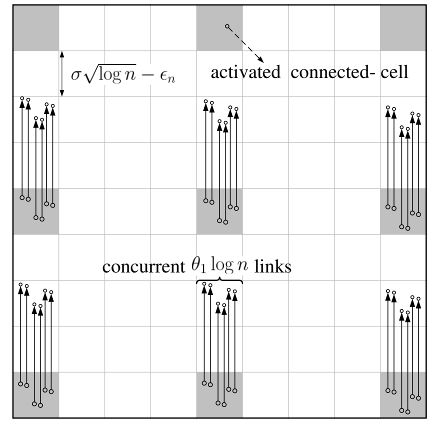


Fig. 2. Transmission scheduling for *second-class highway phase*. Gray *connected-cells* can be scheduled simultaneously. In any time slot, there are $\Theta(\log n)$ concurrent links initiated from every activated *connected-cell*.

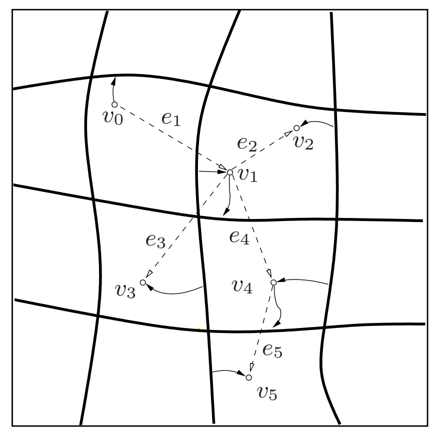


Fig. 3. Overview of Multicast Routing. The EMST is composed by $e_1 \sim e_5$ that are described as dashed lines. The bold solid curves represent the *first-class highways*. v_i drains the packet into the specific *first-class highway* via the *second-class highway* described by the thin solid curves with arrows.

Similarly, there is a gap between the upper bound and lower bound when $n_d : [\frac{n}{(\log n)^3}, \frac{n}{\log n}]$. It is a challenging issue to close the gap by presenting possibly new tight upper bound and designing algorithms to achieve it.

IV. Lower Bound for Random Extended Networks

In this section, we focus on the lower bound of the multicast capacity. We first give a brief overview of our multicast scheme. Subsequently, we give the detailed analysis on the achievable throughput of the multicast scheme.

A. Overview of the multicast strategy

The strategy is based on multihop transmission, pairwise coding and decoding at each hop, and a TDMA scheme. The routing is a hierarchical structure that consists of two general phases: *First-class Phase* and *Second-class Phase*. The former is that the packet is transmitted along a multi-hop path, with a constant distance of single-hop, called *First-class highway* along which the constant rate can be achieved. The latter is that the packet is transmitted along a multi-hop path, with the distance of single-hop of order $\Theta(\sqrt{\log n})$, called *Second-class highway* along which the rate of order $\Omega((\log n)^{-\frac{\alpha}{2}})$ can be achieved. For the two phases, we design independently their transmission scheduling scheme, and we call them *first-class transmission scheduling* and *second-class transmission scheduling*.

First-Class Transmission Scheduling: We use the similar TDMA scheme as [17], [24] and [2] to schedule transmissions in order to limit the number of simultaneous transmissions, which in turn limits the interference. In particular, we divide the region into cells with area $\Theta(1)$, called *percolation-cells*, and divide the time into a sequence of K^2 ($K \geq 3$ and $K \geq 2d+1$) successive slots, see in Fig.1. Each *percolation-cell* is activated during one out of K^2 slots, see Fig.1. Using this TDMA scheme, the rate between any adjacent cells ($d = 1$) can achieve the rate of constant order. (See the detail in [17]). Notice that for the transmission scheduling for *First-class highway phase*, we only use the specific case as $d = 1$ and $K = 3$ of the method in [17].

Second-Class Transmission Scheduling: Unlike the *first-class transmission scheduling*, we divide the region into square cells with area $\Theta(\log n)$ instead of $\Theta(1)$, and call these cells *connected-cells*. We use 16-TDMA scheme to schedule the transmissions in *second-class phase*. Each *connected-cell* is activated during 1 out of 16 slots, see Fig.2. An important technique is the *parallel scheduling*: In any time slot, for each activated *connected-cell*, we can schedule $\Omega(\log n)$ links that initiates from the *connected-cell*, and we guarantee that every links can achieve the rate of $\Omega((\log n)^{-\frac{\alpha}{2}})$. The result is proved in Lemma 7. Indeed, if we adopted the same transmission schedule as the *first-class transmission scheduling*, *i.e.*, scheduled the *percolation-cells*, it can be only guaranteed that each *second-class highway* can sustain a rate of order $\Omega((\log n)^{-1-\frac{\alpha}{2}})$.

Multicast Routing Scheme: The multicast routing is based on multihop scheme and uses the formation of paths percolation across the network. As shown in [17], we construct the *highways* , which is named *First-class highways* in our paper,

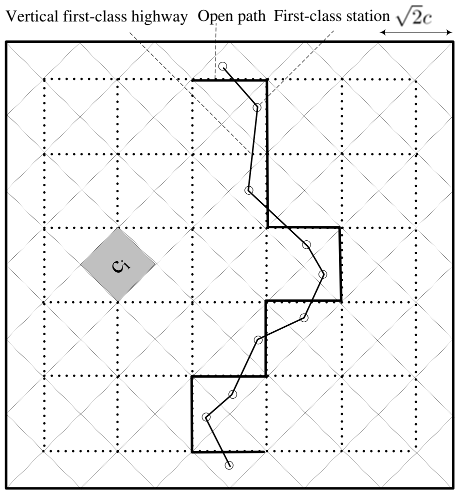


Fig. 4. Bond percolation model. The bold polygonal line represents a *open path* consisting of open edges. A *vertical first-class highway* is described as a polygonal line whose inflexions are called *first-class stations*.

as the backbone of the network. Furthermore, we construct the *Second-class highways*. For each multicast session $\mathcal{M}_k$, we firstly construct the Euclidean minimum spanning tree (EMST) based on the set $U_k = \{v_{k_0}\} \cup \{v_{k_1}, v_{k_2}, \cdots, v_{k_{n_d}}\}$, where v_{k_0} is the source of the multicast session, and $\{v_{k_1}, v_{k_2}, \cdots, v_{k_{n_d}}\}$ is the set of the destinations of the multicast session.

The routing between two vertexes of a link in the EMST, called *communication-pairs*, can be divided into four consecutive phases ([17]). In the first phase, nodes drain the packet into the *horizontal first-class highway(HFH)* via a specific *vertical second-class highway(VSH)*. In the second phase, the packet is carried along the *horizontal first-class highway*. node. Similar to the second phase, the packet is carried along a specific *vertical first-class highway(VFH)* in the third phase. At last, the packet is delivered to the destination node along a specific *horizontal second-class highway (HSH)*. In Fig.6), we shows the schematic representation of the routing between two nodes. For all links of the EMST, we construct the routing using the above scheme, eliminate the unnecessary circles, and merge the same edges. Finally, we obtain the multicast routing graph, see Fig. 3.

B. Multicast throughput of extended networks

Given the construction outlined above, and let all nodes transmit with the constant power P as [17], we might expect the longer hops in the first and last phases of the scheme to have a lower bit-rate, due to the higher power loss along longer distance. However, we need take into account other components that influence the bit-rate, namely, interference, and relay of information from other nodes. We show that when all these components are accounted for, as n_d and n_s are changing, the bottlenecks switch from highway phase to deliver phase (and draining phase), it is different compared with the scenario in [17].

1) First-Class Highway (FHs): We introduce the construction of the *First-Class Highway (FHs)* based on Percolation Theory, and give the result of the density of FHs. Finally, we define the notations for FHs.

Construction of First-Class Highway: We partition the region $\mathcal{A}_n$ into subsquares with constant side length c, as depicted in Fig.4. We call these cells *percolation-cells*. To simplify the description, we call the "*percolation-cell*" as "cell" unless we specialize. Then there are m^2 subsquares, where $m = \lceil \sqrt{n}/\sqrt{2}c \rceil$ (we can adjust the value of c such that $\sqrt{n}/\sqrt{2}c$ is an integer). Denote the grid graph produced by inclined black dotted lines as $\mathcal{C}_n$, see Fig.4 for illustration. Let $N(c_i)$ denote the number of Poisson points inside cell c_i. Thus $N(c_i)$ is a Poisson random variable with parameter c^2, and for all i, the probability that a square c_i contains at least one Poisson point ($N(c_i) \geq 1$) is $p \equiv 1 - e^{-c^2}$. We say a square is open if it contains at least one point, and closed otherwise. Then any square is open with probability p, independently from each other.

We map this model into a discrete *edge-percolation model* on the square grid. We draw an horizontal edge across half of the squares, and a vertical edge across the others, as shown in Fig.4. In this way we obtain a grid graph $\mathcal{D}_n$, described as the dotted grid with horizontal and vertical edges in Fig.4. We say a given edge $\hbar$ in $\mathcal{D}_n$ is open if the inclined subquare in $\mathcal{C}_n$, crossed by $\hbar$, is open, and call a path comprised of edges of subsquares in $\mathcal{D}_n$ open if it contains only open edges. Based on a open crossing path connecting the left side of $\mathcal{A}_n$ with its right side (or connecting the up side of $\mathcal{A}_n$ with its bottom side), depicted in Fig.4, Choosing a node from each cell in $\mathcal{C}_n$ corresponding to the open edges of the open path and connecting those nodes, we finally obtain a routing crossing path. We call those nodes *stations* and call those routing crossing paths *First-Class Highways (FHs)*. It is obvious that the distance between any two adjacent nodes in the routing crossing path is at most $2\sqrt{2}c$.

Rate of First-Class Highway: For the rate along First-Class Highways (FHs) including HFHs and VFHs, we first give some definition.

For a grid square like Fig.1, we denote any cell by a 2-dimension coordinates (x, y), where x and y are the row and column indexes of the cell respectively.

Definition 4 (Layer-Distance): We say the layer-distance between the subsquare (x_1, y_1) and (x_2, y_2) is d when

$$d = \max\{|x_1 - x_2|, |y_1 - y_2|\}$$

Recall Theorem 3 proposed in [17], as a specific case (for $\gamma = 0$) of that theorem 3, we have

Lemma 3: For any integer d, there exists an constant $R(d)$, such that in each *percolation-cell* c_i there is a node that can transmit *w.h.p.*at rate $R(d)$ to any destination located in the cell within *layer-distance* d. Furthermore, as $d \to \infty$, we have $R(d) = \Omega(d^{-\alpha-2})$.

Since the Euclidean distances between any two adjacent nodes on FHs are at most $2\sqrt{2}c$, *i.e.*, $d = 1$, we have the following result as a straightforward corollary of lemma 3.

Corollary 1: The rate along the *First-Class Highways* can achieve the order of $\Omega(1)$.

Density of First-Class Highways: Next we consider the density of *First-Class Highways*, that is, the number of highways in unit area. For a given $\kappa > 0$, we partition the grid graph $\mathcal{C}_n$ into horizontal (vertical) rectangle slabs with the horizontal (vertical) width of m subsquares and the vertical (horizontal) width of $\kappa \log m - \epsilon_m$ subsquares, denoted as R_i^h (R_i^v). Note that we can choose ε_m as the smallest value such that the number of rectangle slabs $m/(\kappa \log m - \epsilon_m)$ in the partition is an integer. It is easy to see that $\epsilon_m = o(1)$ as $m \to \infty$ [17]. Denote the number of disjoint open paths within slab R_i^h (R_i^v), *i.e.*, the number of disjoint horizontal (vertical) first-class highways, as N_i^h (N_i^v). Let $N^h = \min_i N_i^h$, $N^v = \min_i N_i^v$. We have

Lemma 4: ([17]) For all κ and $p \in (5/6, 1)$ satisfying $2 + \kappa \log(6(1-p)) < 0$, there exists a $\delta(\kappa, p)$ such that

$$\lim_{m\to\infty} \Pr(N^h \geq \delta \log m) = 1; \lim_{m\to\infty} \Pr(N^v \geq \delta \log m) = 1.$$

According to the constraints of Lemma 4, we have

$$1 - e^{-c^2} < 5/6 \text{ and } 2 + \kappa(\log 6 - c^2) < 0.$$

To satisfy the conditions, it holds $\kappa > 0$ and $c^2 > \log 6 + 2/\kappa$.

We scale the width of slabs as $\Theta(\log m)$ in order to ensure the slabs are wide enough to contain a whole highway. Notice that $\Theta(\log m)$ is just the threshold satisfying the condition, which can be proved straightforwardly based on the results of [17].

