Optimal Placement of Wavelength Converters in Trees and Trees of Rings*

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Abstract

In wavelength routed optical networks, wavelength converters can potentially reduce the requirement on the number of wavelengths. The problem of placing a minimum number of wavelength converters in a WDM network so that any routing *can* be satisfied using no more wavelengths than if there were wavelength converters at every node was raised in [16] and shown to be NP-complete in general WDM networks. Recently, it was proved in [8] that this problem is as hard as the well-known minimum vertex cover problem. In this paper, we further their study in two topologies that are of more practical concrete relevance to the telecommunications industry: trees and tree of rings. We show that the optimal wavelength converter placement problem in these two practical topologies are tractable. Efficient polynomial-time algorithms are presented.

Keywords: Wavelength routed optical network, wavelength converter, minimum vertex cover, tree, tree of rings.

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1 Introduction

In wavelength routed WDM (wavelength-division multiplexed) optical networks [11] without any wavelength conversion [12], the wavelength assignment must meet the wavelength continuity constraint, i.e., the same wavelength is allocated on all of the links in the path established for a connection [1, 2, ?, 10, 13, 15]. Such constraint can be relaxed when wavelength converters are placed at certain nodes. If a node of the network contains a wavelength converter, any path that passes through this node may change its wavelength. In a network with wavelength converters, the wavelengths are assigned to individual links of all paths, with the restriction that the same wavelength is allocated on all of the links in any subpath that does not pass through a wavelength converter. Clearly wavelength assignments in networks with wavelength converters can sometimes be more efficient (i.e. use fewer wavelengths) than optimal wavelength assignments for the same set of paths when no wavelength converters are available. One extreme example is that if each node contains a wavelength converter, the number of wavelengths required for any routing is reduced down to the natural con*gestion* or load bound, defined to be the maximum number of paths passing through any one link in the network. Another extreme example is that placing a converter at a single arbitrary node in a WDM ring is sufficient to ensure that the number of wavelengths required for any routing is equal to its load [16].

Motivated by this notion, the following question was raised in [16]: Where should the wavelength converters be placed in a WDM network so that any routing can be satisfied using no more wavelengths than if there were wavelength converters at every node? A set S of nodes in a network is defined in [16] to be *sufficient* if, placing converters at the nodes in S, every set of paths can be routed with a number of wavelengths equal to its congestion bound. The *minimum sufficient set problem was* shown to be NP-complete in [16] in general WDM networks. Recently a breakthrough was made in [8] by establishing a tight connection between the minimum sufficient set problem in bi-directed graphs and the minimum vertex cover *problem* in undirected graphs. As a consequence of this connection, a simple 2-approximation algorithm for minimum sufficient set problem in bi-directed graphs was obtained. Furthermore, it's easy to give an approximation-preserving reduction from the minimum vertex cover problem to the minimum sufficient set problem in bi-directed graphs [16]. Since providing an approximation ratio better than 2 for the minimum vertex cover problem is a long-standing open problem, this indicates that improving on the performance guarantee of two for minimum sufficient set problem will be difficult as well.

While the work in [8] provided approximation solutions for general WDM networks, we notice that the topologies of most practical WDM networks are not general. In particular, trees and tree of rings are of more practical concrete relevance to the telecommunications industry. For practical reasons, backbone telecommunication networks need to reflect irregularity of geography, non-uniform clustering of users and traffic, hierarchy of services, dynamic growth, etc. In addition, wide-area multiwavelength technology is evolving around current signal wavelength networking architectures and existing fiber networks. These are mainly SONET rings and tree-like interconnection of such rings [4, 3, 14]. In this paper, we will show that the minimum sufficient set problem in these special topologies can be solved in polynomial time and therefore is not NP-complete. Our algorithms to find the minimum sufficient sets in these topologies are based on the reduction of the minimum sufficient set problem to the minimum vertex cover problem established in [8]. These algorithms are very efficient and easy to implement.

