The 28th International Conference on Distributed Computing Systems

On The Longest RNG Edge of Wireless Ad Hoc Networks

Peng-Jun Wan*, Lixin Wang*, Frances Yao† and Chih-Wei Yi‡
*Department of Computer Science, Illinois Institute of Technology
Emails: wan@cs.iit.edu, wanglix@iit.edu
†Department of Computer Science, City University of Hong Kong
Email: csfyao@cityu.edu.hk
‡Department of Computer Science, National Chiao Tung University
Email: yi@cs.nctu.edu.tw

***Abstract*— Relative neighborhood graph (RNG) has been widely used in topology control and geographic routing in wireless ad hoc networks. Its maximum edge length is the minimum requirement on the maximum transmission radius by those applications of RNG. In this paper, we derive the precise asymptotic probability distribution of the maximum edge length of the RNG on a Poisson point process over a unit-area disk. Since the maximum RNG edge length is a lower bound on the critical transmission radius for greedy forward routing, our result also leads to an improved asymptotic almost sure lower bound on the critical transmission radius for greedy forward routing.**

I. INTRODUCTION

Relative neighborhood graph (RNG) of a finite planar set was originally introduced first by [15] with applications in pattern recognition. It is a bounded-degree planar graph containing the Euclidean minimum spanning tree as a subgraph. Due to its simple construction and maintenance, RNG has found many applications in localized topology control (e.g., [6], [8], [9]) and geographic routing (e.g., [1], [7], [13]) in wireless ad hoc networks. All these applications require the maximum transmission radius of the networking nodes be no shorter than the longest edge in the RNG. While the maximum edge length in the RNG can be computed in polynomial time, little is known about its random behavior when the underlying vertex is a random point set. In this paper, we derive the precise asymptotic distribution of the maximum edge length in the RNG of a Poisson point process over a unit-area disk with density n, which is denoted by $\mathcal{P}_n$. Denote the maximum edge length of a geometric graph G by $\lambda(G)$, and the RNG of a finite planar set V by $RNG(V)$. Let

$$\beta = 1/\sqrt{\frac{2}{3} - \frac{\sqrt{3}}{2\pi}} \approx 1.6.$$

The main result of this paper is stated in the following theorem.

Theorem 1: For any constant ξ, we have

$$\lim_{n\to\infty} \Pr\left[\lambda(RNG(\mathcal{P}_n)) \leq \beta\sqrt{\frac{\ln n + \xi}{\pi n}}\right] = e^{-\frac{\beta^2}{2}e^{-\xi}}.$$

It is interesting to compare the maximum edge length of the RNG with the maximum edge length of the (Euclidean) minimum spanning tree (MST), which is also known the critical transmission radius for connectivity [5], and the maximum edge length of the Gabriel graph (GG) [4], which also has many applications in wireless ad hoc networks. Let $MST(V)$ and $GG(V)$ denote the MST and the GG of finite planar set V. It's well-known that for any finite planar set,

$$MST(V) \subseteq RNG(V) \subseteq GG(V).$$

Thus,

$$\lambda(MST(V)) \leq \lambda(RNG(V)) \leq \lambda(GG(V)).$$

The asymptotic distributions of $\lambda(MST(\mathcal{P}_n))$ and $\lambda(GG(\mathcal{P}_n))$ were derived in [11] (based on an earlier result [2]) and in [16] respectfully. Specifically, for any constant ξ,

$$\lim_{n\to\infty} \Pr\left[\lambda(MST(\mathcal{P}_n)) \leq \sqrt{\frac{\ln n + \xi}{\pi n}}\right] = e^{-e^{-\xi}},$$
$$\lim_{n\to\infty} \Pr\left[\lambda(GG(\mathcal{P}_n)) \leq 2\sqrt{\frac{\ln n + \xi}{\pi n}}\right] = e^{-2e^{-\xi}}.$$

So roughly speaking, the maximum edge length of the RNG (respectfully, GG) of a Poisson point process is asymptotically about 1.6 times (respectfully, twice) its critical transmission radius for connectivity.

Another parameter closely related to the maximum edge length of the RNG is the critical transmission radius for greedy forward routing [3], [14]. In greedy forward routing, each node discards a packet if none of its neighbors is closer to the destination of the packet than itself, or otherwise forwards the packet to the neighbor closest to the destination of the packet. The critical transmission radius of a planar node set V for greedy forward routing, denoted by $\sigma(V)$, is the smallest transmission radius by V which ensures successful delivery of any packets from any source node in V to any destination node in V. Clearly, $\lambda(RNG(V)) \leq \sigma(V)$. It was recently proved in [17] that for any constant $\varepsilon > 0$, it is asymptotically almost sure (abbreviated by a.a.s.) that

$$(1-\varepsilon)\beta\sqrt{\frac{\ln n}{\pi n}} \leq \sigma(\mathcal{P}_n) \leq (1+\varepsilon)\beta\sqrt{\frac{\ln n}{\pi n}}.$$

1063-6927/08 $25.00 © 2008 IEEE
DOI 10.1109/ICDCS.2008.16

This immediately implies that for any constant $\varepsilon > 0$, it is a.a.s. that

$$\lambda\left(RNG\left(\mathcal{P}_n\right)\right) \leq (1+\varepsilon)\,\beta\sqrt{\frac{\ln n}{\pi n}}.$$

In other words, $(1+\varepsilon)\,\beta\sqrt{\frac{\ln n}{\pi n}}$ is an a.a.s. upper bound on $\lambda\left(RNG\left(\mathcal{P}_n\right)\right)$. While this a.a.s. bound is weaker than Theorem 1, it had inspired us to conjecture and then prove Theorem 1. This a.a.s. bound will also be used in the proof of Theorem 1. As the immediate consequence of Theorem 1, a tighter a.a.s. lower bound on $\sigma\left(\mathcal{P}_n\right)$ can be obtained: Suppose that $\lim_{n\to\infty}\xi_n = \infty$ and $\lim_{n\to\infty}\xi_n/\ln n = 0$. Then it is a.s.s. that

$$\sigma\left(\mathcal{P}_n\right) \geq \beta\sqrt{\frac{\ln n - \xi_n}{\pi n}}.$$

