Asymptotic Distribution of The Number of Isolated Nodes in Wireless Ad Hoc Networks with Unreliable Nodes and Links

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***Abstract*— In randomly-deployed wireless ad hoc networks with reliable nodes and links, vanishment of isolated nodes asymptotically implies connectivity of networks. However, in a realistic system, nodes may become inactive, and links may become down. The inactive nodes and down links cannot take part in routing/relaying and thus may affect the connectivity. In this paper, we study the connectivity of a wireless ad hoc network that is composed of unreliable nodes and links by investigating the distribution of the number of isolated nodes in the network. We assume that the wireless ad hoc network consists of n nodes which are distributed independently and uniformly in a unit-area disk or square. Nodes are active independently with probability $0 < p_1 \leq 1$, and links are up independently with probability $0 < p_2 \leq 1$. A node is said to be *isolated* if it doesn't have an up link to an active node. We show that if all nodes have a maximum transmission radius $r_n = \sqrt{\frac{\ln n + \xi}{\pi p_1 p_2 n}}$ for some constant ξ, then the total number of isolated nodes is asymptotically Poisson with mean $e^{-\xi}$ and the total number of isolated active nodes is also asymptotically Poisson with mean $p_1 e^{-\xi}$. In addition, the work can be extended for secure wireless networks which adopt m-composite key predistribution schemes in which a node is said to be *isolated* if it doesn't have a secure link. Let p denote the probability of the event that two neighbor nodes have a secure link. We show that if all nodes have a maximum transmission radius $r_n = \sqrt{\frac{\ln n + \xi}{\pi p n}}$ for some constant ξ, then the total number of isolated nodes is asymptotically Poisson with mean $e^{-\xi}$.**

I. Introduction

A wireless ad hoc network is composed of a collection of wireless devices distributed over a geographic region. A communication session is established either through a single-hop radio transmission if the communication parties are close enough, or through relaying by intermediate devices otherwise. Due to no need for a fixed infrastructure, wireless ad hoc networks can be flexibly deployed at low cost for varying missions. In many applications, the wireless sensors are deployed in a large volume. The sheer large number of devices deployed coupled with the potential harsh environment often hinders or completely eliminates the possibility of strategic device placement, and consequently, random deployment is often the only viable option.

To model a randomly deployed wireless ad hoc network, it is natural to represent the ad hoc devices by a finite random point process over the deployment region [1] [2] [3] [4] [5]. In addition, due to the short transmission range of radio links, two wireless devices can build a communication link only if they are within each other's transmission range. Assume all devices have the same transmission radius r, then the induced network topology is a r-disk graph in which two nodes are joined by an edge if and only if their distance is at most r. This is a variant of the model proposed by Gilbert (1961) [6] and referred as a *random geometric graph*.

The connectivity of a wireless ad hoc networks is an essential problem. The connectivity of random geometric graphs has been studied by Dette and Henze (1989) [7], Penrose (1997) [8], and others [1] [2] [9] [10]. For a uniform n-point process over a unit-area square, Dette and Henze (1989) [7] showed that for any constant ξ, the $\left(\sqrt{\frac{\ln n + \xi}{\pi n}}\right)$-disk graph has no isolated nodes with probability $\exp\left(-e^{-\xi}\right)$ asymptotically. Later, Penrose (1997) [8] established that if a

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random geometric graph induced by a uniform point process or Poisson point process has no isolated nodes, then it is almost surely connected.