The remaining of this paper is arranged as follows. In Section 2, we first introduce some basic terminologies and the reduction from the minimum sufficient set problem to the minimum vertex cover problem. In Section 3, we present a polynomial-time algorithm which finds a minimum sufficient set problem in tree networks. In Section 4, we present a polynomial-time algorithm which finds a minimum sufficient set problem in trees of rings. Finally Section 5 summarizes this paper.

2 Preliminaries

A WDM network in this paper is a bi-directed graph G = (V, E): one for which $(u, v) \in E$ if and only if $(v,u) \in E$. Let $G_s = (V, E')$ denote the *skeleton* of the network G, the undirected graph obtained from G by replacing each bi-directed pair of edges with a single undirected edge. By partial abuse of terminology, we will say a set is sufficient in G_s if and only if it is sufficient in G. A vertex v is referred to as a branching *node* if its degree in G_s is greater than 2, a relay node if its degree in G_s is equal to 2, or a leaf *node* if its degree in G_s is equal to 1. We will assume that G_s is connected and contains at least one branching node, since otherwise G_s is either a path or a cycle, and the minimum sufficient set can be solved trivially. We say that a node of a path P is an *internal node* in this path if it is not one of the two endpoints.

From the graph G_s, we construct another undirected graph $G_c = (V_c, E_c)$, referred to as the *contraction* of the graph G_s, as follows: V_c consists of all branching nodes in G,. For any two branching nodes u and v, (u, v) is an edge in E_c if and only if there exists a path in G_s between u and v such that all internal nodes in this path are relay nodes. Note that G_c may have self-loops, which we retain as part of the graph. The following lemma establishes the connection between the minimum sufficient set in G and the minimum vertex cover in G_c.

Lemma 1 *[8] Any minimum vertex cover* of G_c is also a minimum *sufficient set* of **G.**

A consequence of Lemma 1 is that in order to find a minimum sufficient set in a graph G, one only has to find a minimum vertex cover of its contraction, a potentially simpler undirected graph. In general, the minimum vertex cover problem is NP-complete and it has a 2-approximation algorithm. But it is fixed-parameter tractable: whether a graph has a vertex cover of size at most k can be decided with time $O(f(k) \cdot p(n))$ (see e.g. [7]) where p is a polynomial function. If the graphs are planar, a polynomial-time approximation scheme exists [5, 6]. However, as will show in this paper, when the graph is a tree or a tree of rings, a minimum vertex cover can be found in polynomial time.

3 Minimum Sufficient Set in Trees

In this section, we consider the WDM networks whose underlying topologies are trees. According to Lemma 1, any minimum vertex cover of the contraction tree is also a minimum sufficient set of the original network. As the contraction of any tree is also a tree with the additional property that all internal nodes have degrees of at least three, we only have to identify a minimum vertex cover of the contraction tree. In the next we will present a general polynomial-time algorithm to find minimum vertex covers in forests, a broader set of topologies than trees.

We call an internal node of a forest to be a *leaf-root* if its nodal degree is more than one and one of its neighbors is a leaf node. If a forest has no leaf-root then all edges are isolated and its minimum vertex cover consists of one node from each edge. If a forest contains some leaf-roots, then the next lemma indicates that there is a minimum vertex cover which contains all the leaf-roots.

Lemma 2 *Let G be* a forest. *Then there exists a* minimum *vertex* cover *of G which contains all leaf-roots of* **G.**

Proof. We prove the lemma by contradiction. Assume the lemma were not true. Let C be a minimum vertex cover of G which contains the *most* number of leaf-roots of G, and let u be any leaf-roots of G that is not in C. Let v be any leaf node that is a neighbor of u. Clearly, v must be in C, for otherwise the edge (u, v) would not be covered by C. Consider

$$\mathcal{C}' = (\mathcal{C} - \{v\}) \cup \{u\}.$$

Then $\mathcal{C}'$ is also *a* minimum vertex cover of G, and contains one more leaf-roots of G than C. This contradicts the selection of C. Thus the lemma is true. •

Let G be any forest and $\mathcal{C}$ be any minimum vertex cover of G which contains all leaf-roots of G. Let G' be the graph obtained from G by removing all leaf-roots of G and their incident edges. G' is referred to as the residue of G. Let C' be the vertex set obtained from C by removing all leaf-roots of G. Then G' is also a forest and C' is a minimum vertex cover of G'. Thus we can apply Lemma 2 to G' to get another minimum vertex cover C" which contains all leaf-roots of G'. Based on this observation, we have the following recursive algorithm to find a minimum vertex cover of a forest described in Table 1.