In what follows, $\mathbf{o}$ is origin of the Euclidean plane $\mathbb{R}^2$, and $\mathbb{D}$ is the unit-area (closed) disk centered at $\mathbf{o}$. We assume that $\mathcal{P}_n$ is the Poisson point process over $\mathbb{D}$ with density n. We denote by $\mathcal{X}_n = (X_1, \cdots, X_n)$ the uniform n-point process over $\mathbb{D}$. The symbols $O, o, \sim$ always refer to the limit $n \to \infty$. To avoid trivialities, we tacitly assume n to be sufficiently large if necessary. For simplicity of notation, the dependence of sets and random variables on n will be frequently suppressed. For any set S and positive integer k, the k-fold Cartesian product of S is denoted by S^k. The Euclidean norm of a point x is denoted by $\|x\|$, and the Euclidean distance between two points u and v is denoted by $\|uv\|$. The Lebesgue measure (or area) of a measurable set $A \subset \mathbb{R}^2$ is denoted by $|A|$. The topological boundary of a set $A \subset \mathbb{R}^2$ is denoted by ∂A. The open (respectively, closed) disk of radius r centered at x is denoted by $D(x,r)$ (respectively, $\overline{D}(x,r)$). For any finite planar set V, $K(V)$ denotes the complete (geometric) graph on V which consists of line segments between all pairs of nodes in V.

The remaining of this paper is organized as follows. In Section II, we present several useful geometric results. In Section III, we derive the limits of some relevant integrals. In Section IV, we give the proof for Theorem 1.

II. Geometric Preliminaries

For $x \in \mathbb{D}$, let $t(x)$ denote the distance between x and $\partial\mathbb{D}$, which is equal to $\frac{1}{\sqrt{\pi}} - \|x\|$. For any $0 < \rho < \frac{1}{\sqrt{\pi}}$, define

$$\mathbb{D}_\rho(0) = \left\{x \in \mathbb{D} : t(x) \geq \rho\right\},$$

$$\mathbb{D}_\rho(1) = \left\{x \in \mathbb{D} : \sqrt{\frac{1}{\pi} - \rho^2} \leq t(x) < \rho\right\},$$

$$\mathbb{D}_\rho(2) = \left\{x \in \mathbb{D} : t(x) < \sqrt{\frac{1}{\pi} - \rho^2}\right\}.$$

With this notation, the midpoint of any line segment $xy \subset \mathbb{D}$ is not in $\mathbb{D}_{\|xy\|/2}(2)$. For $x \in \mathbb{D}$ and $0 < \rho < \frac{1}{\sqrt{\pi}}$, define $\theta(x,\rho)$ as follows. If $x \in \mathbb{D}_\rho(0)$, then $\theta(x,\rho) = 2\pi$. If $x \in \mathbb{D}_\rho(2)$, then $\theta(x,\rho) = 0$. If $x \in \mathbb{D}_\rho(1)$, let u and v be the two intersection points of $\partial B(x,\rho)$ and $\partial\mathbb{D}$, and define $\theta(x,\rho) = 2\pi - \angle uxv$ (see Figure 1). We claim that $\rho\theta(x,\rho) \leq 2\pi t(x)$. The claim holds trivially if $x \in \mathbb{D}_\rho(0)$ or $x \in \mathbb{D}_\rho(2)$. So, we consider the case that $x \in \mathbb{D}_\rho(1)$. It's easy to see that $\theta(x,\rho) \leq 4\arcsin\frac{t(x)}{\rho}$. Using the equality $\sin\alpha \geq \frac{2}{\pi}\alpha$ for any $\alpha \in [0,\pi/2]$, we obtain

$$\theta(x,\rho) \leq 4 \cdot \frac{\pi}{2} \cdot \frac{t(x)}{\rho} = \frac{2\pi t(x)}{\rho}.$$

Thus, $\rho\theta(x,\rho) \leq 2\pi t(x)$.

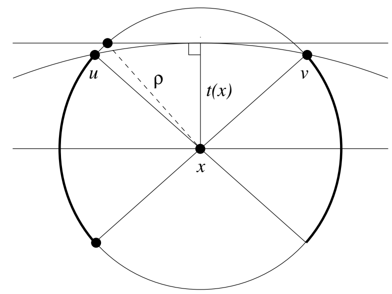

Fig. 1. If $x \in \mathbb{D}_\rho(1)$, then $\theta(x,\rho) = 2\pi - 2\angle uxv$.

The lune of a line segment $e = ab$, denoted by $L(e)$, is the intersection of the disks $D(a,\|ab\|)$ and $D(b,\|ab\|)$; e is called the waist of $L(e)$; the two intersection points of $\partial D(a,\|ab\|)$ and $\partial D(b,\|ab\|)$ are called the vertices of $L(e)$. It's easy to verify that

$$|L(e)| = \pi\|e\|^2/\beta^2.$$

If $e \subset \mathbb{D}$ and the midpoint of e is apart from $\partial\mathbb{D}$ by at least $\frac{\sqrt{3}}{2}\|e\|$, then $L(e) \subset \mathbb{D}$. The next lemma gives a lower bound on $|L(e) \cap \mathbb{D}|$ if otherwise.

Lemma 2: Consider a line segment $e \subset \mathbb{D}$ with midpoint z. If $t(z) \leq \frac{\sqrt{3}}{2}\|e\|$, then

$$|L(e) \cap \mathbb{D}| \geq \frac{1}{2}|L(e)| + \frac{\|e\|}{2}t(z).$$

Proof: Let a and b be the two endpoints of e, and c_1 and c_2 be the two vertices of $L(e)$ with c_1 being farther away from the center of $\mathbb{D}$ (see Figure 2). Then, the half lune abc_2 is fully contained in $\mathbb{D}$. If $c_1 \in \partial\mathbb{D}$, then the triangle abc_1 is contained in $\mathbb{D}$ and its area is $\frac{\|e\|}{2}\|az\| \geq \frac{\|e\|}{2}t(z)$. So, the lemma holds if $c_1 \in \partial\mathbb{D}$. Now assume that $c_1 \notin \partial\mathbb{D}$. Let u be the intersection point of $c_1 z$ and $\partial\mathbb{D}$. Then, the triangle abu is contained in $\mathbb{D}$ and its area is $\frac{\|e\|}{2}\|uz\| \geq \frac{\|e\|}{2}t(z)$. So, the lemma also holds if $c_1 \notin \partial\mathbb{D}$. ■

Two line segments are said to be compatible if the two endpoints of either segment is not contained in the lune of the other segment. The next lemma generalizes Lemma 2 in [17] by taking into account the boundary effect.