However, in a realistic system, nodes may become inactive due to, for example, internal breakdown or being in the monitoring state, and links may become down due to, for example, harsh environment or barriers between nodes. The inactive nodes and down links cannot take part in routing/relaying and thus may affect the connectivity. Recently, Wan and Yi et al [2] [10] showed that if every nodes independently break down with the same probability p, the network is connected with probability $\exp\left(-pe^{-\xi}\right)$ asymptotically. In this paper, based on the work in [2], we study the connectivity of a wireless network with unreliable nodes and links by investigating the number of isolated nodes. We assume nodes are active independently with the same probability p_1 and links are up independently with the same probability p_2. It is referred as a Bernoulli model. In this model, depending on the meaning of the "inactive" nodes, we may have two types of network connectivity: (1) all active nodes form a connected network; and (2) all active nodes form a connected network and each inactive node is adjacent to at least one active node. In both cases, a node is said to be *isolated*, if it doesn't have an up link to an active nodes. The vanishment of isolated nodes is a prerequisite for connectivity. We shall prove that the number of isolated nodes have asymptotic Poisson distributions.

In addition, the work can be extended for secure wireless networks with m-composite key predistribution schemes. In many applications, the wireless sensors network is composed of low cost devices. Due to the limited capacity, traditional security schemes and key management algorithms are too complex and not feasible for such a system. The m-composite key predistribution schemes [11] [12] [13] are proposed to offer security for randomly-deployed wireless sensor networks. In the scheme, K distinct keys are randomly chosen from the key space to form the key pool. A key ring is a k-element subset of the key pool. Before deployed, each node randomly loads a key ring into its memory. Two nodes within each other's transmission range have a secure link if their key rings have at least m common keys. In secure wireless sensor networks, only secure links can participate in the communication task. Hence, the secure wireless network is the graph in which two nodes have an edge if their distance is at most r and they have at least m common keys in their key ring. In other words, the connectivity of the secure network only can consider the secure links. A secure wireless network is said to be connected if all nodes form a connected network by secure links. A node is said to be *isolated*, if it doesn't have a secure link. Similarly, we shall prove that the number of isolated nodes in the secure wireless network have asymptotic Poisson distributions.

In what follows, all integrals considered will be Lebesgue integrals. For any set S and positive integer k, the k-fold Cartesian product of S is denoted by S^k. The disk of radius r centered at x is denoted by $B(x,r)$. The special unit-area disk or square centered at the origin is denoted by Ω. The symbols o and $\sim$ always refer to the limit $n \to \infty$. To avoid trivialities, we tacitly assume n to be sufficiently large if necessary. For simplicity of notation, the dependence of sets and random variables on n will be frequently suppressed.

The remaining of this paper is organized as follows. In Section II, the main results of this paper are given. In Section III, we present several useful geometric results and integrals. In Section IV, we derive the distribution of the number of isolated nodes. Section V is the conclusion. Due to the limitation on the paper length, simulation results are not presented in the paper.

II. Main Results

The approach used in this paper is based on the method used in [2]. We assume that a wireless ad hoc network is represented by a uniform n-point process over Ω. All nodes are associated with a maximal transmission radius r which is a function of n, and two nodes have a link if the distance between them is at most r.

In the Bernoulli model, nodes are active independently with probability p_1 for $0 < p_1 \leq 1$, and links are up independently with probability p_2 for $0 < p_2 \leq 1$. Here p_1 and p_2 can be constants or functions of n. A node is said to be *isolated* if it doesn't have an up link with an active node. We have the following theorem about the total number of isolated (active) nodes.

Theorem 1: Suppose that $\lim_{n\to\infty} p_1 p_2 \ln n = \infty$ and nodes have the same maximum transmission radius $r = \sqrt{\frac{\ln n + \xi}{n p_1 p_2 \pi}}$ for some constant ξ. Then the total number of isolated nodes is asymptotically Poisson with mean $e^{-\xi}$, and the total number of isolated active nodes is also asymptotically Poisson with mean $p_1 e^{-\xi}$.

This work can be extended for secure wireless networks which adopt m-composite key predistribution schemes. In the m-composite key predistribution scheme, the key pool contains K distinct keys which are randomly chosen from the key space, and a key ring is composed of k distinct keys drawn from the key pool. Before deployed, each node randomly loads k distinct keys drawn from the key pool, which is called a key ring, into its memory. After deployed, two nodes within each other's transmission range have a secure link if their key rings have at least m common keys. A node is said to be *isolated* if it doesn't have a secure link.