Notations for FHs: To simplify description, we assume that there are $\delta \log m$ horizontal (vertical) highways in each horizontal (vertical) slab, which does't decrease the derived throughput in order sense. According to lemma 4, we can subdivide every slab into $\delta \log m$ slices with width l, where $l = (\kappa \log m - \varepsilon_m)/\delta \log m$. Hence, we can define a bijective mapping from horizontal slices to *horizontal first-class highways*, denoted as $g^h : S^h \to H^h$, where S^h represents the set of all horizontal slices and H^h represents the set of all *horizontal first-class highways*. Similarly, we can define the bijective mapping from vertical slices to *vertical first-class highways*, denoted as $g^v : S^v \to H^v$, where S^v represents the set of all vertical slices and H^v represents the set of all *vertical first-class highways*. Notice that any slice can and only can project to the *first-class highway* contained by the slab that posses the slice, which ensures that the distance from any points in the slice to the corresponding highway is at most $\kappa \log m - \varepsilon_m$. Based on the two mappings, we can define two functions as $f^h : V \to H^h$ and $f^v : V \to H^v$, where V is the set of all nodes in region $\mathcal{A}_n = [0, \sqrt{n}] \times [0, \sqrt{n}]$. The two functions satisfy the condition that, for a node v and horizontal slice $s_i^h \in S^h$ (or vertical slice $s_i^v \in S^v$), if v belongs to region s_i^h (or s_i^v), then $f^h(v) = g^h(s_i^h)$ (or $f^v(v) = g^v(s_i^v)$). Furthermore, we define two function $\psi^h : H^h \to R^h$ and $\psi^v : H^v \to R^v$, where R^h (R^v) is the set of all *horizontal (vertical) slabs*. $\psi^h(\mathfrak{h}^h)$ ($\psi^v(\mathfrak{h}^v)$) represents *the horizontal (vertical) slab* containing the *horizontal (vertical) first-class highway* $\mathfrak{h}^h$ ($\mathfrak{h}^v$).

2) Second-Class Highway (SHs): We firstly propose the construction of the *second-class highway (SHs)* based on the method called *connected-cell scheduling*, and analyze the density of SHs. Finally, we define the notations of SHs.

Construction of SHs: We partition the region $\mathcal{A}_n$ into subsquares of a side length $\sigma\sqrt{\log n} - \epsilon_n$, as depicted in

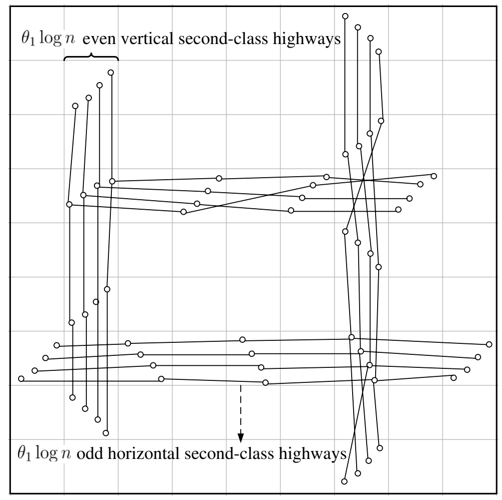


Fig. 5. Construction of *second-class highways.*

Fig.5, where $\sigma > 0$ is a constant and we choose $\epsilon_n > 0$ as the smallest value such that $\sqrt{n}/(\sigma\sqrt{\log n} - \epsilon_n)$ is an integer. It is obvious that $\epsilon_n = o(1)$. We denote these subsquares as $\bar{c}_j$ and call them *connected-cells*. Then there are $n/(\sigma\sqrt{\log n} - \epsilon_n)^2$ connected-cells. Denote the grid graph consisting of the *connected-cells* as $\mathcal{E}_n$, see Fig.5. Let $N(\bar{c}_j)$ be the number of Poisson points inside the *connected-cell* $\bar{c}_j$. Thus $N(\bar{c}_j)$ is a Poisson random variable with parameter $(\sigma\sqrt{\log n} - \epsilon_n)^2$. Furthermore, we define the uniform lower bound of $N(\bar{c}_j)$ as $N_{\bar{c}}$.

To ensure the feasibility of the method to construct SHs, we give the following lemma.

Lemma 5: For any ϱ, $2\varrho > 1 + \log \varrho$ and σ, $\sigma^2 \geq \frac{4\varrho}{(2\varrho - \log \varrho - 1)}$, each connected-cell contains w.h.p no less than $\theta_1 \log n$ nodes, where θ_1 is a constant and $\theta_1 = \frac{\sigma^2}{2\varrho}$.

Proof: Since $(\sigma\sqrt{\log n} - \epsilon_n)^2 > \frac{1}{2}\sigma^2 \log n$, as $n \to \infty$, According to Lemma 2 and Union Bounds, we have

$$\begin{aligned} \Pr(N_{\bar{c}} \leq \tfrac{\sigma^2 \cdot \log n}{2\varrho}) &\leq \tfrac{2n}{\sigma^2 \cdot \log n} \Pr(N(\bar{c}_j) \leq \tfrac{\sigma^2 \cdot \log n}{2\varrho}) \\ &\leq \tfrac{2n}{\sigma^2 \cdot \log n} \frac{n^{\frac{\sigma^2}{2\varrho}}}{n^{\frac{\sigma^2}{2}}} \cdot n^{\frac{\sigma^2 \cdot \log \varrho}{2\varrho}} \\ &= \frac{1}{\sigma^2 \cdot \log n \cdot n^{\frac{\sigma^2}{2} - 1 - \frac{(1+\log \varrho)\sigma^2}{2\varrho}}} \end{aligned} \quad (10)$$

Thus, when we choose ϱ, $2\varrho > 1 + \log \varrho$ and σ, $\sigma^2 \geq \frac{4\varrho}{(2\varrho - \log \varrho - 1)}$, it holds that $\Pr(N_{\bar{c}} \leq \frac{\sigma^2 \cdot \log n}{2\varrho}) \to 0$, when $n \to \infty$. ■

Consider the grid graph $\mathcal{E}_n$, we call each row (column) of $\mathcal{E}_n$ *Row-Slab*(*Column-Slab*), denoted as $\bar{R}_i^h$ ($\bar{R}_i^v$) To simplify the description, in each *Row-Slab*(*Column-Slab*), we give a order index with a natural number for each *connected-cells*. Without the generality, we assign the order index to each *connected-cell* in a *Row-Slab*(*Column-Slab*) by the sequence of "top to bottom"("left to right"), see Fig. 5 for illustration. We call the *connected-cells* with odd (even)indexes *odd-order (even-order) connected-cells*.

According to Lemma 5, each *connected-cell* contains *w.h.p.*at least $\theta_1 \cdot \log n$ nodes, then we can construct the *horizontal second-class highways* in $\bar{R}_i^h$ using the following operations: Firstly, for the $\sqrt{n}/(\sigma\sqrt{\log n} - \epsilon_n)$ *connected-cells* in $\bar{R}_i^h$, choose a node from each *connected-cell*. Secondly, connect the nodes in *odd-order connected-cells* to derive a path called *odd horizontal second-class highway*(OHSH), connect the nodes, named *second-class stations*, in *even-order connected-cells* to derive a path called *even horizontal second-class highway*(EHSH). In a similar way, we can construct the *odd vertical second-class highway*(OVSH) and *even vertical second-class highway* (EVSH).

Density of SHs: We say two second-class highways are disjoint if there is common node contained in them. Next, we consider the density of *second-class highways*, that is, the number of the disjoint *second-class highways* in unit area. Denote the number of disjoint *second-class highways* within a *row-slab* (*column-slab*) $\bar{R}_i^h$ ($\bar{R}_i^v$) as $\bar{N}_i^h$ ($\bar{N}_i^v$). Let $\bar{N}^h = \inf \bar{N}_i^h$, $\bar{N}^v = \inf \bar{N}_i^v$. According to Lemma 5, each connected-cell contains at least $\theta_1 \log n$ nodes. Since the *second-class highways* include *odd second-class highways* and *even second-class highways*, the following lemma is a straightforward result.

Lemma 6: For any ϱ, $2\varrho > 1 + \log \varrho$ and σ, $\sigma^2 \geq \frac{4\varrho}{(2\varrho - \log \varrho - 1)}$, there exists a $\theta_1 = \frac{\sigma^2}{2\varrho}$ such that

$$\lim_{n\to\infty} \Pr(\bar{N}^h \geq 2\theta_1 \log n) = 1; \lim_{n\to\infty} \Pr(\bar{N}^v \geq 2\theta_1 \log n) = 1.$$

Parallel Scheduling of SHs: We adopt 16-TDMA scheme to schedule the transmissions in *second-class highways*. The main trick here is: Instead of scheduling only one link in each activated cell in each time slot, we consider scheduling a set of links which initiate from the same connected-cell together. Specially, after we partition the deployment region into connected-cells, we further divide time into a sequence of 16 successive slots. In each time slot, we consider disjoint sets of connected-cells that are allowed to be activated simultaneously, as depicted in Fig. 2. Notice that if a connected-cell is activated, $2\theta_1 \log n$ links that initiates from the *connected-cell* can transmit simultaneously. Obviously, compared with only scheduling one link in each connected cell, this modification increase the total bit-rate by order of $\log n$ if the total interference is still bounded. So can we prove that the total interference is still bounded? Fortunately, the proof of the following lemma give us a positive answer. We further prove that, the data rate of any *second-class highway* can achieve the order of $\Omega((\log n)^{-\frac{\alpha}{2}})$.

Rate of SHs: It is easy to see that the distances of every hop in the *second-class highways* are at most $\sqrt{10}\cdot(\sigma\sqrt{\log n} - \epsilon_n)$ and at least $\sigma\sqrt{\log n} - \epsilon_n$. For the rate along the *second-class highway*, we give the following lemma.

Lemma 7: Along each *second-class highway*, the achievable rate is of order $\Omega(1/(\log n)^{-\frac{\alpha}{2}})$.

Proof: Considering any link in the *second-class highway* in any time slot, since the length of the link is at least $\sigma\sqrt{\log n} - \epsilon_n$, we obtain the sum of interferences to the receivers as:

$$\begin{aligned} I(n) \leq\ & P\cdot(\theta_1 \log n - 1)\cdot l(\sigma\sqrt{\log n} - \epsilon_n) \\ & + \sum_{i=1}^{n} 8iP(\theta_1 \log n)\cdot l((4i-3)\cdot(\sigma\sqrt{\log n} - \epsilon_n)) \\ \leq\ & P\cdot 2^{\alpha}\theta_1\sigma^{-\alpha}(\log n)^{1-\frac{\alpha}{2}}\cdot\left(1 + \lim_{n\to\infty}\sum_{i=1}^{n}\frac{8i}{(4i-3)^{\alpha}}\right) \end{aligned}$$

Algorithm 1 Construction of EMST

Input: The set of nodes U_k

Output: $EMST(U_k)$.

1: In the initial state, all nodes of U_k are isolated, then there are $n_d + 1$ connected components.

2: For $i = 1 : n_d$

(1) Partition the deployment region $\mathcal{A}_n = [0, \sqrt{n}] \times [0, \sqrt{n}]$ into at most $n_d + 1 - i$ square cells, each with side length $\sqrt{n}/\lfloor\sqrt{n_d + 1 - i}\rfloor$;

(2) Find a cell that contains two nodes of U_k that are from two different connected components. By connecting the pair of nodes, we merge the two connected components.

The latest limitation is obviously converges to a constant when $\alpha > 2$. On the other hand, since the length of the link is at most $\sqrt{10}(\sigma\sqrt{\log n} - \epsilon_n)$, the signal $S(n)$ at the receiver can be bounded as

$$S(n) \geq Pl(\sqrt{10}(\sigma\sqrt{\log n} - \epsilon_n)) \geq P\cdot 10^{-\frac{\alpha}{2}}\sigma^{-\alpha}(\log n)^{-\frac{\alpha}{2}}$$

Then, by Equation (1) and the 16-TDMA scheme, we have that the achievable rate along the *second-class highway* is

$$\bar{R}(n) = \frac{1}{16}\log\left(1 + \frac{S(n)}{N_0 + I(n)}\right)$$

By $\alpha > 2$ and $N_0 > 0$, we have $\frac{S(n)}{N_0 + I(n)} \to 0$, as $n \to \infty$. Hence, $\bar{R}(n) = \Omega((\log n)^{-\frac{\alpha}{2}})$. ■

Notations for SHs: To simplify description, we assume that there are $2\theta_1 \log n$ *horizontal (vertical) second-class highways* in each $\bar{R}_j^h$ ($\bar{R}_j^v$) that include $\theta_1 \log n$ *odd horizontal (vertical) second-class highways* and $\theta_1 \log n$ *even horizontal (vertical) second-class highways*, which does't decrease the derived throughput in order sense. According to Lemma 6, we can subdivide every *row-slab* (*column-slab*) $\bar{R}_j^h$ ($\bar{R}_j^v$) into $2\theta_1 \log n$ slices with width $\bar{l}$ and length $\sqrt{n}$, where $\bar{l} = \sigma/(2\theta_1\sqrt{\log n})$. We call these derived slices as *row-slices* (*column-slices*). Hence, we can define a bijective mapping from *row slices* to *horizontal second-class highways*, denoted as $\bar{g}^h : \bar{S}^h \to \bar{H}^h$, where $\bar{S}^h$ represents the set of all *row slices* and $\bar{H}^h$ represents the set of all *horizontal second-class highways*. Similarly, we can define the bijective mapping from *column slices* to *vertical second-class highways*, denoted as $\bar{g}^v : \bar{S}^v \to \bar{H}^v$, where $\bar{S}^v$ represents the set of all *vertical slices* and $\bar{H}^v$ represents the set of all *vertical second-class highways*. Notice that any slice can and only can project to the highway contained by the slab that passes the slice, which ensures that the distance from any points in the slice to the corresponding highway is at most $\sigma\sqrt{\log n}$. Based on the two mappings, we can define two functions as $\bar{f}^h : V \to \bar{H}^h$ and $\bar{f}^v : V \to \bar{H}^v$, where V is the set of all nodes in region $\mathcal{A}_n = [0, \sqrt{n}] \times [0, \sqrt{n}]$. The two functions satisfy the condition that, for a node v and horizontal slice $\bar{s}_i^h \in \bar{S}^h$ (or vertical slice $\bar{s}_i^v \in \bar{S}^v$), if v belongs to region $\bar{s}_i^h$ (or $\bar{s}_i^v$), then $\bar{f}^h(v) = \bar{g}^h(\bar{s}_i^h)$ (or $\bar{f}^v(v) = \bar{g}^v(\bar{s}_i^v)$).