Lemma 3: Suppose that e_1 and e_2 are two compatible segments in $\mathbb{D}$ satisfying that $\|e_1\|, \|e_2\| \in [R/2, R]$ for some $R \leq \frac{1}{200\sqrt{\pi}}$. Let z_1 and z_2 be the midpoints of e_1 and e_2 respectfully. If $\|z_1 z_2\| \leq \sqrt{3}R$ and z_1 is farther away from the center of $\mathbb{D}$ than z_2, then

$$|(L(e_2) \setminus L(e_1)) \cap \mathbb{D}| \geq 0.0029R\,\|z_1 z_2\|.$$

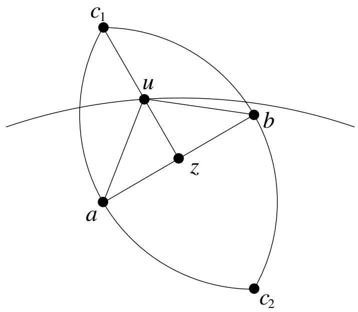

Fig. 2. $L(e) \cap \mathbb{D}$ contains the triangle abc_2 and the half lune abu.

The proof of this lemma is very lengthy and complicated. We omit the proof in this conference version due to the limitation on the space.

For any line segment e, we define

$$\nu(e) = |L(e) \cap \mathbb{D}|.$$

For any geometric graph H, define

$$\nu(H) = |(\cup_{e \in H} L(e)) \cap \mathbb{D}|,$$

and $\chi(H)$ to be indicator for all edges of H are pairwise compatible. An edge $e \in E$ is called an outermost edge of H if its midpoint is the nearest to $\partial\mathbb{D}$. For any finite planar set V and any positive number r, the r-disk graph of V is a geometric graph over V in which there is an edge between two nodes if and only if their distance is at most r. The next lemma generalizes Lemma 3 to the multiple lunes.

Lemma 4: Suppose that H is a geometric graph over a finite subset of $\mathbb{D}$ with at least two edges satisfying that (1) $\chi(H) = 1$, (2) all the edges have length between $R/2$ and R for some $R \leq \frac{1}{200\sqrt{\pi}}$, and (3) the midpoints of its edges induce a connected $\sqrt{3}R$-disk graph. Let e be an outermost edge of H, and ℓ be the largest distance between the midpoint of e and the midpoints of other edges of H. Then,

$$\nu(H) \geq \nu(e) + 0.0029R\ell.$$

Proof: Let e' be the edge of H whose midpoint is the farthest from the midpoint of e. Let $P = z_1 z_2 \cdots z_k$ be the min-hop path between the midpoint z_1 of e and the midpoint z_k of e' in the $\sqrt{3}R$-disk graph over the midpoints of the edges in H. For each $1 \leq i \leq k$, let e_i be the edge of H whose midpoint in z_i. Then, $e_1 = e$ and $e_k = e'$. For each $2 \leq j \leq k$, let H_j denote the subgraph of H consisting of the edges e_i for $1 \leq i \leq j$. We will prove by induction on j with $2 \leq j \leq k$ that

$$\nu(H_j) \geq \nu(e_1) + 0.0029R \sum_{i=1}^{j-1} \|z_i z_{i+1}\|. \tag{1}$$

By Lemma 3, the inequality (1) holds when $j = 2$. Since P is the min-hop path, $\|z_1 z_3\| > \sqrt{3}R$ and $L(e_3)$ is disjoint from $L(e_1)$. Thus,

$$\begin{aligned}
\nu(H_3) &\geq \nu(e_1) + \nu(e_3) \\
&\geq \nu(e_1) + \frac{1}{2}|L(e_3)| \geq \nu(e_1) + \frac{\pi \|e_3\|^2}{2\beta^2} \\
&\geq \nu(e_1) + \frac{\pi R^2}{8\beta^2} \geq \nu(e_1) + \frac{\pi}{16\sqrt{3}\beta^2} R \cdot 2\sqrt{3}R \\
&\geq \nu(e_1) + 0.044R \sum_{i=1}^{2} \|z_i z_{i+1}\|.
\end{aligned}$$

Hence, the inequality (1) holds when $j = 3$. Next, assume that $j > 3$. Since $L(e_j)$ is disjoint from each $L(e_i)$ with $1 \leq i \leq j-2$, we have

$$\nu(H_j) \geq \nu(H_{j-2}) + \nu(e_j).$$

By the induction hypothesis, we have

$$\begin{aligned}
\nu(H_j) &\geq \nu(e_1) + 0.0029R \sum_{i=1}^{j-3} \|z_i z_{i+1}\| + \frac{\pi R^2}{8\beta^2} \\
&\geq \nu(e_1) + 0.0029R \sum_{i=1}^{j-3} \|z_i z_{i+1}\| + 0.044R \cdot 2\sqrt{3}R \\
&> \nu(e_1) + 0.0029R \sum_{i=1}^{j-1} \|z_i z_{i+1}\|.
\end{aligned}$$

Thus, the inequality (1) holds. By the principle of induction, the inequality (1) holds for every $2 \leq j \leq k$.

Since $\nu(H) \geq \nu(H_k)$ and

$$\sum_{i=1}^{k-1} \|z_i z_{i+1}\| \geq \|z_1 z_k\| = \ell,$$

the lemma holds. ■

III. Integral Ingredients

In this section, we derive the asymptotic values of several integrals. We will frequently change the integral variables using a technique introduced in [16]. Consider a tree topology on k planar points $x_1, x_2, \cdots, x_k$, and assume without loss of generality that $x_{k-1}x_k$ is an edge in this tree. Let z_{k-1}, ρ, and ω be the midpoint, half-length and the slope of $x_{k-1}x_k$ respectively. We root the tree at x_k. For $1 \leq i \leq k-2$, let z_i be the midpoint of the edge between x_i and its parent in such rooted tree. Then, we replace $x_1, x_2, \cdots, x_k$ by $z_1, \cdots, z_{k-1}, \rho, \omega$. The Jacobian determinant of this change is $4^{k-1}\rho$.