Let q_i denote the probability of the event that two key rings have exactly i common keys. If two key rings have exactly i common keys, the second one contains i keys from the k keys of the first one and $k-i$ keys from the remaining $K-k$ keys not of the first one. Therefore,

$$q_i = \frac{\binom{k}{i}\binom{K-k}{k-i}}{\binom{K}{k}}.$$

Let p denote the probability of the event that two nodes (or key rings) have at least m common keys and q denote the probability of the event that two key rings have at most $m-1$

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common keys. Then,

$$\begin{aligned} q &= q_0 + q_1 + \cdots + q_{m-1} \\ p &= 1 - q \end{aligned} \tag{1}$$

We have the following theorem about the total number of isolated nodes in the secure wireless network.

Theorem 2: In m-composite key predistribution schemes, let p be given by Eq. (1). If $\lim_{n\to\infty} p \ln n = \infty$ and nodes have the same maximum transmission radius $r = \sqrt{\frac{\ln n + \xi}{\pi p n}}$ for some constant ξ, then the total number of isolated nodes is asymptotically Poisson with mean $e^{-\xi}$.

III. Preliminaries

We adopt notations and terminologies used in [2]. For completeness, we also give their definitions here. Most lemmas in this section also can be found corresponding ones in [2] but with some extension.

Let r be the transmission radius of the nodes. For any finite set of nodes $\{x_1, \cdots, x_k\}$ in Ω, we use $G_r(x_1, \cdots, x_k)$ to denote the r-disk graph over $\{x_1, \cdots, x_k\}$ in which there is an edge between two nodes if and only if their distance is at most r. For any positive integers k and m with $1 \le m \le k$, let C_{km} denote the set of $(x_1, \cdots, x_k) \in \Omega^k$ satisfying that $G_{2r}(x_1, \cdots, x_k)$ has exactly m connected components. For any set $S \subseteq \Omega$ and $r > 0$, the r-neighborhood of S is the set $\bigcup_{x\in S} B(x, r) \cap \Omega$. We use $v_r(S)$ to denote the area of the r-neighborhood of S, and sometimes by slightly abusing the notation, to denote the r-neighborhood of S itself.

In the remaining of this section, we give the limits of several integrals. Similar lemmas can be found in [2].

Lemma 3: If $\lim_{n\to\infty} p \ln n = \infty$ and $r = \sqrt{\frac{\ln n+\xi}{\pi p n}}$ for some constant ξ, then

$$n \int_{\Omega} (1 - p v_r(x))^{n-1} dx \sim e^{-\xi}.$$

Lemma 4: If $\lim_{n\to\infty} p \ln n = \infty$ and $r = \sqrt{\frac{\ln n+\xi}{\pi p n}}$ for some constant ξ, then for any fixed integer $k \ge 2$,

$$n^k \int_{C_{k1}} (1 - p v_r(x_1, x_2, \cdots, x_k))^{n-k} \prod_{i=1}^{k} dx_i = o(1).$$

Lemma 5: Let $\lim_{n\to\infty} p \ln n = \infty$ and $r = \sqrt{\frac{\ln n+\xi}{\pi p n}}$ for some constant ξ. Then for any fixed integers $2 \le m < k$.

$$n^k \int_{C_{km}} (1 - p v_r(x_1, x_2, \cdots, x_k))^{n-k} \prod_{i=1}^{k} dx_i = o(1).$$

Lemma 6: Let $\lim_{n\to\infty} p \ln n = \infty$ and $r = \sqrt{\frac{\ln n+\xi}{\pi p n}}$ for some constant ξ. Then for any fixed integer $k \ge 2$,

$$n^k \int_{C_{kk}} (1 - p v_r(x_1, x_2, \cdots, x_k))^{n-k} \prod_{i=1}^{k} dx_i \sim e^{-k\xi}.$$

IV. Asymptotic Distribution of The Number of Isolated Nodes

Theorem 1 and 2 will be proved by using *Brun's sieve* in the form described, for example, in [14], Chapter 8, which is an implication of the Bonferroni inequalities.