In the algorithm **MVC_Forest**, we select all leaf-roots in a single recursive step. Another variation is to select only one leaf-root in a step. Such leaf-node can be chosen such that the removal of it and its incident edges from a tree G results in another tree. Such selection could potentially simplify the implementation. For both approaches, suitable data structures must be carefully designed in order to achieve faster running time. The details are omitted over here.

Algorithm: MVC_Forest
Input: *a forest* **G;**
Output: a minimum *vertex cover of* **G;**
begin
  if G has no leaf *root*
    return *the set consisting of one* node *from each edge;*
  else
    add all leaf-roots of G to *the vertex* cover *of* **G;**
    orm the residue *graph G'of G;*
    find the minimum vertex cover of G' recursively;
    add them to the vertex cover of **G;**
end

Table 1: Recursive algorithm to find a minimum vertex cover of a forest.

4 Minimum Sufficient Set in Trees of Rings

The tree of rings, illustrated in Figure 1, is a widely-used interconnection topology in the telecommunications industry. In this topology, each node is within a ring and these rings are interconnected via a tree-like topology. It's easy to see that the contraction of any tree of rings is also a tree of ring. As the minimum vertex cover of the contraction graph provides an optimal sufficient set of the original graph, we will provide a polynomial-time algorithm that finds the minimum vertex cover in an arbitrary tree of rings.

It's well-known that there are at least two leaf-nodes in any tree. Similarly, one can show that in any tree of rings, there exist at least two rings in which all nodes have degree of two except one whose degree is three. Such rings are referred to as leaf-rings. The only node in a leaf-ring whose degree is three is called a *bridging-node.* Suppose that a leaf-ring contains m nodes. Then any minimum vertex cover contains at least $\lceil \frac{m}{2} \rceil$ nodes in this leaf-ring. In the next we show that there is always a minimum vertex cover which contains the bridging node of any leaf ring.

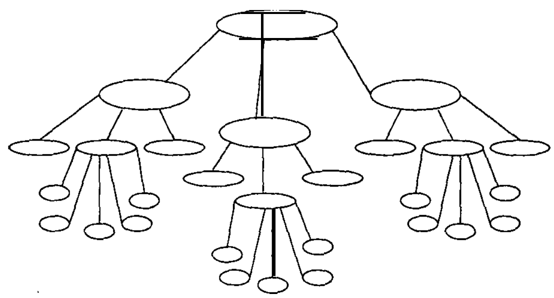

Figure 1: An example of tree of rings.

Lemma 3 *Let G be* a tree *of* rings. *Then* there ex*ists a* minimum vertex *cover of G which contains all bridging-nodes of all leaf-rings in G.*

Proof. We prove the lemma by contradiction. Assume the lemma were not true. Let C be a minimum vertex cover of G which contains the *most* number of bridging-nodes of G, and let u be any bridging-nodes of G that is not in C. Let R be the leaf-ring containing the bridging-node u. Then R can't be a self-loop, for otherwise u must be in any vertex cover of G and thus in C too. So R contains m $\geq$ 2 nodes, say $u = v_1, v_2, \cdots, v_m$ in the clockwise order. Clearly C must contain at least $\lceil \frac{m}{2} \rceil$ nodes in R, i.e.,

$$|\mathcal{C} \cap R| \geq \left\lceil \frac{m}{2} \right\rceil .$$

Consider

$$\mathcal{C}' = (\mathcal{C} - R) \cup \{v_i : 1 \leq i \leq m, i \text{ is odd}\} .$$