Fix a constant ξ and a sequence (ξ_n) with $\xi_n = o(\ln n)$ and $\xi_n \to \infty$. Let

$$\begin{aligned}
r_n &= \beta\sqrt{\frac{\ln n + \xi}{\pi n}}, \\
R_n &= \beta\sqrt{\frac{\ln n + \xi_n}{\pi n}}, \\
R'_n &= 1.1\beta\sqrt{\frac{\ln n}{\pi n}}.
\end{aligned}$$

Then, for sufficiently large n,we have $r_n < R_n < R'_n < 2r_n$. Define

$$\Omega = \left\{(x_1, x_2) \in \mathbb{D}^2 : r_n < \|x_1x_2\| \le R_n\right\},$$
$$\Omega' = \left\{(x_1, x_2) \in \mathbb{D}^2 : R_n < \|x_1x_2\| \le R'_n\right\}.$$

Lemma 5: The following are true:

$$\frac{n^2}{2}\int\limits_{\Omega} e^{-nv(x_1x_2)}dx_1dx_2 \sim \frac{\beta^2}{2}e^{-\xi},$$
$$\frac{n^2}{2}\int\limits_{\Omega'} e^{-nv(x_1x_2)}dx_1dx_2 = o(1).$$

Proof: Let $\rho = \rho(x_1, x_2)$ be the half-length of x_1x_2, and $z = z(x_1, x_2)$ be the midpoint of x_1x_2. Let Ω_1 be the set of $(x_1, x_2) \in \Omega$ satisfying that $z \in \mathbb{D}_{\sqrt{3}\rho}(0)$, and let $\Omega_2 = \Omega \setminus \Omega_1$. First, we calculate the integration over Ω_1. If $(x_1, x_2) \in \Omega_1$, $L_{x_1x_2}$ is fully contained in $\mathbb{D}$ and $v(x_1x_2) = \frac{4}{\beta_0}\pi\rho^2$. Changing the integration variable x_1 and x_2 by z, ρ, and the slope of x_1x_2 yields

$$\begin{aligned}
&\frac{n^2}{2}\int\limits_{\Omega_1} e^{-nv(x_1x_2)}dx_1dx_2 \\
&= 4\pi n^2 \int_{\frac{r_n}{2}}^{\frac{R_n}{2}} e^{-\frac{4}{\beta^2}n\pi\rho^2}\rho d\rho \int\limits_{\mathbb{D}_{\sqrt{3}\rho}(0)} dz \\
&\sim 4\pi n^2 \int_{\frac{r_n}{2}}^{\frac{R_n}{2}} e^{-\frac{4}{\beta^2}n\pi\rho^2}\rho d\rho \\
&= -\frac{\beta^2}{2} n e^{-\frac{4}{\beta^2}n\pi\rho^2}\Big|_{\frac{r_n}{2}}^{\frac{R_n}{2}} \sim \frac{\beta^2}{2}e^{-\xi}.
\end{aligned}$$

Next, we calculate the integration over Ω_2. Let $t = t(z)$ be the distance between z and $\partial\mathbb{D}$. By Lemma 2, we have

$$v(x_1x_2) \ge \frac{2}{\beta^2}\pi\rho^2 + \rho t$$

and

$$\rho\theta(z, \rho) \le 2\pi t.$$

Changing the integration variable as above yields

$$\begin{aligned}
&\frac{n^2}{2}\int\limits_{\Omega_2} e^{-nv(x_1x_2)}dx_1dx_2 \\
&\le \frac{n^2}{2}\int\limits_{\Omega_2} e^{-n\left(\frac{2}{\beta^2}\pi\rho^2+\rho t\right)}dx_1dx_2. \\
&= 2n^2 \int_{\frac{r_n}{2}}^{\frac{R_n}{2}} d\rho \int\limits_{\mathbb{D}_{\sqrt{3}\rho}(1)\setminus\mathbb{D}_\rho(2)} e^{-n\left(\frac{2}{\beta^2}\pi\rho^2+\rho t\right)}\rho\theta(z, \rho)\,dz \\
&= 2n^2 \int_{\frac{r_n}{2}}^{\frac{R_n}{2}} d\rho \int\limits_{\mathbb{D}_{\sqrt{3}\rho}(1)\setminus\mathbb{D}_\rho(2)} e^{-n\left(\frac{2}{\beta^2}\pi\rho^2+\rho t\right)}\rho\theta(z, \rho)\,dz \\
&\le 4\pi n^2 e^{-\frac{1}{2\beta^2}n\pi r_n^2} \int_{\frac{r_n}{2}}^{\frac{R_n}{2}} d\rho \int\limits_{\mathbb{D}} e^{-n\rho t}t dz \\
&= O(1)\, n^{1.5} \int_{\frac{r_n}{2}}^{\frac{R_n}{2}} d\rho \int_0^{1/\sqrt{\pi}} e^{-n\rho t}t dt \\
&\le O(1)\, n^{1.5} \int_{\frac{r_n}{2}}^{\frac{R_n}{2}} d\rho \int_0^{\infty} e^{-n\rho t}t dt \\
&= O(1)\frac{1}{\sqrt{n}} \int_{\frac{r_n}{2}}^{\frac{R_n}{2}} \rho^{-2}d\rho \le O(1)\frac{1}{\sqrt{n}} r_n^{-2}R_n \\
&= O(1)\frac{1}{\sqrt{n}r_n} = O(1)\frac{1}{\sqrt{\ln n}} = o(1).
\end{aligned}$$

Therefore,

$$\frac{n^2}{2}\int\limits_{\Omega} e^{-nv_{x_1x_2}}dx_1dx_2 \sim \frac{\beta^2}{2}e^{-\xi}.$$