Theorem 7: Let $B_1, \cdots, B_n$ be events and Y be the number of B_i that hold. Suppose that for any set $\{i_1, \cdots, i_k\} \subseteq \{1, \cdots, n\}$

$$\Pr(B_{i_1} \wedge \cdots \wedge B_{i_k}) = \Pr(B_1 \wedge \cdots \wedge B_k),$$

and there is a constant μ so that for any fixed k

$$n^k \Pr(B_1 \wedge \cdots \wedge B_k) \sim \mu^k.$$

Then Y is also asymptotically Poisson with mean μ.

A. Networks with Bernoulli Nodes and Links

In the Bernoulli model, for applying Theorem 7, let B_i be the event that X_i is isolated for $1 \le i \le n$ and Y be the number of B_i that hold. Then Y is exactly the number of isolated nodes. Similarly, let B_i' be the event that X_i is isolated and active for $1 \le i \le n$ and Y' be the number of B_i' that hold. Then Y' is exactly the number of isolated active nodes. Obviously, for any set $\{i_1, \cdots, i_k\} \subseteq \{1, \cdots, n\}$,

$$\begin{aligned} \Pr(B_{i_1} \wedge \cdots \wedge B_{i_k}) &= \Pr(B_1 \wedge \cdots \wedge B_k), \\ \Pr\left(B_{i_1}' \wedge \cdots \wedge B_{i_k}'\right) &= \Pr(B_1' \wedge \cdots \wedge B_k'). \end{aligned}$$

In addition,

$$\Pr(B_1' \wedge \cdots \wedge B_k') = (p_1)^k \Pr(B_1 \wedge \cdots \wedge B_k).$$

Thus, in order to prove Theorem 1, it suffices to show that if $r = \sqrt{\frac{\ln n+\xi}{\pi p_1 p_2 n}}$ for some constant ξ, then for any fixed k,

$$n^k \Pr(B_1 \wedge \cdots \wedge B_k) \sim e^{-k\xi}. \tag{2}$$

The proof of this asymptotic equality will use the following two lemmas. For convenience, let $q_1 = 1 - p_1$ and $q_2 = 1 - p_2$.

Lemma 8: For any $x \in \Omega$,

$$\Pr(B_1 \mid X_1 = x) = (1 - p_1 p_2 v_r(x))^{n-1}.$$

Proof: For any $x \in \Omega$, let N_1 and N_2 denote the number of active nodes and the number of inactive nodes of $X_2, \cdots, X_n$ within $v_r(X_1)$ respectively. There are exactly N_1 links between X_1 and those N_1 active nodes. If X_1 are isolated, all of those N_1 links must be down. So

$$\begin{aligned} &\Pr(B_1 \mid N_1 = i, N_2 = j) \\ &= \Pr\left(\begin{array}{l} \text{all links of } X_1 \text{ to active} \\ \text{nodes are down} \end{array} \middle| \begin{array}{l} N_1 = i, \\ N_2 = j \end{array} \right) \\ &= (q_2)^i, \end{aligned}$$

and

$$\begin{aligned} &\Pr(N_1 = i, N_2 = j \mid X_1 = x) \\ &= \binom{n-1}{i, j} (1 - v_r(x))^{n-1-i-j} (p_1 v_r(x))^i (q_1 v_r(x))^j. \end{aligned}$$