Then C' is also a vertex cover of G, and

$$\begin{aligned} |\mathcal{C}'| &= |\mathcal{C} - R| + \left\lceil \frac{m}{2} \right\rceil \\ &= |\mathcal{C}| - |\mathcal{C} \cap R| + \left\lceil \frac{m}{2} \right\rceil \\ &\leq |\mathcal{C}| . \end{aligned}$$

Thus C' is also a minimum vertex cover of G. On the other hand, C' contains one more bridging-nodes of G than C. This contradicts the selection of C. Thus the lemma is true. ■

The proof of Lemma 3 suggests a minimum vertex covering of a leaf-ring. Suppose that a leaf-ring R contains more m nodes, say $v_1, v_2, \ldots, v_m$ in the clockwise order in which v_1 is its bridging node. The set of vertices

$$\{v_i : 1 \leq i \leq m, i \text{ is odd}\}$$

is called a canonical vertex cover of the leaf-ring R. Then Lemma 3 indicates that for any leaf-ring R, there is a minimum vertex cover C of G which contains its canonical vertex cover. Let $G - R$ denote the graph obtained from G by removing all node in R and their incident edges. Let C -R denote the vertex set obtained from C by removing the canonical vertex cover of R. Then $G - R$ is also a tree of rings and $C - R$ is a minimum vertex cover of G'. On the other hand, the union of the canonical vertex cover of R and any minimum vertex cover of $G - R$ is also a minimum vertex cover of G. Based on this observation, we have the following incremental algorithm to find the minimum vertex cover of a tree of rings listed in Table 2.

```
Algorithm: MVC_Tree_Rings
Input: a tree of rings G;
Output: a minimum vertex cover of G;
begin
  C = 0; // the vertex cover of G
  while G is not empty
    find a leaf-ring R in G;
    add the canonical vertex cover of R to C;
    G = G - R ;
  output c;
end
```

Table 2: Recursive algorithm to find a minimum vertex cover of a tree of rings.

The algorithm **MVC_Tree_Rings** removes one leaf-ring at each incremental step and add its canonical vertex cover. With carefully selected data structures and implementation, its run-time can be linear in the network size. The details are omit over here.

From the algorithm **MVC_Tree_Rings**, **we** can explicitly count the cardinality of any minimum vertex cover of a tree of rings.

Theorem 4 *Let* $G = (V, E)$ *be a tree of rings, and* k *be the number of odd-sized rings in* G. *Then the cardinality of any minimum vertex cover of* G *is* $\frac{|V|+k}{2}$.

Proof. Let $R_1, R_2, \cdots, R_\ell$ be the component rings in G. According to our above algorithm, each ring will eventually become a leaf-ring and its canonical vertex cover will be added to the minimum vertex cover of G. Note that the canonical vertex covers of different component rings are disjoint. Thereby, the cardinality of any minimum vertex cover of G is

$$\begin{aligned}\sum_{i=1}^{\ell} \left\lceil \frac{|R_i|}{2} \right\rceil &= \sum_{i:|R_i| \text{ is even}} \frac{|R_i|}{2} + \sum_{i:|R_i| \text{ is odd}} \frac{|R_i|+1}{2} \\ &= \sum_{i=1}^{\ell} \frac{|R_i|}{2} + \frac{k}{2} \\ &= \frac{1}{2}\sum_{i=1}^{\ell} |R_i| + \frac{k}{2} \\ &= \frac{|V|}{2} + \frac{k}{2} \\ &= \frac{|V|+k}{2}.\end{aligned}$$

■

As a corollary of Theorem 4, the cardinality of any minimum sufficient set is at least half of the number of branching-nodes. In order to reduce the number of converters needed, the topology should be carefully designed. For an example, we can choose the topology such that the number of branching nodes in each component ring to be even.

5 Summary

The minimum sufficient set problem is in general NP-complete and is as hard as the minimum vertex cover problem. However, the underlying topologies of most WDM networks in the telecommunications industry are built around trees, rings, and trees of rings. For these topologies, this paper showed that both minimum sufficient problem and the minimum vertex cover problem can be solved in polynomial time. Efficient algorithms have been provided for the minimum vertex cover problem in these topologies which in turn are used to solve the minimum sufficient set problem in these topologies.

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