Note that $\Omega \cup \Omega'$ consists of $(x_1, x_2) \in \mathbb{D}^2$ satisfying that $r_n < \|x_1x_2\| \le R'_n$. Using the same argument as above, we can show that

$$\frac{n^2}{2}\int\limits_{\Omega\cup\Omega'} e^{-nv(x_1x_2)}dx_1dx_2 \sim \frac{\beta^2}{2}e^{-\xi}.$$

Thus, the second asymptotic equality in the lemma holds. ∎

A topology with numbered vertices is specified by a collection of the pairs of the indices of the numbered vertices. For any topology τ on m numbered vertices and a planar set U of m numbered points, we denote by $\tau(U)$ the graph on U with topology τ. Suppose that τ is a topology with m numbered vertices and without isolated vertices. We denote by $\Gamma(\tau)$ the set of $\mathbf{x} = (x_1, \cdots, x_m) \in \mathbb{D}^m$ satisfying that the length of each edge in $\tau(\mathbf{x})$ is more than r_n but at most R_n. Note that for each $\mathbf{x} \in \Gamma(\tau)$, the $\sqrt{3}R_n$-disk graph on the midpoints of the edges in any connected component of $\tau(\mathbf{x})$ is connected. Thus, the $\sqrt{3}R_n$-disk graph on the midpoints of the edges in $\tau(\mathbf{x})$ has no more connected components than $\tau(\mathbf{x})$ itself. For any positive integer l no more than the number of connected components of τ, we denote by $\Gamma_l(\tau)$ the set of $\mathbf{x} \in \Gamma(\tau)$ such that the $\sqrt{3}R_n$-disk graph on the midpoints of the edges in $\tau(\mathbf{x})$ has l connected components. For any positive integer k, we denote by C_k the forest on $2k$ numbered vertices $v_1, v_2, \cdots, v_{2k}$ which consists of k edges $v_{2i-1}v_{2i}$ for $1 \le i \le k$. Then, C_k has k tree components, each consisting of a single edge, and $\Gamma(C_k) = \Omega^k$.

Lemma 6: For any fixed integer $k \ge 2$,

$$\left(\frac{n^2}{2}\right)^k \int\limits_{\Gamma_1(C_k)} \chi(C_k(\mathbf{x}))\, e^{-n\nu(C_k(\mathbf{x}))}d\mathbf{x} = o(1).$$

Proof: For each $\mathbf{x} = (x_1, \cdots, x_{2k}) \in \Gamma_1(C_k)$, let z_i and ρ_i be the midpoint and half-length of $x_{2i-1}x_{2i}$ respectively. We denote by S the set of $\mathbf{x} = (x_1, \cdots, x_{2k}) \in \Gamma_1(C_k)$ satisfying that x_1x_2 is the outermost edge in $C_k(\mathbf{x})$ and z_2 is the farthest from z_1. It suffices to prove

$$n^{2k}\int\limits_{S} \chi(C_k(\mathbf{x}))\, e^{-n\nu(C_k(\mathbf{x}))}d\mathbf{x} = o(1).$$

By Lemma 4, for any $\mathbf{x} = (x_1, \cdots, x_{2k}) \in S$ with $\chi(C_k(\mathbf{x})) = 1$,

$$\nu(C_k(\mathbf{x})) \geq \nu(x_1 x_2) + cR_n \|z_1 z_2\|$$

for some constant c. So, it is sufficient to show that

$$n^{2k} \int\limits_S e^{-n(\nu(x_1x_2)+cR_n\|z_1z_2\|)} d\mathbf{x} = o(1).$$

For each $2 \leq i \leq k$, we replace x_{2i-1} and x_{2i} by z_i, ρ_i and the slope of $x_{2i-1}x_{2i}$. Note that for any $3 \leq i \leq k$, $z_i \in \overline{D}(z_1, \|z_2 z_1\|)$. Thus,

$$\begin{aligned}
& n^{2k} \int\limits_S e^{-n(\nu(x_1x_2)+cR_n\|z_1z_2\|)} \prod_{i=1}^{2k} dx_i \\
& \leq O(1)\, n^{2k} \left(\int\limits_\Omega e^{-n\nu(x_1x_2)} dx_1 dx_2 \right) \left(\int_{\frac{r_n}{2}}^{\frac{R_n}{2}} \rho d\rho \right)^{k-1} \\
& \quad \cdot \left(\int\limits_{\mathbb{R}^2} e^{-ncR_n\|z_2z_1\|} dz_2 \right) \left(\int\limits_{\overline{D}(z_1,\|z_2z_1\|)} dz \right)^{k-2} \\
& \sim O(1)\, n^{2k-2} \left(R_n^2 - r_n^2\right)^{k-1} \\
& \quad \cdot \int\limits_{\mathbb{R}^2} e^{-ncR_n\|z_2z_1\|} \|z_2z_1\|^{2(k-2)} dz_2 \\
& \leq O(1)\, n^{2k-2} \left(R_n^2 - r_n^2\right)^{k-1} \int_0^\infty e^{-ncR_nt} t^{2k-3} dt \\
& = O(1) \frac{n^{2k-2}\left(R_n^2 - r_n^2\right)^{k-1}}{(nR_n)^{2(k-1)}} \\
& = O(1) \left(\frac{nR_n^2 - nr_n^2}{nR_n^2} \right)^{k-1} \\
& = O(1) \left(\frac{\xi_n - \xi}{\ln n} \right)^{k-1} = o(1),
\end{aligned}$$

where the second asymptotic equality follows from Lemma 5, and the last equality is based on $\xi_n = o(\ln n)$. ∎

Lemma 7: For any fixed integers $2 \leq l < k$.

$$\left(\frac{n^2}{2}\right)^k \int\limits_{\Gamma_l(C_k)} \chi(C_k(\mathbf{x})) e^{-n\nu(C_k(\mathbf{x}))} d\mathbf{x} = o(1).$$