3) Multicast Routing Scheme: Considering the multicast session $\mathcal{M}_k$, $k = 1, 2, \cdots n_s$, we denote the set of nodes as $U_k = \{v_{k_0}\} \cup \{v_{k_1}, v_{k_2}, \cdots v_{k_{n_d}}\}$, where v_{k_0} is the source node and $\{v_{k_1}, v_{k_2}, \cdots v_{k_{n_d}}\}$ is the set of destinations. We firstly

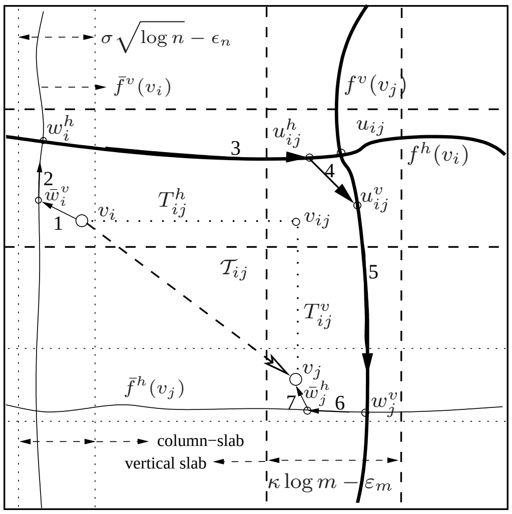


Fig. 6. Routing between the communication-pairs. Two bold solid curves represent the *first-class highways* $f^h(v_i)$ and $f^v(v_j)$. Two thin solid curves represent the *second-class highways* $\bar{f}^v(v_i)$ and $\bar{f}^h(v_j)$. The path consisting of the solid curves with arrows represents the real routing of $v_i \to v_j$. T_{ij}^h and T_{ij}^v denote the horizontal and vertical edges of the right-angled triangle $\mathcal{T}_{ij}(v_i v_{ij} v_j)$, they just serve as the auxiliaries for the proof of lemma 9.

construct the EMST spanning the set of nodes U_k using the similar method in [11](Algorithm 1). Based on $EMST(U_k)$, we propose Algorithm 2 to construct the multicast routing graph $\mathcal{G}(U_k)$. To simplify the notation, we denote the multicast session $\mathcal{M}_k$ as $U_k = \{v_0\} \cup \{v_1, v_2, \cdots v_{n_d}\}$ and reaffirm that HSH (VSH) and HFH (VFH) are respectively the abbreviations of *Horizontal (Vertical) Second-Class Highway* and *Horizontal (Vertical) First-Class Highway* in the following Algorithm 2.

During the realization of the routing between each communication-pairs (denoted as $v_i \to v_j$) in EMST, there are seven phases that v_i transmits the packet to v_j (Step 1 of Algorithm 2):

1) *Second-Class Draining Phase*-(SDr-Phase)
In this phase, v_i drains the packet to the SVH $\bar{f}^v(v_i)$ by a single hop with distance of $\Theta(\sqrt{\log n})$.

2) *Vertical Second-Class Highway Phase* -(VSH-Phase)
In this phase, packet is transmitted to the HFH $f^h(v_i)$ along the SVH $\bar{f}^v(v_i)$.

3) *Horizontal First-Class Highway Phase* -(HFH-Phase)
In this phase, the packet is transmitted along the HFH $f^h(v_i)$.

4) *First-Class Turning Phase* -(FTu-Phase)
In this phase, the packet is carried from the HFH $f^h(v_i)$ to the VFH $f^v(v_j)$ by a single short hop with a constant distance.

5) *Vertical First-Class Highway Phase* -(VFH-Phase)
In this phase, the packet is transmitted along the VFH $f^v(v_j)$;

6) *Horizontal Second-Class Highway Phase* -(HSH-Phase)
In this phase, the packet is delivered into the HSH $\bar{f}^h(v_j)$.

7) *Second-Class Deliver Phase* -(SDe-Phase)
In this phase, the packet is delivered from the HSH $\bar{f}^h(v_j)$ to v_j by a single hop with distance of $\Theta(\sqrt{\log n})$.

Intuitively, when the bottleneck of the whole routing locates in VSH-Phase and HSH-Phase, the not poorer performance of throughput can be derived by the scheme only based on the second-class highways system. At this point, we need only adopt the *second-class transmission scheduling*. The multicast routing scheme, denoted as $\digamma'$, is described in Algorithm 3.

Algorithm 2 Multicast Routing Scheme $\digamma$

Input: The multicast session $\mathcal{M}_k$ and $EMST(U_k)$.
Output: A multicast routing graph $\mathcal{G}(U_k)$.

1: For each link $v_i \to v_j$ of $EMST(U_k)$, implement the following sub-steps to realize the routing $v_i \to v_j$.
(1) By a single hop, v_i drains the packet into the VSH $\bar{f}^v(v_i)$ via $\bar{w}_i^v$ that is the closest *second-class station* in $\bar{f}^v(v_i)$ with the distance of $|v_i\bar{w}_i^v| = \Theta(\sqrt{\log n})$to v_i.
(2) Along the VSH $\bar{f}^v(v_i)$, the packet is drained to the HFH $f^h(v_i)$ via w_i^h that is the closest *first-class station* (Fig. 4) to the intersection point of $\bar{f}^v(v_i)$ and $f^h(v_i)$.
(3) The packet is transmitted along $f^h(v_i)$ to u_{ij}^h that is the closest station on $f^h(v_i)$ to u_{ij}, where u_{ij} denotes the intersection point of $f^h(v_i)$ and $f^v(v_j)$;
(4) By a single short hop, the packet is transported from u_{ij}^h to u_{ij}^v that is the closest station on $f^v(v_j)$ to u_{ij}.
(5) The packet is transmitted along $f^h(v_i)$ to w_j^v that is the closest *first-class station* to the intersection point of the HSH $\bar{f}^h(v_j)$ and the VFH $f^v(v_j)$.
(6) Along the HSH $\bar{f}^h(v_j)$, the packet is delivered to $\bar{w}_j^h$ that is the closest *second-class station* in $\bar{f}^h(v_j)$ with the distance of $|v_j\bar{w}_j^h| = \Theta(\sqrt{\log n})$ to v_j.
(7) By a single hop, $\bar{w}_j^h$ delivers the packet to v_j.

2: Consider the next link of $EMST(U_k)$ (go to step 1), until all the links in $EMST(U_k)$ are checked.

3: Considering the resulted routing graph, we merge the same edges (hops), remove those circles which have no impact on the connectivity of the communications for $EMST(U_k)$. Finally, we obtain the multicast routing graph $\mathcal{G}(U_k)$.

C. Analysis of multicast throughput

For the seven phases in Algorithm 2, we successively analyze the achievable total rate and relay burden of each *percolation-cell* in every phase. In Phase 3 and Phase 5 (HFH-Phase and VFH-Phase), the packet is both transmitted along the *first-class highways* (FHs) . From Lemma 9, we know a constant rate can be achieved along FHs. For Phase 4 (FTu-Phase), the analysis of rate and relay burden is similar to that of Phase 3 and Phase 5, because the single hop in Phase 4 has no difference from the hops in the HFH and VFH. Thus, we do not individually analyze Phase 3, and call generally Phases 3,4 and 5 as *first-class highway phase* (FH-Phase). For these three phases, we state the Lemma 9. Before stating the lemma, we recall a result proposed in [11] and [24].

Lemma 8: For EMST produced by Algorithm 1, we have

$$\|EMST(U_k)\| \le 2\sqrt{2}\sqrt{n_d}\sqrt{n}$$

where $k = 1, 2, \cdots, n_s$ and $\|EMST(U_k)\|$ is the total distance of all links in $EMST(U_k)$.

To facilitate the expression, we define a sequence of directed edges sets $\Pi_k = \{e_{ij} |\ e_{ij} = v_i v_j \in EMST(U_k)\}$, where

TABLE I
DEFINED EVENTS AND RANDOM VARIABLES

Events	Events Definition	Variables	Variables Definition
$A(k,t)$	During FH-Phase, multicast session $\mathcal{M}_k$ passes through a given *first-class station* v_t^*.	$I(A)$	The indicator function of the event A. Namely, it takes value 1 if A happens and 0 otherwise.
$A_{ij}^h(k,t)$	During HFH-Phase, the routing from v_i to v_j passes by a given *first-class station* v_t^*.	X_t	The number of multicast sessions routed through a given *first- class station* v_t^* during *first-class highway phase*.
$A_{ij}^v(k,t)$	During VFH-Phase, the routing from v_i to v_j passes by v_t^*.	X	The uniform upper bound of X_t.
$B(k,t)$	a given poisson node is in a region with area $\min\{n, \\|\mathcal{S}(k,t)\\|\}$ (see Equation (17)).	Y_t	$\sum_{k=1}^{n_s} I(B(k,t))$.
$\bar{A}^h(k,t)$	During HSH-Phase, multicast session $\mathcal{M}_k$ passes through a given *second-class station* $\bar{v}_t^*$.	$\tilde{Y}_t$	The upper bound of Y_t.
$\bar{A}^v(k,t)$	During VSH-Phase, multicast session $\mathcal{M}_k$ passes through a given *second-class station*,tab-events-variables $\bar{v}_t^*$.	$\bar{X}_t^v$	The number of multicast sessions routed through a given *second- class station* $\bar{v}_t^*$ during VSH Phase.
$\bar{A}_i^h(k,t)$	During HSH-Phase, the routing of the link in $EMST(U_k)$ initiated from v_i passes through $\bar{v}_t^*$.	$\bar{X}^h$	The uniform upper bound of $\bar{X}_t^h$.
$\bar{A}_j^v(k,t)$	During VSH-Phase, the routing of the link in $EMST(U_k)$ intended to v_j passes through $\bar{v}_t^*$.	$\bar{X}^v$	The uniform upper bound of $\bar{X}_t^v$.

Algorithm 3 Multicast Routing Scheme $\digamma'$

Input: The multicast session $\mathcal{M}_k$ and $EMST(U_k)$.
Output: A multicast routing graph $\bar{\mathcal{G}}(U_k)$.

1: For each link $v_i \to v_j$ of $EMST(U_k)$, implement the following sub-steps to realize the routing $v_i \to v_j$.
(1) By a single hop, v_i drains the packet into the VSH $\bar{f}^v(v_i)$ via $\bar{w}_i^v$ that is the closest *second-class station* to v_i on $\bar{f}^v(v_i)$ with $|v_i\bar{w}_i^v| = \Theta(\sqrt{\log n})$.
(2) The packet is transmitted along $\bar{f}^v(v_i)$ to $\bar{u}_{ij}^v$ that is the closest second-class station on $\bar{f}^v(v_i)$ to $\bar{u}_{ij}$, where $\bar{u}_{ij}$ denotes the intersection point of $\bar{f}^v(v_i)$ and $\bar{f}^h(v_j)$;
(3) By a single hop, the packet is transported from $\bar{u}_{ij}^v$ to $\bar{u}_{ij}^h$ that is the closest station on $\bar{f}^h(v_j)$ to $\bar{u}_{ij}$.
(4)The packet is transmitted along HSH $\bar{f}^h(v_j)$ to $\bar{w}_j^h$ that is the closest *second-class station* to v_j on $\bar{f}^h(v_j)$ with $|v_j\bar{w}_j^h| = \Theta(\sqrt{\log n})$.
(5) By a single hop, $\bar{w}_j^h$ delivers the packet to v_j.
2: Consider the next link of $EMST(U_k)$ (go to step 1), until all the links in $EMST(U_k)$ are checked.
3: Use the same method as step 3 of $\digamma$ to obtain the multicast routing graph $\bar{\mathcal{G}}(U_k)$.