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Thus

$$\begin{aligned}&\Pr(B_1 \mid X_1 = x)\\&= \sum_{i+j=0}^{n-1} \begin{array}{l}\Pr(B_1 \mid N_1 = i, N_2 = j)\cdot\\ \Pr(N_1 = i, N_2 = j \mid X_1 = x)\end{array}\\&= \sum_{i+j=0}^{n-1} \begin{array}{l}(q_2)^i \binom{n-1}{i,j}(1-v_r(x))^{n-1-i-j}\cdot\\ (p_1 v_r(x))^i (q_1 v_r(x))^j\end{array}\\&= (1-p_1p_2v_r(x))^{n-1}.\end{aligned}$$

Therefore, the lemma is proved. ■

Lemma 9: For any $k \geq 2$ and $(x_1, \cdots, x_k) \in \Omega^k$,

$$\begin{aligned}&\Pr(B_1 \wedge \cdots \wedge B_k \mid X_i = x_i, 1 \leq i \leq k)\\&\leq (1-p_1p_2v_r(x_1,\cdots,x_k))^{n-k}.\end{aligned}$$

In addition, the equality is achieved for $(x_1, \cdots, x_k) \in C_{kk}$.

Proof: For any $(x_1, \cdots, x_k) \in \Omega^k$, let N_1 and N_2 be the number of active nodes and the number of inactive nodes of $X_{k+1}, \cdots, X_n$ within $v_r(X_1, \cdots, X_k)$ respectively. There are at least N_1 links between $X_1, \cdots, X_k$ and those N_1 active nodes. If $X_1, \cdots, X_k$ are isolated, all of those links must be down. So

$$\begin{aligned}&\Pr(B_1 \wedge \cdots \wedge B_k \mid N_1 = i, N_2 = j)\\&= \Pr\left(\begin{array}{l}\text{links of } X_1,\cdots,X_k \text{ to}\\ \text{active nodes are down}\end{array}\middle| \begin{array}{l}N_1 = i,\\ N_2 = j\end{array}\right)\\&\leq (q_2)^i.\end{aligned}$$

Thus,

$$\begin{aligned}&\Pr(B_1 \wedge \cdots \wedge B_k \mid X_i = x_i, 1 \leq i \leq k)\\&= \sum_{i+j=0}^{n-k} \begin{array}{l}\Pr(B_1 \wedge \cdots \wedge B_k \mid N_1 = i, N_2 = j)\cdot\\ \Pr(N_1 = i, N_2 = j \mid X_i = x_i \text{ for } 1 \leq i \leq k)\end{array}\\&\leq \sum_{i+j=0}^{n-k} \begin{array}{l}(q_2)^i \binom{n-k}{i,j}(1-v_r(x_1,\cdots,x_k))^{n-k-i-j}\cdot\\ (p_1v_r(x_1,\cdots,x_k))^i (q_1v_r(x_1,\cdots,x_k))^j\end{array}\\&= (1-p_1p_2v_r(x_1,\cdots,x_k))^{n-k}.\end{aligned}$$

For any $(x_1, \cdots, x_k) \in C_{kk}$,

$$\begin{aligned}&\Pr(B_1 \wedge \cdots \wedge B_k \mid X_i = x_i, 1 \leq i \leq k)\\&= \Pr\left(\begin{array}{l}\forall 1 \leq i \leq k,\ X_i \text{ has no up links}\\ \text{to active nodes of } X_{k+1},\cdots,X_n\end{array}\right)\\&= \sum_{m_1+\cdots+m_k=0}^{n-k} \Pr\left(\begin{array}{l}\forall 1 \leq i \leq k,\ v_r(x_i) \text{ contains}\\ m_i \text{ active nodes, } m_i' \text{ inactive}\\ \text{nodes, and links of } X_i \text{ to}\\ \text{active nodes are down}\end{array}\right)\\&= \sum_{\substack{m_1+\cdots+m_k+\\ m_1'+\cdots+m_k'=0}}^{n-k} \binom{n-k}{m_1,\cdots,m_k,m_1',\cdots,m_k'}\\&\quad\cdot\left(\prod_{i=1}^{k}(q_2p_1v_r(x_i))^{m_i}\right)\left(\prod_{i=1}^{k}(q_1v_r(x_i))^{m_i'}\right)\\&\quad\cdot(1-v_r(x_1,\cdots,x_k))^{n-k-\sum_{i=1}^{k}(m_i+m_i')}\\&= (1-p_1p_2v_r(x_1,\cdots,x_k))^{n-k}.\end{aligned}$$