Proof: For any nontrivial l-partition $\Pi = \{P_1, P_2, \cdots, P_l\}$ of $\{1, 2, \cdots, k\}$, let $S(\Pi)$ denote the set of $\mathbf{x} = (x_1, \cdots, x_{2k}) \in \Gamma_l(C_k)$ such that for each $1 \leq j \leq l$, the set $\{z_i : i \in P_j\}$ is a connected components of the $\sqrt{3}R_n$-disk graph on $z_1, z_2, \cdots, z_k$. Then $\Gamma_l(C_k)$ is the union of $S(\Pi)$ over all nontrivial l-partitions Π of $\{1, 2, \cdots, k\}$. So, it is sufficient to show that for any l-partition Π of $\{1, 2, \cdots, k\}$,

$$n^{2k} \int\limits_{S(\Pi)} \chi(C_k(\mathbf{x})) e^{-n\nu(C_k(\mathbf{x}))} d\mathbf{x} = o(1).$$

Now fix an l-partition $\Pi = \{P_1, P_2, \cdots, P_l\}$ of $\{1, 2, \cdots, k\}$, and let $p_j = |P_j|$ for $1 \leq j \leq l$. Then,

$$S(\Pi) \subseteq \prod_{j=1}^{l} \Gamma_1(C_j).$$

For any $\mathbf{x} = (x_1, x_2, \cdots, x_{2k}) \in S(\Pi)$, let $\mathbf{x}^{(j)}$ denote the subsequence of (x_{2i-1}, x_{2i}) with $i \in P_j$ for $1 \leq j \leq l$. Then,

$$\begin{aligned}
\nu(C_k(\mathbf{x})) &= \sum_{j=1}^{l} \nu\left(C_j\left(\mathbf{x}^{(j)}\right)\right) \\
\chi(C_k(\mathbf{x})) &\leq \prod_{j=1}^{l} \chi\left(C_j\left(\mathbf{x}^{(j)}\right)\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
& n^{2k} \int\limits_{S(\Pi)} \chi(C_k(\mathbf{x})) e^{-n\nu(C_k(\mathbf{x}))} d\mathbf{x} \\
& \leq n^{2k} \int\limits_{S(\Pi)} \prod_{j=1}^{l} \chi\left(C_j\left(\mathbf{x}^{(j)}\right)\right) e^{-n\nu\left(C_j\left(\mathbf{x}^{(j)}\right)\right)} d\mathbf{x} \\
& \leq n^{2k} \int\limits_{\prod_{j=1}^{l} \Gamma_1(C_j)} \prod_{j=1}^{l} \chi\left(C_j\left(\mathbf{x}^{(j)}\right)\right) e^{-n\nu\left(C_j\left(\mathbf{x}^{(j)}\right)\right)} d\mathbf{x} \\
& = \prod_{j=1}^{l} \left(n^{2p_j} \int\limits_{\Gamma_1(C_j)} \chi\left(C_j\left(\mathbf{x}^{(j)}\right)\right) e^{-n\nu\left(C_j\left(\mathbf{x}^{(j)}\right)\right)} d\mathbf{x}^{(j)} \right) \\
& = o(1),
\end{aligned}$$

where the last equality follows from Lemma 6 and the fact that at least one $p_j \geq 2$. ∎

Lemma 8: For any fixed integer $k \geq 2$,

$$\left(\frac{n^2}{2}\right)^k \int\limits_{\Gamma_k(C_k)} e^{-n\nu(C_k(\mathbf{x}))} d\mathbf{x} \sim \left(\frac{\beta^2}{2} e^{-\xi}\right)^k.$$

Proof: For any $\mathbf{x} = (x_1, \cdots, x_{2k}) \in \Gamma_k(C_k)$,

$$\nu(C_k(\mathbf{x})) = \sum_{i=1}^{k} \nu(x_{2i-1}x_{2i}).$$

Thus,

$$\begin{aligned}
& \left(\frac{n^2}{2}\right)^k \int\limits_{\Gamma_k(C_k)} e^{-n\nu(C_k(\mathbf{x}))} d\mathbf{x} \\
& = \left(\frac{n^2}{2}\right)^k \int\limits_{\Gamma_k(C_k)} e^{-n\sum_{i=1}^{k} \nu(x_{2i-1}x_{2i})} d\mathbf{x} \\
& = \left(\frac{n^2}{2}\right)^k \int\limits_{\Gamma(C_k)} e^{-n\sum_{i=1}^{k} \nu(x_{2i-1}x_{2i})} d\mathbf{x} \\
& \quad - \sum_{l=1}^{k-1} \left(\frac{n^2}{2}\right)^k \int\limits_{\Gamma_l(C_k)} e^{-n\sum_{i=1}^{k} \nu(x_{2i-1}x_{2i})} d\mathbf{x}.
\end{aligned}$$

We shall show that the first term is asymptotically equal to $\left(\frac{\beta^2}{2}e^{-\xi}\right)^k$, and the second term is vanishing. Indeed,

$$\left(\frac{n^2}{2}\right)^k \int\limits_{\Gamma(C_k)} e^{-n\sum_{i=1}^k \nu(x_{2i-1}x_{2i})} d\mathbf{x}$$
$$= \prod_{i=1}^k \left(\frac{n^2}{2}\int\limits_\Omega e^{-n\nu(x_{2i-1}x_{2i})} dx_{2i-1}dx_{2i}\right) \sim \left(\frac{\beta^2}{2}e^{-\xi}\right)^k.$$

where the last equality follows from Lemma 5. For any $\mathbf{x} = (x_1, \cdots, x_{2k}) \in \Gamma_1(C_k)$, if x_1x_2 is the outermost edge in $C_k(\mathbf{x})$ and z_2 is the farthest from z_1, it can be proved that

$$\sum_{i=1}^k \nu(x_{2i-1}x_{2i}) \geq \nu(x_1x_2) + cR_n \|z_1z_2\|.$$

Following the same argument in Lemma 6, we can show that

$$\left(\frac{n^2}{2}\right)^k \int\limits_{\Gamma_1(C_k)} e^{-n\sum_{i=1}^k \nu_{x_{2i-1}x_{2i}}} d\mathbf{x} = o(1).$$

Then, following the same argument in Lemma 7, we can show that for any $2 \leq l \leq k-1$,

$$\left(\frac{n^2}{2}\right)^k \int\limits_{\Gamma_l(C_k)} e^{-n\sum_{i=1}^k \nu_{x_{2i-1}x_{2i}}} d\mathbf{x} = o(1).$$