$k = 1, 2, \cdots, n_s$.

Lemma 9: Along the *first-class highway*, packets can be relayed at rate *w.h.p.*,

$$\begin{cases} \Omega(n/(n_s\Gamma(n,n_d))) & \text{when } n_s\Gamma(n,n_d)/n = \Omega(\log n) \\ \Omega(1/\log n) & \text{when } n_s\Gamma(n,n_d)/n = O(\log n) \end{cases} \tag{11}$$

where $\Gamma(n,n_d) = \min_{order}\{\sqrt{nn_d} + n_d\log n,\ n\}$.

Proof: According to Corollary 1, the rate along the highways can be achieved of a constant order. So we need only to prove the maximum relay burden of the *first-class stations* on *first-class highways* is *w.h.p.*,

$$\begin{cases} O(n_s\Gamma(n,n_d)/n) & \text{when } n_s\Gamma(n,n_d)/n = \Omega(\log n) \\ O(\log n) & \text{when } n_s\Gamma(n,n_d)/n = O(\log n) \end{cases} \tag{12}$$

with $\Gamma(n,n_d) = \min_{order}\{\sqrt{nn_d} + n_d\log n,\ n\}$.

Given a *first-class station* v_t^* passed by a *horizontal first-class highway* $\mathfrak{f}^{\mathfrak{h}}$ or a vertical highway $\mathfrak{f}^{\mathfrak{v}}$, we define a random

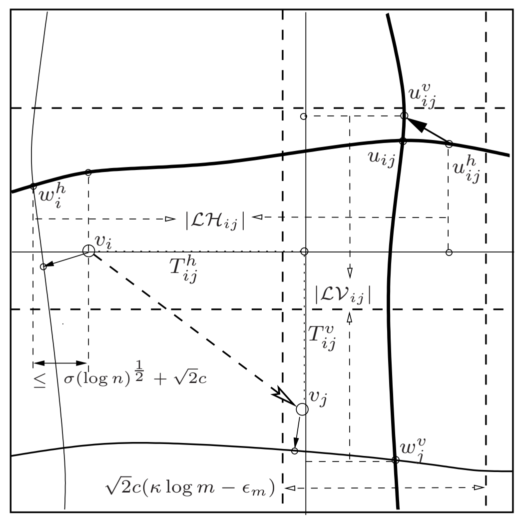


Fig. 7. The *horizontal span distance* and *vertical span distance* of $v_i \to v_j$ are denoted as $|\mathcal{LH}_{ij}|$ and $|\mathcal{LV}_{ij}|$ respectively, where $|\mathcal{LH}_{ij}| = |w_i^h u_{ij}^h|_{\text{h}}$ and $|\mathcal{LV}_{ij}| = |w_j^v u_{ij}^v|_{\text{v}}$. Here $|w_i^h v_i|_{\text{h}}, |w_j^v v_j|_{\text{v}} \leq \sigma\sqrt{\log n} + \sqrt{2}c$.

variable X_t as Table I. Note that we finally consider the uniform upper bound of X_t for stations, denoted as X.

1. Firstly, we define an event $A(k,t)$, and for any edge $e_{ij} \in \Pi_k$, define event $A_{ij}^h(k,t)$ and $A_{ij}^v(k,t)$ as in Table I. Obviously, we have

$$\Pr(A(k,t)) = \Pr(\bigcup_{e_{ij}\in\Pi_k}(A_{ij}^h(k,t) \cup A_{ij}^v(k,t)))$$

From union bounds, we obtain

$$\Pr(A(k,t)) \leq \sum_{e_{ij}\in\Pi_k}(\Pr(A_{ij}^h(k,t)) + \Pr(A_{ij}^v(k,t))) \tag{13}$$

2. Secondly, we consider the bounds of $\Pr(A_{ij}^h(k,t))$ and $\Pr(A_{ij}^v(k,t))$.

Considering the *first-class highway phase* (FH-Phase) of routing for $v_i \to v_j$, we denote the maximum horizontal distance (called *horizontal span distance*) that data are transmitted, *i.e.*, $|w_i^h u_{ij}^h|_{\text{h}}$, as $|\mathcal{LH}_{ij}|$. Similarly, we define $|w_j^v u_{ij}^v|_{\text{v}}$ the $|\mathcal{LV}_{ij}|$, where $|\cdot|_{\text{h}}$ and $|\cdot|_{\text{v}}$ represent respectively the horizontal and vertical Euclid distance that are defined as:

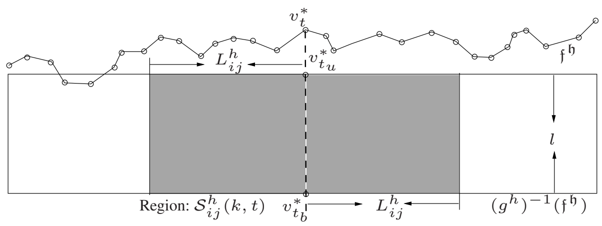


Fig. 8. The slice $(g^h)^{-1}(\mathfrak{f}^{\mathfrak{h}})$ and region $\mathcal{S}_{ij}^h(k,t)$. Here $(g^h)^{-1}(\mathfrak{f}^{\mathfrak{h}})$ is the horizontal slice that corresponds to the *first-class highway* $\mathfrak{f}^{\mathfrak{h}}$. $\mathcal{S}_{ij}^h(k,t)$ is the shadowed region with width of l and length of $2L_{ij}^h$, where $L_{ij}^h = |T_{ij}^h| + \sqrt{2}c(\kappa\log m - \varepsilon_m) + \sigma\sqrt{\log n} + \sqrt{2}c$. Obviously, $|\mathcal{LH}_{ij}| \le L_{ij}^h$.

In the 2-dimension plane, for any two nodes u_1 and u_2 denoting by the coordinates (x_1, y_1) and (x_2, y_2) respectively, $|u_1u_2|_{\mathrm{h}} = |x_1 - x_2|$ and $|u_1u_2|_{\mathrm{v}} = |y_1 - y_2|$.

Recall that the station v_t^* is passed by the horizontal highway $\mathfrak{f}^{\mathfrak{h}}$ or the vertical highway $\mathfrak{f}^{\mathfrak{v}}$.

Project the station v_t^* vertically on the two horizontal boundaries of the slice $(g^h)^{-1}(\mathfrak{f}^{\mathfrak{h}})$, and we obtain two projections $\bar{v}_{t_u}^*$ and $\bar{v}_{t_b}^*$, see in Fig.8. Let

$$L_{ij}^h = |T_{ij}^h| + \sqrt{2}c(\kappa\log m - \varepsilon_m) + \sigma\sqrt{\log n} + \sqrt{2}c \quad (14)$$

where $|\cdot|$ represents the Euclid length of a line segment or Euclid distance between two nodes. Then, From the illustration in Fig. 7, It is obvious that $|\mathcal{LH}_{ij}| \le L_{ij}^h$. Consider the region $\mathcal{S}_{ij}^h(k,t)$ (See in Fig.8), and since $|\mathcal{LH}_{ij}| \le L_{ij}^h$, we have

Proposition 1: If $A_{ij}^h(k,t)$ happens, then the poisson point v_i locates in the region $\mathcal{S}_{ij}^h(k,t)$.

In a similar way, based on the highway $\mathfrak{f}^{\mathfrak{v}}$ and the *first-class slice* $(g^v)^{-1}(\mathfrak{f}^{\mathfrak{v}})$, we construct the region $\mathcal{S}_{ij}^v(k,t)$, and obtain that $|\mathcal{LV}_{ij}| \le L_{ij}^v$ where

$$L_{ij}^v = |T_{ij}^v| + \sqrt{2}c(\kappa\log m - \varepsilon_m) + \sigma\sqrt{\log n} + \sqrt{2}c \quad (15)$$

hence, we have the following result,

Proposition 2: If $A_{ij}^v(k,t)$ happens, then the poisson point v_j locates in the region $\mathcal{S}_{ij}^v(k,t)$.

Define two events as $B_{ij}^h(k,t)$: A given poisson node is in the region $\mathcal{S}_{ij}^h(k,t)$(See Fig. 8 for illustration). $B_{ij}^v(k,t)$: A given poisson node is in the region $\mathcal{S}_{ij}^v(k,t)$. From proposition 1 and 2, we obtain

$$\Pr(A_{ij}^h(k,t)) \le \Pr(B_{ij}^h(k,t));\ \Pr(A_{ij}^v(k,t)) \le \Pr(B_{ij}^v(k,t)) \quad (16)$$

3. Finally, we consider the upper bounds of X_t and X. Define a region $\mathcal{S}(k,t) \subseteq \mathcal{A}_n$ with area of

$$\|\mathcal{S}(k,t)\| = \sum_{e_{ij}\in\Pi_k} (\|\mathcal{S}_{ij}^h(k,t)\| + \|\mathcal{S}_{ij}^v(k,t)\|)\} \quad (17)$$

where $\mathcal{A}_n$ is the global deployment region and $\|\mathcal{S}\|$ represents the area of the region $\mathcal{S}$.

Accordingly, we can define event $B(k,t)$: a given poisson node locates in a region with area $\min\{n, \|\mathcal{S}(k,t)\|\}$.

By the inequality (16) and the definition of $B(k,t)$, we have

$$\Pr(A(k,t)) \le \Pr(B(k,t)) \quad (18)$$

For events $A(k,t)$ and $B(k,t)$, define their indicator variables as $I(A(k,t))$ and $I(B(k,t))$, where the random variables $I(A)$ takes value 1 if A happens and 0 otherwise.

Recall the definition of X_t, it holds that

$$X_t = \sum\nolimits_{k=1}^{n_s} I(A(k,t)).$$

Let $Y_t = \sum_{k=1}^{n_s} I(B(k,t)$, By inequality (18) we have

$$\Pr(X_t \ge x) \le \Pr(Y_t \ge x), \quad \text{for any } x \ge 0 \quad (19)$$

The random variable Y_t represents the number of nodes located in the region with area $\min\{n, \|\mathcal{S}(k,t)\|\}$ according to a p.p.p of density $\frac{n_s}{n}$. So it follows Poisson distribution with the expectation of

$$\lambda = \frac{n_s}{n}\min\{n, \|\mathcal{S}(k,t)\|\}$$

Next, we consider the upper bound of $\|\mathcal{S}(k,t)\|$. Recall Equations (14) and (15) and (17), we have

$$\begin{aligned}\|\mathcal{S}(k,t)\| &= \sum\nolimits_{e_{ij}\in\Pi_k} (\|\mathcal{S}_{ij}^h(k,t)\| + \|\mathcal{S}_{ij}^v(k,t)\|)\} \\ &= \sum\nolimits_{e_{ij}\in\Pi_k} 2l(L_{ij}^h + L_{ij}^v) \\ &= 2l\sum\nolimits_{e_{ij}\in\Pi_k} (|T_{ij}^h| + |T_{ij}^v|) + \varpi\end{aligned} \quad (20)$$

where $\varpi = 4\sqrt{2}lc(\kappa\log m - \varepsilon_m + 1)n_d + 4l\sigma n_d\sqrt{\log n}$. Since $m = n/\sqrt{2}$, we have as $n \to \infty$,

$$\varpi < 4\sqrt{2}lc \cdot n_d \cdot \frac{3}{2}\kappa\log n = 3\sqrt{2}lc\kappa \cdot n_d \cdot \log n \quad (21)$$

Then, Equation (20) turns into,

$$\|\mathcal{S}(k,t)\| \le 3\sqrt{2}lc\kappa \cdot n_d \cdot \log n + 2l\sum_{e_{ij}\in\Pi_k} (|T_{ij}^h| + |T_{ij}^v|) \quad (22)$$

Since $|T_{ij}^h| + |T_{ij}^v| \le \sqrt{2}|v_iv_j|$ and by inequality (22), we have

$$\|\mathcal{S}(k,t)\| \le 3\sqrt{2}lc\kappa n_d\log n + 2\sqrt{2}l(|EMST(U_k)| \quad (23)$$

From lemma 8, we get $\|EMST(U_k)\| \le 2\sqrt{2}\sqrt{n_d}\sqrt{n}$. Thus,

$$\|\mathcal{S}(k,t)\| < 8l\sqrt{nn_d} + 3\sqrt{2}lc\kappa n_d\log n \le C_1(\sqrt{nn_d} + n_d\log n) \quad (24)$$

where $C_1 = \max\{8l, 3\sqrt{2}lc\kappa\}$ is a constant.