Therefore, the lemma is proved. ■

Now we are ready to prove the asymptotic equality (2). From Lemma 8 and Lemma 3,

$$n\Pr(B_1) = n\int_{\Omega}(1-p_1p_2v_r(x))^{n-1}dx \sim e^{-\xi}.$$

So the asymptotic equality (2) is true for $k = 1$. Now we fix $k \geq 2$. From Lemma 9, Lemma 4 and Lemma 5,

$$\begin{aligned}&n^k\Pr\left(B_1 \wedge \cdots \wedge B_k \text{ and } (X_1,\cdots,X_k) \in \Omega^k \setminus C_{kk}\right)\\&\leq n^k\int_{\Omega^k\setminus C_{kk}}(1-p_1p_2v_r(x_1,\cdots,x_k))^{n-k}\prod_{i=1}^{k}dx_i\\&= o(1).\end{aligned}$$

From Lemma 9 and Lemma 6,

$$\begin{aligned}&n^k\Pr(B_1 \wedge \cdots \wedge B_k \text{ and } (X_1,\cdots,X_k) \in C_{kk})\\&= n^k\int_{C_{kk}}(1-p_1p_2v_r(x_1,\cdots,x_k))^{n-k}\prod_{i=1}^{k}dx_i\\&\sim e^{-k\xi}.\end{aligned}$$

Thus, the asymptotic equality (2) is also true for any fixed $k \geq 2$. This completes the proof of Theorem 1.

B. Secure Wireless Networks

In secure wireless networks, for applying Theorem 7, let B_i be the event that X_i is isolated for $1 \leq i \leq n$ and Y be the number of B_i that hold. Then Y is exactly the number of isolated nodes. Obviously, for any set $\{i_1, \cdots, i_k\} \subseteq \{1, \cdots, n\}$,

$$\Pr(B_{i_1} \wedge \cdots \wedge B_{i_k}) = \Pr(B_1 \wedge \cdots \wedge B_k).$$

Thus, in order to prove Theorem 2, it suffices to show that if $r = \sqrt{\frac{\ln n + \xi}{\pi p n}}$ for some constant ξ, then for any fixed k,

$$n^k\Pr(B_1 \wedge \cdots \wedge B_k) \sim e^{-k\xi}. \quad (3)$$

The proof of this asymptotic equality will use the following two lemmas. For convenience, let $q = 1 - p$. (Here p is the probability of the event that two key rings have at least m common keys.)

Lemma 10: For any $x \in \Omega$,

$$\Pr(B_1 \mid X_1 = x) = (1 - pv_r(x))^{n-1}.$$

Proof: For any $x \in \Omega$, let N denote the number of nodes of $X_2, \cdots, X_n$ within $v_r(X_1)$. If X_1 is isolated, all X_1's neighbors may have at most $m-1$ keys that are also in the key ring of X_1. For X_1's neighbors, the event is independent

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and identical. Thus,

$$\begin{aligned}&\Pr\left(B_1 \mid X_1 = x\right)\\&= \sum_{i=0}^{n-1} \Pr\left(X_1 \text{ is isolated} \mid N = i\right) \Pr\left(N = i \mid X_1 = x\right)\\&= \sum_{i=0}^{n-1} q^i \binom{n-1}{i} \left(1 - v_r\left(x\right)\right)^{n-1-i} v_r\left(x\right)^i\\&= \left(1 - v_r\left(x\right) + q v_r\left(x\right)\right)^{n-1} = \left(1 - p v_r\left(x\right)\right)^{n-1}.\end{aligned}$$

Therefore, the lemma is proved. ■

Lemma 11: For any $k \geq 2$ and $(x_1, \cdots, x_k) \in \Omega^k$,

$$\begin{aligned}&\Pr\left(B_1 \wedge \cdots \wedge B_k \mid X_i = x_i, 1 \leq i \leq k\right)\\&\leq \left(1 - p v_r\left(x_1, \cdots, x_k\right)\right)^{n-k}.\end{aligned}$$

In addition, the equality is achieved for $(x_1, \cdots, x_k) \in C_{kk}$.