Thus, the lemma holds. ■

Lemma 9: Let F be a forest on m numbered vertices with maximum degree at least two and minimum degree at least one. Then,

$$n^m \int\limits_{\Gamma(F)} \chi(F(\mathbf{x})) e^{-n\nu(F(\mathbf{x}))} d\mathbf{x} = o(1).$$

Proof: Let κ be the number of tree components of F. Then, $m \geq \kappa + 2$, and F has exactly $m - \kappa$ edges denoted by $e_1, \cdots, e_{m-\kappa}$. For any $\mathbf{x} = (x_1, \cdots, x_m) \in \Gamma(F)$, let z_i denote the middle point of e_i in $F(\mathbf{x})$ for each $1 \leq i \leq m-\kappa$. We first show that

$$n^m \int\limits_{\Gamma_1(F)} \chi(F(\mathbf{x})) e^{-n\nu(F(\mathbf{x}))} d\mathbf{x} = o(1).$$

For any pair of distinct integers p and q between 1 and $m - \kappa$, let S_{pq} denote the set of $\mathbf{x} = (x_1, \cdots, x_m) \in \Gamma_1(F)$ satisfying that e_p is an outermost edge in $F(\mathbf{x})$ and z_q is the farthest from z_p among all $z_1, \cdots, z_{m-\kappa}$. Then, it suffices to prove for any such p and q,

$$n^m \int\limits_{S_{pq}} \chi(F(\mathbf{x})) e^{-n\nu(F(\mathbf{x}))} d\mathbf{x} = o(1).$$

Fix a pair of distinct integers p and q between 1 and $m-\kappa$. Let p' and p'' be the indices of the two endpoints of the edges e_p. Then, for any $\mathbf{x} = (x_1, \cdots, x_m) \in S_{pq}$ with $\chi(F(\mathbf{x})) = 1$,

$$\nu(F(\mathbf{x})) \geq v(x_{p'}x_{p''}) + cR_n \|z_pz_q\|$$

for some constant $c > 0$. Thus, we only need to show that

$$n^m \int\limits_{S_{pq}} e^{-n\left(v\left(x_{p'}x_{p''}\right)+cR_n\|z_pz_q\|\right)} d\mathbf{x} = o(1).$$

We change the integral variables $x_1, \cdots, x_m$ as follows. For the tree component containing e_p, we replace the x_i's in this tree by the midpoints of the edges in this tree except z_p and $x_{p'}$, $x_{p''}$ (both of which are kept). For any other tree component, we use the method introduced at the beginning of this section: pick an arbitrary edge as the rooted edge. We replace x_i's in this tree by the midpoints of all the edges in this tree together with the half-length and slope of the root edge. Such change of integration variables yields

$$\begin{aligned}
& n^m \int\limits_{S_{pq}} e^{-n\left(v\left(x_{p'}x_{p''}\right)+cR_n\|z_pz_q\|\right)} d\mathbf{x} \\
& \leq O(1)\, n^m \left(\int\limits_\Omega e^{-nvv\left(x_{p'}x_{p''}\right)} dx_{p'}dx_{p''}\right) \\
& \quad \cdot \left(\int_{\frac{r_n}{2}}^{\frac{R_n}{2}} \rho d\rho\right)^{\kappa-1} \left(\int\limits_{\mathbb{R}^2} e^{-ncR_n\|z_pz_q\|} dz_q\right) \\
& \quad \cdot \left(\int\limits_{\overline{D}(z_p,\|z_pz_q\|)} dz\right)^{m-\kappa-2} \\
& \sim O(1)\, n^{m-2}\left(R_n^2 - r_n^2\right)^{\kappa-1} \\
& \quad \cdot \int\limits_{\mathbb{R}^2} e^{-ncR_n\|z_pz_q\|} \|z_pz_q\|^{2(m-\kappa-2)} dz_q \\
& \leq O(1)\, n^{m-2}\left(R_n^2 - r_n^2\right)^{\kappa-1} \\
& \quad \cdot \int_0^\infty e^{-ncR_n\mu} \mu^{2(m-\kappa)-3} d\mu \\
& = O(1) \frac{n^{m-2}\left(R_n^2 - r_n^2\right)^{\kappa-1}}{(nR_n)^{2(m-\kappa-1)}} \\
& = O(1) \frac{\left(nR_n^2 - nr_n^2\right)^{\kappa-1}}{\left(nR_n^2\right)^{m-\kappa-1}} \\
& = O(1) \frac{(\xi_n - \xi)^{\kappa-1}}{\ln^{m-\kappa-1} n} = o(1),
\end{aligned}$$

where the asymptotic equality follows from Lemma 5, and the last equality follows from $\xi_n = o(\ln n)$ and $m - \kappa - 1 \geq 1$.

Following the same decomposition argument as in the proof of Lemma 7, we can show that for any $2 \leq l \leq \kappa$,

$$n^m \int\limits_{\Gamma_l(F)} \chi(F(\mathbf{x})) e^{-n\nu(F(\mathbf{x}))} d\mathbf{x} = o(1)..$$

Thus, the lemma holds. ■

IV. Proof for Theorem 1

We first give a brief overview on our approach to prove Theorem 1. Let M_n denote the number of edges in $RNG(\mathcal{P}_n)$

longer than r_n but not shorter than R_n, M'_n denote the number of edges in $RNG(\mathcal{P}_n)$ longer than R_n but not shorter than R'_n, and M''_n denote the number of edges in $RNG(\mathcal{P}_n)$ longer than R'_n. Then, $\lambda(RNG(\mathcal{P}_n)) \leq r_n$ if and only if $M_n + M'_n + M''_n = 0$. According the discussion in Section I, $M''_n = 0$ is a.a.s.. In Lemma 12, we will prove that $E[M'_n] = o(1)$, which implies that $M'_n = 0$ is a.a.s. by Markov's inequality. In Lemma 13, we will prove that M_n is asymptotically Poisson with mean $\frac{\beta^2}{2}e^{-\xi}$. Consequently,

$$\begin{aligned}&\lim_{n\to\infty} \Pr[\lambda(RNG(\mathcal{P}_n)) \leq r_n] \\ &= \lim_{n\to\infty} \Pr[M_n + M'_n + M''_n = 0] \\ &= \lim_{n\to\infty} \Pr[M_n = 0] = e^{-\frac{\beta^2}{2}e^{-\xi}}.\end{aligned}$$

Two key techniques used in our proof are the Palm theory for Poisson processes (see, e.g., Theorem 1.6 in [12]) and the Brun's sieve (see, e.g., Theorem 10 in [16]), which are stated below.