From the inequality (24), we obtain the uniform upper bound of $\|\mathcal{S}(k,t)\|$, independent of k and t, denoted as $\Gamma(n, n_d)$. That is,

$$\|\mathcal{S}(k,t)\| \le \varphi = \Gamma(n, n_d) = \min\{C_1(\sqrt{nn_d} + n_d\log n),\ n\}$$

Hence, an upper bound of Y_t, denoted as $\tilde{Y}_t$, follows Poisson distribution with parameter $\lambda = \frac{n_s}{n}\varphi$.

Case 1 When $n_s\varphi/n = \Omega(\log n)$, by lemma 1, we get

$$\begin{aligned}\Pr(Y_t \ge 2n_s\varphi/n) &\le \Pr(\tilde{Y}_t \ge 2n_s\varphi/n) \\ &\le \frac{e^{-n_s\varphi/n}(en_s\varphi/n)^{2n_s\varphi/n}}{(2n_s\varphi/n)^{(2n_s\varphi/n)}} \\ &= (4/e)^{-n_s\varphi/n}\end{aligned} \quad (25)$$

Recall the inequality (19), we obtain

$$\Pr(X_t \ge 2n_s\varphi/n) \le \Pr(Y_t \ge 2n_s\varphi/n) \le (4/e)^{-n_s\varphi/n}$$

By union bounds and the fact that there are at most n first-class stations, we have,

$$\begin{aligned}&\Pr(X \ge 2n_s\varphi/n) \le n\Pr(X_t \ge 2n_s\varphi/n) \\ &\le n(e/4)^{n_s\varphi/n} \to 0, \quad (n \to \infty).\end{aligned} \quad (26)$$

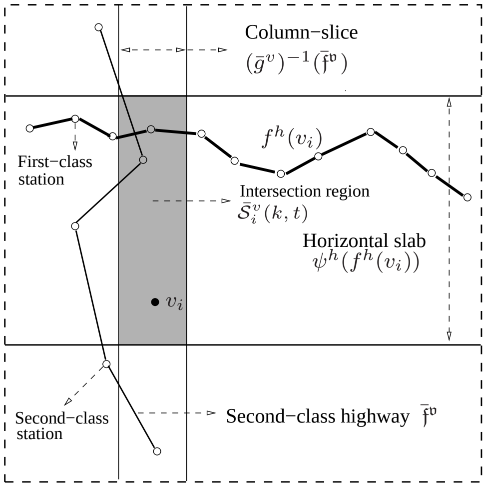


Fig. 9. The region $\bar{\mathcal{S}}_i^v(k,t)$ is the intersection of the *column-slice* $(\bar{g}^v)^{-1}(\bar{\mathfrak{f}}^{\mathfrak{v}})$ and the *horizontal slab* $\psi^h(f^h(v_i))$.

Case 2 When $n_s\varphi/n = O(\log n)$, and

$$\lim_{n\to\infty} \tfrac{n_s\varphi}{n\log n} = f_1,\ f_1 \text{ is a nonnegative constant.}$$

Let $z_1 = e^2\max\{f_1, 1\}$, by union bounds, the inequality (19) and lemma 1, we have, as $n \to \infty$,

$$\Pr(X \ge z_1 \log n) \le (e^{(1-f_1)/z_1+1} f_1/z_1)^{z_1 \log n} \to 0. \quad (27)$$

To sum up case 1 and case 2, we prove Equation (12) and complete the proof of the lemma. ■

Subsequently, we consider the *Vertical Second-Class Phase* and *Horizontal Second-Class Phase*, *i.e.*, Phase 2 and Phase 6. Note that the *Vertical Second-Class Phase* is not symmetrical with the *Horizontal Second-Class Phase* due to the tree structure of $EMST$. Generally, we call these two phases as *second-class phase*.

Lemma 10: In *vertical second-class phase*, each session can transmit packets with rate of order

$$\begin{cases} \Omega(\frac{n}{n_s n_d (\log n)^{\frac{\alpha+1}{2}}}) & \text{when } n_s n_d = \Omega(n\sqrt{\log n}) \\ \Omega(\frac{1}{(\log n)^{1+\frac{\alpha}{2}}}) & \text{when } n_s n_d = O(n\sqrt{\log n}) \end{cases} \quad (28)$$

Proof: According to Lemma 7, we know that, the achievable rate is of order $\Omega((\log n)^{-\frac{\alpha}{2}})$ along the *second-class highways*. So we only need to prove the maximum relay burden of the *second-class stations* during *second-class highway phase* is *w.h.p.*,

$$\begin{cases} O(\frac{n_s\cdot n_d\cdot\sqrt{\log n}}{n}) & \text{when } n_s n_d = \Omega(n\sqrt{\log n}) \\ O(\log n) & \text{when } n_s n_d = O(n\sqrt{\log n}) \end{cases} \quad (29)$$

Given a *second-class station* $\bar{v}_t^*$ passed by a *vertical second-class highway* $\bar{\mathfrak{f}}^{\mathfrak{v}}$, we define a random variable $\bar{X}_t^v$ as Table I. Note that we finally consider the uniform upper bound of $\bar{X}_t^v$ for all *second-class stations*, denoted as $\bar{X}^v$.

For any $v_i \in U_k$, we define an event $\bar{A}_i^v(k,t)$ as Table I. Denote the intersection of the *column-slice* $(\bar{g}^v)^{-1}(\bar{\mathfrak{f}}^{\mathfrak{v}})$ and the *horizontal slab* $\psi^h(f^h(v_i))$ as $\bar{\mathcal{S}}_i^v(k,t)$, then the area of $\bar{\mathcal{S}}_i^v(k,t)$ equals to

$$\begin{aligned} \|\bar{\mathcal{S}}_i^v(k,t)\| &= \frac{\sigma}{(2\theta_1\sqrt{\log n})}\cdot(\kappa\log m - \varepsilon_m) \\ &= \tfrac{\kappa\sigma}{4\theta_1}\sqrt{\log n} - \frac{\sigma}{2\theta_1\sqrt{\log n}}(\kappa\log 2c + \varepsilon_m). \end{aligned}$$

To simplify the description, we can denote $\|\bar{\mathcal{S}}_i^v(k,t)\|$ as $\eta(n) = \Theta(\sqrt{\log n})$. Obviously, the following proposition holds.

Proposition 3: The poisson node v_i locates in the region $\bar{\mathcal{S}}_i^v(k,t)$ if the event $\bar{A}_i^v(k,t)$ happens.

Denote the set of leaf nodes in $EMST(U_k)$ as $\hat{U}_k$, and denote the set $U_k - \hat{U}_k$ as $\check{U}_k$. So, we have

$$\Pr(\bar{A}^v(k,t)) = \Pr(\bigcup\nolimits_{v_i\in\check{U}_k} \bar{A}_i^v(k,t)) \le \Pr(\bigcup\nolimits_{v_i\in U_k} \bar{A}_i^v(k,t))$$

From union bounds, we obtain

$$\Pr(\bar{A}^v(k,t)) \le \sum_{i=1}^{n_d} \Pr(\bar{A}_i^v(k,t)) \quad (30)$$

Accordingly, we define event $\bar{B}_i^v(k,t)$: a given poisson node locates in the region $\bar{\mathcal{S}}_i^v(k,t)$. Hence,

$$\Pr(\bar{A}_i^v(k,t)) \le \Pr(\bar{B}_i^v(k,t)) \quad (31)$$

Let $\bar{B}^v(k,t) = \bigcup_{v_i\in U_k} \bar{B}_i^v(k,t)$. Then, the event $\bar{B}^v(k,t)$ means that a poisson node locates in a region with area $n_d\cdot\eta(n)$, and by Equation (31) we have

$$\Pr(\bar{A}^v(k,t)) \le \Pr(\bar{B}^v(k,t)) = \sum_{i=1}^{n_d} \Pr(\bar{B}_i^v(k,t)) \quad (32)$$

According to the definition of $\bar{X}_t^v$, we know that $\bar{X}_t^v = \sum_{k=1}^{n_s} I(\bar{A}^v(k,t))$. Let $\Upsilon_t = \sum_{k=1}^{n_s} I(\bar{B}^v(k,t))$, by inequality (32) we have

$$\Pr(\bar{X}_t^v \ge x) \le \Pr(\Upsilon_t \ge x), \quad \text{for any } x \ge 0 \quad (33)$$

The random variable Υ_t represents the number of nodes located in the region with area $n_d\cdot\eta(n)$ according to a p.p.p of intensity n_s/n. So it follows Poisson with $\lambda = n_s n_d \eta(n)/n$.

Case 1 When $n_s n_d \eta(n)/n = \Omega(\log n)$, From lemma 1 and union bounds, we obtain

$$\begin{aligned} \Pr(\bar{X}^v \ge 2n_s n_d\eta(n)/n) &\le n\cdot\Pr(\bar{X}_t^v \ge 2n_s n_d\eta(n)/n) \\ &\le n\cdot(e/4)^{n_s n_d\eta(n)/n} \to 0,\ \text{as } n\to\infty. \end{aligned}$$

Case 2 When $n_s n_d\eta(n)/n = O(\log n)$, and

$$\lim_{n\to\infty} \frac{n_s n_d \eta(n)}{n\log n} = f_2, f_2 \text{ is a nonnegtive constant.}$$

Let $z_2 = e^2\max\{f_2, 1\}$, we have

$$\Pr(\bar{X}^v \ge z_2\log n) \le n\cdot(e^{(1-f_2)/z_2+1} f_2/z_2)^{z_2\log n} \to 0.$$

To sum up case 1 and case 2, we prove the result (29) and complete the proof of the lemma. ■

With similar but indifferent way to Lemma 10, we consider the *Horizontal Second-Class Phase*.

Lemma 11: In *horizontal second-class phase*, each session can transmit packets with rate of order described in Equation (28).

Proof: Above all, notice that, unlike unicast case, the deliver phase is not symmetrical with the draining phase due to the tree structure of $EMST$. We define the events $\bar{A}^h(k,t)$ and $\bar{A}_j^h(k,t)$ for any $v_j \in U_k$ as Table I. Let $U_k' = U_k - \{v_{k_0}\}$, we have

$$\Pr(\bar{A}^h(k,t)) = \Pr(\bigcup_{v_j \in U_k'} \bar{A}_j^h(k,t)) \le \Pr(\bigcup_{v_j \in U_k} \bar{A}_j^h(k,t))$$

By union bounds, $\Pr(\bar{A}^h(k,t)) \le \sum_{i=1}^{n_d} \Pr(\bar{A}_i^h(k,t))$. Hereafter, we can adopt the similar way to Lemma 10 to complete the proof for this lemma. ■

In the *second-class draining phase* and *second-class deliver phase*, *i.e.*, Phase 1 and Phase 7, like the Phases 2 and 6, we use 16-TDMA scheme to schedule the links with distance of $\Theta(\sqrt{\log n})$ in parallel, by which we make every link achieve the rate of $\Omega((\log n)^{-\frac{\alpha}{2}})$. On the other hand, There is no relay burden for the nodes in Phase 1 and Phase 7 due to the method of single-hop, thus, it is easy to obtain the following result.

Lemma 12: For all seven phases, it holds that Phases 1 and 7 must not be the bottleneck of the whole routing as long as Phases 2 and 6 are not the bottleneck.

We consider the bottleneck of the whole routing scheme that can be regarded as the per-session multicast throughput. For the *Horizontal (Vertical) First-Class Highway Phase* and *First-Class Turning Phase* (Phases 3, 4 and 5), we have the result described in Lemma 9. For the *Vertical Second-Class Draining Phase* and *Horizontal Second-Class Highway Phase* (Phases 2 and 6), we have the result described in Lemma 10 and Lemma 11. For the *Second-Class Draining Phase* and *Second-Class Deliver Phase* (Phases 1 and 7), we have the result described in Lemma 12.

Consider all phases of routing $\digamma$, we obtain,

Lemma 13: By the multicast routing $\digamma$ combining with the first-class and second-class transmission schedulings, the per-session throughput is achieved of order
If $n_d = O(n/\sqrt{\log n})$,

$$\begin{cases} \Omega(\frac{1}{(\log n)^{1+\frac{\alpha}{2}}}) & \text{when } n_s : [1, \frac{n \log n}{\Gamma}] \\ \Omega(\min_{order}\{\frac{n}{n_s \Gamma}, \frac{1}{(\log n)^{1+\frac{\alpha}{2}}}\}) & \text{when } n_s : [\frac{n \log n}{\Gamma}, \frac{n\sqrt{\log n}}{n_d}] \\ \Omega(\min_{order}\{\frac{n}{n_s \Gamma}, \frac{n}{n_s n_d (\log n)^{\frac{\alpha+1}{2}}}\}) & \text{when } n_s : [\frac{n\sqrt{\log n}}{n_d}, n] \end{cases}$$

If $n_d = \Omega(n/\sqrt{\log n})$,

$$\begin{cases} \Omega(\frac{n}{n_s n_d (\log n)^{\frac{\alpha+1}{2}}}) & \text{when } n_s : [1, n\sqrt{\log n}/n_d] \\ \Omega(\frac{1}{(\log n)^{1+\frac{\alpha}{2}}}) & \text{when } n_s : [n\sqrt{\log n}/n_d, n] \end{cases}$$

where Γ is defined as Equation (35).