Proof: For any $(x_1, \cdots, x_k) \in \Omega^k$, let N denote the number of nodes of $X_{k+1}, \cdots, X_n$ within $v_r(X_1, \cdots, X_k)$. Each of those N nodes is neighbor to at least one of $X_1, \cdots, X_k$, but the link is not secured. Therefore, we have $\Pr(B_1 \wedge \cdots \wedge B_k \mid N = i) \leq q^i$. Thus,

$$\begin{aligned}&\Pr\left(B_1 \wedge \cdots \wedge B_k \mid X_i = x_i, 1 \leq i \leq k\right)\\&= \sum_{i=0}^{n-k} \begin{array}{l}\Pr\left(B_1 \wedge \cdots \wedge B_k \mid N = i\right) \cdot\\ \Pr\left(N = i \mid X_i = x_i \text{ for } 1 \leq i \leq k\right)\end{array}\\&\leq \sum_{i=0}^{n-k} \begin{array}{l}q^i \binom{n-k}{i} \left(1 - v_r\left(x_1, \cdots, x_k\right)\right)^{n-k-i}\\ v_r\left(x_1, \cdots, x_k\right)^i\end{array}\\&= \left(1 - v_r\left(x_1, \cdots, x_k\right) + q v_r\left(x_1, \cdots, x_k\right)\right)^{n-k}\\&= \left(1 - p v_r\left(x_1, \cdots, x_k\right)\right)^{n-k}.\end{aligned}$$

For any $(x_1, \cdots, x_k) \in C_{kk}$, each of those N nodes has exactly one neighbor among $X_1, \cdots, X_k$. Therefore, we have $\Pr(B_1 \wedge \cdots \wedge B_k \mid N = i) = q^i$ and

$$\begin{aligned}&\Pr\left(B_1 \wedge \cdots \wedge B_k \mid X_i = x_i, 1 \leq i \leq k\right)\\&= \left(1 - p v_r\left(x_1, \cdots, x_k\right)\right)^{n-k}.\end{aligned}$$

Therefore, the lemma is proved. ■

The asymptotic equality (3) can be proved by applying the same argument used for the Bernoulli model but replacing Lemma 8 and 9 by Lemma 10 and 11. Thus, we complete the proof of Theorem 2.

V. Conclusions

In this paper, the connectivity of wireless networks in which nodes and links are not reliable is investigated by the distribution of the number of isolated nodes in the networks. We assume a wireless network is composed of a collection of wireless sensors represented by a uniform n-point process over the unit-area disk or square. In the Bernoulli model, nodes are active independently with probability $0 < p_1 \leq 1$, and links are up independently with probability $0 < p_2 \leq 1$. We show that if all nodes have the same transmission radius $r_n = \sqrt{\frac{\ln n + \xi}{\pi p_1 p_2 n}}$ for some constant ξ, then the total number of isolated nodes is asymptotically Poisson with mean $e^{-\xi}$ and the total number of isolated active nodes is also asymptotically Poisson with mean $p_1 e^{-\xi}$. In the m-composite key predistribution schemes, let p denote the probability of the event that two neighbor nodes have a secure link. We show that if all nodes have the same transmission radius $r_n = \sqrt{\frac{\ln n + \xi}{\pi p n}}$ for some constant ξ, then the total number of isolated nodes is asymptotically Poisson with mean $e^{-\xi}$. The problem whether vanishment of isolated nodes almost surely implies connectivity of networks or not is still open.

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