As mentioned above, when the bottleneck of the whole routing locates in VSH-Phase and HSH-Phase in multicast routing $\digamma$, better performance of throughput can be derived by routing $\digamma'$ than that derived by routing $\digamma$. Subsequently, we consider the throughput achieved by $\digamma'$.

Lemma 14: By the multicast routing $\digamma'$ and second-class transmission scheduling, the per-session throughput is achieved of order

$$\begin{cases} \Omega(\frac{n}{(\log n)^{\frac{\alpha}{2}} n_s \cdot \phi(n,n_d)}) & \text{when } n_s \cdot \phi(n,n_d)/n = \Omega(\log n) \\ \Omega(1/(\log n)^{1+\frac{\alpha}{2}}) & \text{when } n_s \cdot \phi(n,n_d)/n = O(\log n) \end{cases}$$

where $\phi(n,n_d) = \min_{order}\{\sqrt{nn_d}/\sqrt{\log n} + n_d,\ n\}$.

TABLE II
PER-SESSION THROUGHPUT FOR *Random Extended Networks*

Range of n_d	Order of $\lambda(n)$
$[1,\ \frac{n}{(\log n)^3}]$	$\begin{cases} \lambda_1(n) & \text{if } n_s : [1, \frac{n \log n}{\Gamma}] \\ \min_{order}\{\lambda_1(n), \lambda_2(n)\} & \text{if } n_s : [\frac{n \log n}{\Gamma}, \frac{n \log n}{\Psi}] \\ \min_{order}\{\lambda_2(n), \lambda_3(n)\} & \text{if } n_s : [\frac{n \log n}{\Psi}, n] \end{cases}$
$[\frac{n}{(\log n)^3},\ \frac{n}{(\log n)^2}]$	$\begin{cases} \lambda_1(n) & \text{if } n_s : [1, \frac{n \log n}{\Phi}] \\ \min_{order}\{\lambda_1(n), \lambda_2(n)\} & \text{if } n_s : [\frac{n \log n}{\Phi}, \frac{n \log n}{\Psi}] \\ \min_{order}\{\lambda_2(n), \lambda_3(n)\} & \text{if } n_s : [\frac{n \log n}{\Psi}, n] \end{cases}$
$[\frac{n}{(\log n)^2},\ n]$	$\begin{cases} \lambda_1(n) \text{ if } n_s : [1, \frac{n \log n}{\Phi}] \\ \lambda_4(n) \text{ if } n_s : [\frac{n \log n}{\Phi}, n] \end{cases}$

Proof: Similar to routing scheme $\digamma$, the throughput in SDr-Phase and SDe-Phase is not less than that in VSH-Phase and VSH-Phase, which means that the bottleneck of the whole routing locates on VSH-Phase and VSH-Phase. According to Lemma 7, the rate along the second-class highways can be achieved of $\Omega((\log n)^{-\frac{\alpha}{2}})$. On the other hand, using the similar way to the proof of Lemma 9, we can obtain that the maximum relay burden of the *second-class stations* on *second-class highways* is *w.h.p.*,

$$\begin{cases} O(n_s \phi(n,n_d)/n) & \text{when } n_s \phi(n,n_d)/n = \Omega(\log n) \\ O(\log n) & \text{when } n_s \phi(n,n_d)/n = O(\log n) \end{cases} \quad (34)$$

with $\phi(n,n_d) = \begin{cases} \Theta(\sqrt{\frac{n_d n}{\log n}}) & \text{when } n_d = O(\frac{n}{\log n}) \\ \Theta(n_d) & \text{when } n_d = \Omega(\frac{n}{\log n}) \end{cases}$
Considering both sides, we complete the proof. ■

D. General Result for Extended Networks

Combining Lemma 13 with Lemma 14, we obtain the general result in Theorem 6. To simplify the description, let

$$\lambda_1(n) := \frac{1}{(\log n)^{1+\frac{\alpha}{2}}} \qquad \lambda_2(n) := \frac{n}{n_s \Gamma}$$
$$\lambda_3(n) := \frac{n}{n_s n_d (\log n)^{\frac{\alpha+1}{2}}} \qquad \lambda_4(n) := \frac{n}{(\log n)^{\frac{\alpha}{2}} n_s \Phi}$$

and let $\Psi := n_d \cdot \sqrt{\log n}$,

$$\Gamma := \begin{cases} \Theta(\sqrt{n_d n}) & \text{when } n_d : [1, \frac{n}{(\log n)^2}] \\ \Theta(n_d \log n) & \text{when } n_d : [\frac{n}{(\log n)^2}, \frac{n}{\log n}] \\ \Theta(n) & \text{when } n_d : [n/\log n, n] \end{cases}$$
$$\Phi := \begin{cases} \Theta(\sqrt{\frac{n_d n}{\log n}}) & \text{when } n_d : [1, n/\log n] \\ \Theta(n_d) & \text{when } n_d : [n/\log n, n] \end{cases} \quad (35)$$

Theorem 6: The achievable per-session throughput for *random extended networks* is of order $\Omega(\lambda(n))$ as in Table II.

Based on Theorem 6, we get Theorem 1 by assuming that $n_s = \Theta(n)$.

V. LOWER BOUND FOR RANDOM DENSE NETWORKS

In this section, we consider the model where nodes are distributed according to a Poisson point process of intensity n over a square of unit area $\mathcal{A}_1$. Similar to the case of the *random extended network model* (RenM), the routing is also hierarchical structure that consists of two general phases: *First-class Phase* and *Second-class Phase*. The former is that the packet is transmitted along the *first-class highways*(FHs). The latter is that the packet is drained into the *first-class highways* along the *second-class highways*(SHs). The results in *First-class Phase* are similar to that derived under (RenM), while the results in *Second-class Phase* are different to that of (RenM).

A. First-class Phase

In order to build FHs and implement the first-class transmission scheduling, we set edge of the *percolation-cell* as the width $c/\sqrt{n}$, by which we obtain the same number of *percolation-cells* as in the *Random extended network*. The average number of nodes in each *percolation-cell* is also the same, namely c^2. Therefore, all the percolation results above still hold for this model, and we can find as many *first-class highways* as above. That is, the density of FHs is the same as that in *extended network model* (described in Lemma 4). Recall Theorem 4 in [17], we obtain the rate along FHs as follows.

Lemma 15: For any integer $d > 0$, there exists an constant $R(d) > 0$, such that in each *percolation-cell* c_i there is a node that can transmit *w.h.p.*at rate $R(d)$ to any destination located in the cell within *layer-distance* d. Furthermore, as $d \to \infty$, we have $R(d) = \Omega(d^{-2})$.

From Lemma 15, we obtain easily that the rate along the *first-class highways* can is achieved as the order of $\Omega(1)$. Then, with a similar proof to Lemma 9 under the *extended network* case, we have

Lemma 16: In the *random dense network*, packets can be relayed along FHs at rate *w.h.p.*,

$$\begin{cases} \Omega(n/(n_s\Gamma)) \text{ when } n_s\Gamma/n = \Omega(\log n) \\ \Omega(1/\log n) \text{ when } n_s\Gamma/n = O(\log n) \end{cases} \tag{36}$$

where $\Gamma = \min_{order}\{\sqrt{nn_d} + n_d \log n,\ n\}$.

B. Second-class Phase

Under the *random dense network model* (RdnM), we set edge of the *connected-cell* as the width $(\sigma\sqrt{n} - \epsilon_n)/\sqrt{n}$. Then, we obtain the same number of *connected-cells* and the same average number of nodes in each *connected-cells* as that in the *random extended network*. We construct the *Row-Slabs*(or *Column-Slabs*), denoted as $\bar{R}_i^h$ (or $\bar{R}_i^v$), by using the similar method under RenM. Next, we demonstrate whether the technique of *parallel scheduling* in RenM can improve the performance of throughput under the RdnM. Let the number of parallel links initiated from each *connected-cell* in any slot as an integer $\pi(n) \geq 1$, where $\pi(n) = O(\log n)$ because the number of SHs in each *Row-Slab*(or *Column-Slab*) is of order $O(\log n)$. We call the scheme $\pi(n)$*-parallel transmission scheduling*. Then, we have

Lemma 17: Using $\pi(n)$*-parallel transmission scheduling*, the achievable rate along each SH is of order $\Omega(1/\pi(n))$.

Proof: Considering any link in the *second-class highway* in any time slot, since the length of the link is at least $(\sigma\sqrt{\log n} - \epsilon_n)/\sqrt{n}$, we obtain the sum of interferences to the receivers as:

$$\begin{aligned} I(n) &\leq P \cdot (\pi(n) - 1) \cdot l((\sigma\sqrt{\log n} - \epsilon_n)/\sqrt{n}) \\ &\quad + \sum_{i=1}^{n} 8iP \cdot \pi(n) \cdot l((4i-3) \cdot (\sigma\sqrt{\log n} - \epsilon_n)/\sqrt{n}) \\ &\leq 2^\alpha\sigma^{-\alpha}P \cdot \pi(n) \cdot (\tfrac{n}{\log n})^{\frac{\alpha}{2}} \cdot \left(1 + \lim_{n\to\infty}\sum_{i=1}^{n} \tfrac{8i}{(4i-3)^\alpha}\right) \end{aligned}$$

The latest limitation is obviously converges to a constant when $\alpha > 2$. On the other hand, since the length of the link is at most $\sqrt{10}((\sigma\sqrt{\log n} - \epsilon_n)/\sqrt{n})$, the signal $S(n)$ at the receiver can be bounded as $S(n) \geq P \cdot 10^{-\frac{\alpha}{2}}\sigma^{-\alpha}(n/\log n)^{\frac{\alpha}{2}}$. Then, by Equation (1) and the 16-TDMA scheme, we have that the achievable rate along the *second-class highway* is

$$\bar{R}(n) = \frac{1}{16}\log\left(1 + \frac{S(n)}{N_0 + I(n)}\right) = \Omega(\log(1 + \frac{1}{\pi(n)}))$$

Hence, $\bar{R}(n) = \Omega(1/\pi(n))$ which completes the proof. ■

From Lemma 17, we know although the traffic burden of each *horizontal second-class highway* (HSF) (or *vertical second-class highway* (VSH)) is linear inverse ratio to the number of HSHs (or VSHs) constructed in each *Row-Slab*(or *Column-Slab*), there is indeed a linear proportion by inversion between the rate along each HSH (or VSHs) and the number of parallel links to be scheduled in each *connected-cell* during any time slot, which means no improvement in throughput can be derived by the *parallel scheduling*. Hence, unlike in RenM, we only build one HSH (or VSH) in each *Row-Slab* (or *Column-Slab*) by choosing randomly one node from each *connected-cell* and connecting them. Likewise, we call those nodes *second-class stations*. We denote the HSH (or VSH) in the *Row-Slab* $\bar{R}_i^h$ (or *Column-Slab* $\bar{R}_i^v$) as $\bar{f}_i^h$ (or $\bar{f}_i^v$). Then, the packets to be transmitted (or received) by any node v in $\bar{R}_i^h$ (or $\bar{R}_i^v$) are all drained into (or delivered from) the *first-class highway* $f^h(v)$ (or $f^v(v)$) along VSH $\bar{f}_i^h$ (or HSH $\bar{f}_i^h$). Subsequently, we first show that each SH can sustain a constant aggregated rate (Lemma 18), which is used in conjunction with the burden of each SH to obtain the per-session throughput during *second-class phase* in Lemma 19.

Lemma 18: Using a 9-TDMA scheme and setting every node only directly communicate with the node in its adjacent *connected-cell*, each SH can sustain a aggregated rate of $\Omega(1)$.

Proof: Firstly, we obtain the sum of interferences to the receivers as:

$$\begin{aligned} I(n) &\leq \sum_{i=1}^{n} 8i \cdot P \cdot l((3i-2) \cdot (\sigma\sqrt{\log n} - \epsilon_n)/\sqrt{n}) \\ &\leq 2^\alpha\sigma^{-\alpha}P \cdot \left(1 + \lim_{n\to\infty}\sum_{i=1}^{n} \tfrac{8i}{(3i-2)^\alpha}\right) \cdot (\tfrac{n}{\log n})^{\frac{\alpha}{2}} \end{aligned}$$

The latest limitation is obviously converges.

Secondly, we get the signal at the receiver can be bounded as $S(n) \geq P \cdot 5^{-\frac{\alpha}{2}}\sigma^{-\alpha}(n/\log n)^{\frac{\alpha}{2}}$, because the length of the link is at most $\sqrt{5}((\sigma\sqrt{\log n} - \epsilon_n)/\sqrt{n})$, Then, by Equation (1) and the 9-TDMA scheme, we have that the achievable rate along the *second-class highway* is $\bar{R}(n) = \frac{1}{9}\log\left(1 + \frac{S(n)}{N_0 + I(n)}\right) = \Omega(1)$. The proof is completed. ■

Lemma 19: In *second-class phase*, each session can transmit packets along the SHs with rate of order

$$\begin{cases} \Omega(\frac{n}{n_s n_d (\log n)^{\frac{3}{2}}}) & \text{when } n_s = \Omega(\frac{n \log n}{\Psi_d}) \\ \Omega(\frac{1}{\log n}) & \text{when } n_s = O(\frac{n \log n}{\Psi_d}) \end{cases} \tag{37}$$

where $\Psi_d = n_d(\log n)^{\frac{3}{2}}$.

Proof: According to Lemma 18, we know that, the achievable aggregated rate along the *second-class highways* is of order $\Omega(1)$. So we only need to prove the maximum relay burden of the *second-class stations* during *second-class highway phase* is *w.h.p.*,

$$\begin{cases} O(\frac{n_s \cdot n_d \cdot (\log n)^{\frac{3}{2}}}{n}) & \text{when } n_s n_d = \Omega(n/\sqrt{\log n}) \\ O(\log n) & \text{when } n_s n_d = O(n/\sqrt{\log n}) \end{cases} \tag{38}$$

Because the density of SHs in RdnM is sparser than that in RenM with a factor $\Theta(\log n)$, it holds that there is naturally a factor $\Theta(\log n)$ between the maximum relay burden of the *second-class stations* during *second-class highway phase* under RdnM and that under RenM. According to the similar routing schemes of two models and Equation (29), we obtain the result of Equation (38) which completes the proof. ∎

Combining Lemma 16 and Lemma 19, we obtain the following result.

Lemma 20: By the multicast scheme combining on first-class phase and second-class phase, the per-session throughput for *random extended networks* is achieve of order

If $n_d = O(\frac{n}{(\log n)^3})$, $\begin{cases} 1/\log n & \text{if } n_s : [1, \frac{n \log n}{\Gamma}] \\ \Omega(\frac{n}{n_s \cdot \sqrt{n n_d}}) & \text{if } n_s : [\frac{n \log n}{\Gamma}, n] \end{cases}$

If $n_d = \Omega(\frac{n}{(\log n)^3})$, $\begin{cases} 1/\log n & \text{if } n_s : [1, \frac{n \log n}{\Psi_d}] \\ \Omega(\frac{n}{n_s n_d (\log n)^{\frac{3}{2}}}) & \text{if } n_s : [\frac{n \log n}{\Psi_d}, n] \end{cases}$

Similar to the case of RenM, when the bottleneck of the whole routing locates in VSH-Phase and HSH-Phase, better performance of throughput can be derived by the routing only based on *second-class highway system*, denoted as F'', than that derived by routing combining *second-class phase* and *first-class phase*. In F'', the routing of each link in EMST is constructed based on one corresponding VSH and HSH. Subsequently, we consider the throughput achieved by F''.

Lemma 21: By the multicast routing F'' and second-class transmission scheduling, the per-session throughput is achieved of order

$$\begin{cases} \Omega(\frac{n}{n_s \Phi_d}) & \text{when } n_s \cdot \Phi_d = \Omega(n \log n) \\ \Omega(1/\log n) & \text{when } n_s \cdot \Phi_d = O(n \log n) \end{cases}$$

where $\Phi_d = \min_{order}\{\sqrt{n \cdot n_d \cdot \log n},\ n\}$.

Proof: From Lemma 18, the rate along the second-class highways can be achieved of $\Omega(1)$. On the other hand, because there is only one SH in each *column-slab* or *row-slab*, using the similar but more simple process to the proof of Lemma 9, we can obtain that the maximum relay burden of *second-class stations* on *second-class highways* is *w.h.p.*,

$$\begin{cases} O(n_s \Phi_d / n) & \text{when } n_s \cdot \Phi_d = \Omega(n \log n) \\ O(\log n) & \text{when } n_s \cdot \Phi_d = O(n \log n) \end{cases}$$

Hence, we complete the proof. ∎

C. General Result for Dense Networks

Let $\Phi_d = \begin{cases} \Theta(\sqrt{n \cdot n_d \cdot \log n}) & \text{when } n_d = O(n/\log n) \\ \Theta(n) & \text{when } n_d = \Omega(n/\log n) \end{cases}$

and $\Psi_d = n_d(\log n)^{\frac{3}{2}}$. Combining Lemma 20 with Lemma 21, we obtain the following general result.

Theorem 7: The achievable per-session throughput for *random dense networks* is of order $\Omega(\lambda_d(n))$ described in Table III.

TABLE III
PER-SESSION THROUGHPUT FOR *Random Dense Networks*

Range of n_d	Order of $\lambda_d(n)$
$[1,\ \frac{n}{(\log n)^3}]$	$\begin{cases} \Omega(1/\log n) & \text{if } n_s : [1, \frac{n \log n}{\Phi_d}] \\ \Omega(\frac{n}{n_s \cdot \sqrt{n n_d}}) & \text{if } n_s : [\frac{n \log n}{\Phi_d}, n] \end{cases}$
$[\frac{n}{(\log n)^3},\ \frac{n}{(\log n)^2}]$	$\begin{cases} \Omega(1/\log n) & \text{if } n_s : [1, \frac{n \log n}{\Phi_d}] \\ \Omega(\frac{n}{n_s n_d (\log n)^{\frac{3}{2}}}) & \text{if } n_s : [\frac{n \log n}{\Phi_d}, n] \end{cases}$
$[\frac{n}{(\log n)^2},\ \frac{n}{\log n}]$	$\begin{cases} \Omega(1/\log n) & \text{if } n_s : [1, \frac{n \log n}{\Psi_d}] \\ \Omega(\frac{\sqrt{n}}{n_s \sqrt{n_d \log n}}) & \text{if } n_s : [\frac{n \log n}{\Psi_d}, n] \end{cases}$
$[\frac{n}{\log n},\ n]$	$\begin{cases} \Omega(1/\log n) & \text{if } n_s : [1, \frac{n \log n}{\Psi_d}] \\ \Omega(\frac{1}{n_s}) & \text{if } n_s : [\frac{n \log n}{\Psi_d}, n] \end{cases}$

Based on Theorem 7, we obtain Theorem 4 by assuming that $n_s = \Theta(n)$.

VI. UPPER BOUND FOR MULTICAST CAPACITY

In this section, we consider the upper bound both for *random extended networks* and *random dense networks* with the assumption $n_s = \Theta(n)$ under Gaussian channel model. Notice that we revoke the representative meaning of all variables and number labels in above sections, unless we explicitly use them.

A. Random Dense Networks

Based on the technique called *arena* exploited in [25], Keshavarz-Haddad *et al.* have proposed the upper bound of the multicast capacity for *dense networks* in [23]. That is,

Lemma 22: The per-session multicast capacity for *dense networks* is at most of order

$$\begin{cases} O(\frac{1}{\sqrt{n_d n}}) & \text{when } n_d : [1, \frac{n}{(\log n)^2}] \\ O(\frac{1}{n_d \cdot \log n}) & \text{when } n_d : [\frac{n}{(\log n)^2}, \frac{n}{\log n}] \\ O(\frac{1}{n}) & \text{when } n_d : [\frac{n}{\log n}, n] \end{cases}$$

B. Random Extended Networks

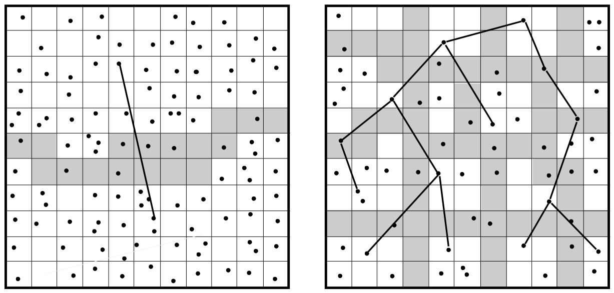
In this subsection, we give an upper bound for multicast capacity for *extended networks*.

Firstly, by partitioning the region $\mathcal{A}_n = [0, \sqrt{n}] \times [0, \sqrt{n}]$ into cells with a constant side length g, we obtain a grid graph $\mathcal{F}_n$ consisting of $\frac{n}{g^2}$ cells. Based on the grid graph $\mathcal{F}_n$, we propose a result for arbitrary multicast trees.

Lemma 23: Given multicast session $\mathcal{M}_k$, let T_k be the multicast tree for $\mathcal{M}_k$ and $C(T_k)$ denote the number of cells used in T_k, then we have

$$C(T_k) = \Omega(\frac{1}{g} \cdot |EMST_{\mathcal{M}_k}|)$$

when $n_d = O(\frac{n}{g^2})$, where $|EMST_{\mathcal{M}_k}|$ denotes the length of Euclidean Minimum Spanning Tree spanning $\mathcal{M}_k$.



(a) quasi-closed cut (b) quasi-closed net

Fig. 10. Grey cells are the quasi-closed cells which contain at most Δ nodes.

Proof: We prove this lemma using some existing results under protocol model, especially the area argument [11]. For the sake of our proof, assume that every node has an artificial "transmission radius" r such that each node v can only communicate with other nodes in its transmission range (a transmitting disk with its center at v and radius r). In addition, we define the area covered by a tree T as the union of its nodes' transmitting disks. Then by showing a lower bound on the area of the region covered by any multicast tree T, we can give the desired lower bound on the number of cells it crosses.

The result of Lemma 11 in [24] applies for the *constant range* network model, it can be implemented a straightforward extension for the *extended network* model as following: In protocol model, the area of the region $D(T) = \Omega(r\sqrt{n_d n})$, when $n_d = O(\frac{n}{r^2})$. Here r denotes the transmission range of each node in protocol model and $D(T)$ denotes the region covered by all transmitting disks of all transmitting nodes (internal nodes of T) in the any multicast tree T. Unfortunately, this result can not help us directly, since in our model, each node has no fixed transmission range r. Instead, any pair of nodes can communicate with each other even though the data rate may be very small. Based on the original network under Gaussian channel model, we construct a new network under protocol model as follows.

1) Set the transmission range of each node as g, *i.e.*, the side length of the cell in $\mathcal{F}_n$.
2) Add some artificial "additional relay nodes" v_a such that any pair of nodes have enough relay nodes along its link to make sure that the minimum number of cells the routing path crosses under protocol model is no more than the number of cells the direct link will cross in Gaussian channel model. Notice that v_a cannot be selected as source or receivers, they can only act as relay nodes.

Let T be any multicast tree in original network under Gaussian channel model and T_p denote the corresponding multicast tree(spanning the same multicast session) constructed on this network under protocol model. We have two important observations here:

1) Our preceding two modifications will not affect the proof for Lemma 11 in [24]. In other words, the lower bound on $|D(T_p)|$ still holds,
2) Furthermore, any link in Gaussian channel model can be simulated by using these artificial "additional relay nodes" in the protocol model such that the number of cells it will cross is not increased. So the lower bound of $C(T)$ is no smaller than the lower bound of $C(T_p)$.

By Lemma 11 in [24], we get $D(T_p) = \Omega(g\sqrt{n_d n})$ for $g = r$. Since one transmitting disk can cover no more than 4 cells. We have $C(T_p) = \Omega(\frac{\sqrt{n_d n}}{g})$. Hence,

$$C(T) = \Omega(\frac{\sqrt{nn_d}}{g}), \text{ when } n_d = O(\frac{n}{g^2}).$$

By $|EMST| = O(\sqrt{nn_d})$, the proof is completed. ■

We partition the region $\mathcal{A}_n$ into cells with constant side length c, we obtain a grid graph $\mathcal{C}_n$ consisting of $m^2 = \frac{n}{c^2}$ cells. We focus on those cells containing only a constant number of nodes, and give the following definition.

Definition 5: We say a cell is *quasi-closed cell* if it contains at most Δ nodes, here Δ is some constant. As illustrated in Figure 10, we call a path of cells *quasi-closed cut* if it contains only quasi-closed cells and crosses from left to right side of $\mathcal{A}_n$. Furthermore, we define the length of a quasi-closed cut as the total number of cells it contains.

According to the results in [17] and Lemma 2, we can choose c large enough such that $\Omega(m)$ quasi-closed cuts can be partitioned into a number of disjoint groups each with $\lceil \delta \log m \rceil$ disjoint quasi-closed cuts, and each group is constraint in a slab of size $m \times (\kappa \log m - \epsilon_m)$, for all $\kappa > 0$, δ small enough, and a non-zero small ϵ_m such that the side length of each slab is an integer. The same is true when we partition the square into vertical slabs with side length $m \times (\kappa \log m - \epsilon_m)$. Notice that all of the horizontal and vertical stripes together partition $\mathcal{A}_n$ into *super-cells* with side length $c \cdot (\kappa \log m - \epsilon_m)$.

Lemma 24: When $n_d = O(\frac{n}{(\log n)^2})$ and $n_s = \Theta(n)$, with probability at least $1 - 2e^{-n_s c_8^2/32}$, the per-session data rate that can be supported using any routing strategy, due to the congestion in some quasi-closed cell, is $O(\frac{1}{n_s} \cdot \frac{\sqrt{n}}{\sqrt{n_d}})$.

Proof: Our proof is to analyze the load of some cells. We use L to denote the total load of all cells. By Lemma 23, we have, there exist a constant c_1 such that

$$L \geq \sum_{i=1}^{n_s} \frac{c_1}{c} \cdot \frac{|EMST(\mathcal{M}_k)|}{\kappa \log m - \epsilon_m}$$

Since $\Pr(\sum_{i=1}^{n_s} |EMST(\mathcal{M}_k)| \geq n_s c_2 \sqrt{nn_d}/2) \geq 1 - 2e^{-n_s c^2/32}$, from Lemma 23, we get

$$\Pr(L \geq n_s c_3 \sqrt{n_d} \frac{m}{\log m}) \geq 1 - 2e^{-n_s c^2/32}.$$

for some constant c_3. Here we use $\mathbb{L}$ to denote the total number of flows crossing some super-cell. Notice that here "crossing" means visiting and leaving. We get,

$$\Pr(\mathbb{L} \geq L - n_s n_d = \frac{n_s c_3 \sqrt{n_d} m}{2 \log m} - n_s n_d) \geq 1 - 2e^{-n_s c^2/32}.$$

We can easily obtain that any multicast routing tree will cross at least $\lceil \delta \log m \rceil$ quasi-closed cuts if it crosses three super-cells. Denoted by $\mathbb{L}'$ the total number of flows crossing some quasi-closed cut. We have $\mathbb{L}' \geq \frac{\mathbb{L}}{3} \times \lceil \delta \log m \rceil$. It follows that, with probability at least $1 - 2e^{-n_s c^2/32}$, the total load of all quasi-closed cell is at least

$$\frac{n_s c_3 \sqrt{n_d} \frac{m}{\log m}/2 - n_s n_d}{3} \times \lceil \delta \log m \rceil.$$

Then by pigeonhole principle, with probability at least $1-2e^{-n_s c^2/32}$, there is at least one quasi-closed cell, that will be used by at least

$$\frac{\frac{n_s c_3 \sqrt{n_d}\frac{m}{\log m}/2 - n_s n_d}{3} \times \lceil \delta \log m \rceil}{m^2}$$

flows which is of order $\Omega(\frac{n_s\sqrt{n_d}}{\sqrt{n}})$ when $n_d = O((\frac{m}{\log m})^2)$. Then with probability at least $1-2e^{-n_s c^2/32}$, the per-session data rate that can be supported using any routing strategy, due to the congestion in some quasi-closed cell, is at most

$$O(\frac{\sqrt{n}}{\theta_2 n_s \sqrt{n_d}}) = O(\frac{1}{n_s}\cdot\frac{\sqrt{n}}{\sqrt{n_d}}), \tag{39}$$

when the number of receivers n_d per-session is at most $O(\frac{n}{\log^2 n})$. This finishes the proof of the theorem. ■

Furthermore, we will derive another upper bound on multicast capacity using other arguments approaches similar to [23]. The basic idea is to show that, for a random network topology, a cluster of nodes exists, that is relatively isolated from the rest of the nodes. The separation distance is at the same order of the size of the isolated cluster of nodes. Consequently, the average rate of the information that can be sent/received by the nodes of the cluster is limited. Specifically, we can extend the result of Theorem 2 in [23] from *dense networks* case into *extended networks* case.

Lemma 25: Under Gaussian channel model, the per-session multicast capacity for *extended networks* is at most of order

$$\begin{cases} O(\frac{n}{n_s n_d}(\log n)^{-\frac{\alpha}{2}}), & \text{if } n_d = O(\frac{n}{\log n}) \\ O(\frac{1}{n_s}(\log n)^{1-\frac{\alpha}{2}}), & \text{if } n_d = \Omega(\frac{n}{\log n}) \end{cases} \tag{40}$$

Proof: As in [23], it holds that there exists *w.h.p.*, a cluster of $\Theta(\log n)$ nodes inside a cell of size $h=\frac{1}{3}\sqrt{\log n}$, and the cluster is separated from the rest of nodes with distance at least $\frac{1}{3}\sqrt{\log n}$. Let $\mathfrak{p}$ be the probability that at least one node in this isolated cluster is a source or terminal (destination) of a multicast session. We can show that

$$\mathfrak{p} = \begin{cases} \Theta(\frac{n_d \log n}{n}) & \text{when } n_d = O(\frac{n}{\log n}) \\ \Theta(1) & \text{when } n_d = \Omega(\frac{n}{\log n}) \end{cases} \tag{41}$$

Obviously, the maximum link rate that can be supported for any link uv with u inside this isolated cluster and v outside of this cluster is at most $\log(1+\frac{P\cdot h^{-\alpha}}{N_0}) = \Theta(h^{-\alpha})$ since $h\to\infty$ when $n\to\infty$. Notice that there are $\Theta(\log n)$ nodes inside this isolated cluster. Consequently, the total data rate that can be transmitted from/to this cluster is at most $\Theta(\log n)\cdot\Theta(h^{-\alpha}) = \Theta((\log n)^{1-\frac{\alpha}{2}})$ since each node inside the cluster can not receive from multiple nodes. Define the number of flows that have receivers inside this isolated cluster as a random variable ϑ. Since there are $n_s=\omega(1)$, ϑ follows a Poisson distribution with an expectation of $\mathfrak{p}n_s$. Using Lemma 1 and the similar method to Lemma 5 we can prove that with high probability, there are $\mathfrak{p}n_s/2$ flows that will have receivers inside this isolated cluster. Hence, we obtain that the minimum per-session multicast data rate is at most of order $O((\log n)^{1-\frac{\alpha}{2}}/(\mathfrak{p}n_s))$. Combining with (41), we complete the proof. ■

The preceding upper bound on multicast is derived by analyzing an isolated cluster of nodes. For the *random extended network*, by the result in [26], the nearest neighbor graph has *w.h.p.*, an edge of length $\Theta(\sqrt{\log n})$. By exploring this long edge, we can derive another upper bound on multicast capacity.

Lemma 26: Under Gaussian channel model, the per-session multicast capacity for *extended networks* is at most of order $O(\frac{n}{n_s n_d}(\log n)^{-\frac{\alpha}{2}})$ when $n_d=\omega(\sqrt{n})$.

Proof: Assume that the longest edge in the nearest neighbor graph of the random network is uv. Then for node v, the probability $\mathfrak{p}$ that it is chosen as a terminal of a given multicast flow is $\mathfrak{p}=\frac{n_d}{n}$. It is easy to show that, with high probability, the number of multicast flows that will choose the node v as a terminal is at least $n_s\mathfrak{p}/2$ when $n_d=\omega(\sqrt{n})$. Observe that the total data rate that node v can receive is at most $R(v)=O((\log n)^{-\frac{\alpha}{2}})$ since the shortest link incident at node v is at least $\Theta(\sqrt{\log n})$. Then we have the minimum per-session multicast data rate is at most of order $O(R(v)/(n_s\mathfrak{p}))$, which completes the proof. ■

Combining Lemma 24, Lemma 25, Lemma 26 and the assumption $n_s=\Theta(n)$, we obtain Theorem 2.

VII. Literature Reviews

In this section, we mainly review the *networking–theoretic* capacity bounds for wireless network. We classify them in terms to the diversity of sessions.

Unicast Sessions: Gupta and Kumar [1] studied the *unicast* capacity for dense network under the *threshold-based channel* model. They show that classical multihop architectures with conventional single-user decoding and forwarding of packets can achieve the per-session throughput at most of order $O(1/\sqrt{n})$, and that a scheme of nearest neighbor communication can achieve a throughput of order $\Theta(1/\sqrt{n\log n})$. Later, Franceschetti *et al.* [17] showed the per-session throughput for *random extended networks* and *random dense networks* can both be achieved of order $O(1/\sqrt{n})$. Note that their results are derived under the *Gaussian Channel* model. The interesting is that Xie and Kumar [3] have shown that the *information-theoretic* upper bound of unicast capacity for *extended networks* is also of order $O(1/\sqrt{n})$ when the power path loss exponent $\alpha>6$, which means that the classic multihop scheme is in fact order-optimal for $\alpha>6$. In fact, Xie and Kumar [18] successively improved the threshold on α for which multihop is order-optimal from 6 to 4.

Broadcast Sessions: Under the *threshold-based channel* model, Keshavarz-Haddad *et al.* [12] studied the broadcast capacity of an arbitrary network. They showed that the per-session broadcast capacity is only of $\Theta(1/n)$. The same bound is proposed in [27]. In [28], Keshavarz-Haddad *et al.* studied the broadcast capacity with dynamic power adjustment for *physical model*. Under the Gaussian Channel model, Zheng [2], [29] proved that the per-session broadcast capacity for *extended networks* is $\frac{1}{n}(\log n)^{-\frac{\alpha}{2}}$. The gap between the results of [2] and that of [12] means that for *extended networks* the assumption of the *threshold-based channel* model that each successful transmission can sustain a constant rate W is over-optimistic, because the value of W is dependent on n under the more reasonable and realistic channel models. The same effect could occur in unicast and multicast sessions.

Multicast Sessions: Earlier, Jacquet and Rodolakis [30] studied the scaling properties of multicast for random wireless networks. They showed that the maximum rate at which a node can transmit multicast data at rate of $O(1/\sqrt{n_d n \log n})$ order. Recently, Li *et al.* [11] and Shakkottai *et al.* [13] proposed results for multicast throughput of networks, respectively. Li *et al.* showed that, assuming that the number of multicast sessions is $n_s = \Omega(\log n_d \cdot \sqrt{n \log n / n_d})$ ([24]), for *random networks*, the per-session capacity of n_s multicast session is $\Theta(1/\sqrt{n_d n \log n})$ when $n_d = O(n/\log n)$, and is $\Theta(1/n)$ when $n_d = \Omega(n/\log n)$. Shakkottai's result can be regarded as a specific case of Li's ([24]). They studied the multicast capacity of random networks when the number of multicast sources is n^{ε} for some $\varepsilon > 0$, and the number of receivers per multicast session is $n^{1-\varepsilon}$.All above results for multicast capacity is derived under the *threshold-based channel* model. Most recently, Li *et al.* ([31]) studied the multicast capacity of random networks under Gaussian Channel Model. They show that, when $n_d = O(\frac{n}{(\log n)^{2\alpha+6}})$ and $n_s = \Omega(n^{\frac{1}{2}+\theta})$, the per-session multicast throughput can be achieved of order $\Omega(\frac{\sqrt{n}}{n_s\sqrt{n_d}})$, where $\theta > 0$ is any positive real number. Keshavarz-Haddad *et al.* [25] proposed a technique called *arena* that is a novel tool to study the upper bounds of capacity for wireless networks. Successively, they [23] studied the multicast capacity for *dense networks*. They also sketched schemes and estimated that the capacity achieved by their method.

VIII. Conclusion

In this paper, we focus on the *networking-theoretic* multicast capacity bounds for both *random extended networks* (REN) and *random dense networks* (RDN) under *Gaussian Channel Model*. Based on percolation theory, we propose two multicast schemes for REN and derive the achievable throughput taking account of all n_s and n_d. We show that under the assumption of $n_s = \Theta(n)$, the per-session multicast throughput derived by our scheme is order-optimal when $n_d = O(\frac{n}{(\log n)^{\alpha+1}})$ or $n_d = \Omega(\frac{n}{\log n})$. When the schemes are extended to *random dense networks*, we analyze the difference between REN and RDN in terms of capacity and adapt the schemes for RDN. We show that for RDN, the per-session multicast throughput derived by our scheme is order-optimal when $n_d = O(\frac{n}{(\log n)^3})$ or $n_d = \Omega(\frac{n}{\log n})$. There are still gaps between the lower bounds and upper bounds of multicast capacity for some ranges of n_d, *i.e.*, $n_d : [\frac{n}{(\log n)^3}, \frac{n}{\log n}]$ for RDN and $n_d : [\frac{n}{(\log n)^{\alpha+1}}, \frac{n}{\log n}]$ for REN. An interesting and challenging issue is to close the gaps on multicast capacity by presenting possibly new tighter upper bounds, and lower bounds, and designing corresponding algorithms to achieve the asymptotic multicast capacity.

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