Theorem 10: Suppose that $h(U,V)$ is a bounded measurable function defined on all pairs of the form (U,V) with V being a finite planar set and U being a subset of V. Then any positive integer k,

$$\mathbf{E}\left[\sum_{U\subseteq\mathcal{P}_n,|U|=k} h(U,\mathcal{P}_n)\right] = \frac{n^k}{k!}\mathbf{E}[h(\mathcal{X}_k, \mathcal{X}_k \cup \mathcal{P}_n)].$$

Theorem 11: Suppose that N is a non-negative integer random variable, and $B_1, \cdots, B_N$ are N Bernoulli random variables. If there is a constant μ such that for every fixed positive integer k,

$$\mathbf{E}\left[\sum_{I\subseteq\{1,\cdots,N\},|I|=k} \prod_{i\in I} B_i\right] \sim \frac{1}{k!}\mu^k,$$

then $\sum_{i=1}^{N} B_i$ is asymptotically Poisson with mean μ.

Now, we apply Palm theory to show that $E[M'_n]$ is vanishing.

Lemma 12: $E[M'_n] = o(1)$.

Proof: For any edge $e \in K(\mathcal{P}_n)$, define $B'(e)$ to be the Bernoulli random variable which equals to one if and only if $e \in RNG(\mathcal{P}_n)$ and $R_n < \|e\| \leq R^*_n$. Then $M'_n = \sum_{e\in K(\mathcal{P}_n)} B'(e)$. Let $\mathcal{X}_2 = \{X_1X_2\}$ and define B'_1 to be the Bernoulli random variable which equals to one if and only if $X_1X_2 \in RNG(\mathcal{X}_{2k} \cup \mathcal{P}_n)$ and $R_n < \|X_1X_2\| \leq R^*_n$. By treating each edge of $K(\mathcal{P}_n)$ as a subset of two points in $\mathcal{P}_n$ and with the application of Theorem 10, we have

$$\mathbf{E}[M'_n] = \mathbf{E}\left[\sum_{e\in K(\mathcal{P}_n)} B'(e)\right] = \frac{n^2}{2}\mathbf{E}[B'_1].$$

By Lemma 5,

$$\begin{aligned}&\frac{n^2}{2}\mathbf{E}[B'_1] = \frac{n^2}{2}\int_{\Omega'} \Pr[B'_1 = 1 \mid \mathcal{X}_2 = \mathbf{x}]\, d\mathbf{x} \\ &= \frac{n^2}{2}\int_{\Omega'} e^{-n\nu(x_1x_2)} dx_1 dx_2 = o(1).\end{aligned}$$

Therefore, $E[M'_n] = o(1)$. ■

Next, we apply the Brun's seive together with the Palm theory to prove M_n is asymptotically Poisson.

Lemma 13: M_n is asymptotically Poisson with mean $\frac{\beta^2}{2}e^{-\xi}$.

Proof: For any edge $e \in K(\mathcal{P}_n)$, define $B(e)$ to be the Bernoulli random variable which equals to one if and only if $e \in RNG(\mathcal{P}_n)$ and $r_n < \|e\| \leq R_n$. Then $M_n = \sum_{e\in K(\mathcal{P}_n)} B(e)$. For any subgraph H of $K(\mathcal{P}_n)$, define $B(H) = \prod_{e\in H} B(e)$. Denote by $\mathcal{T}_m$ the set of topologies on m numbered vertices in which there are exactly k edges and no vertex is isolated. Denote by $k^* = \left\lceil \frac{1+\sqrt{1+4k^2}}{2} \right\rceil$. Then, $\mathcal{T}_m = \emptyset$ unless $k^* \leq m \leq 2k$. For any topology τ on m numbered vertices and a planar set U of m numbered points, we denote by $\tau(U)$ the graph on U with topology τ. By Theorem 11, we only need to prove that

$$\begin{aligned}&\mathbf{E}\left[\sum_{m=k^*}^{2k} \sum_{U\subset\mathcal{P}_n,|U|=m} \sum_{\tau\in\mathcal{T}_m} B(\tau(U))\right] \\ &\sim \frac{1}{k!}\left(\frac{\beta^2}{2}e^{-\xi}\right)^k. \qquad (2)\end{aligned}$$

For each $e \in K(\mathcal{X}_m)$, define $B_m(e)$ to be the Bernoulli random variable which equals to one if and only if $e \in RNG(\mathcal{X}_{2k} \cup \mathcal{P}_n)$ and $r_n < \|e\| \leq R_n$. For any subgraph H of $K(\mathcal{X}_m)$, define $B_m(H) = \prod_{e\in H} B_m(e)$. By Theorem 10,

$$\begin{aligned}&\mathbf{E}\left[\sum_{m=k^*}^{2k} \sum_{U\subset\mathcal{P}_n,|U|=m} \sum_{\tau\in\mathcal{T}_m} B(\tau(U))\right] \\ &= \sum_{m=k^*}^{2k} \frac{n^m}{m!}\mathbf{E}\left[\sum_{\tau\in\mathcal{T}_m} B_m(\tau(\mathcal{X}_m))\right].\end{aligned}$$

We will prove that

$$\frac{n^{2k}}{(2k)!}\mathbf{E}\left[\sum_{\tau\in\mathcal{T}_{2k}} B_{2k}(\tau(\mathcal{X}_{2k}))\right] \sim \frac{1}{k!}\left(\frac{\beta^2}{2}e^{-\xi}\right)^k, \qquad (3)$$

and for each $\tau \in \mathcal{T}_m$ with $k^* \leq m < 2k$

$$n^m \mathbf{E}[B_m(\tau(\mathcal{X}_m))] = o(1). \qquad (4)$$

These asymptotic equalities imply the asymptotic equality (2) immediately.

We first prove the asymptotic equality (3). Since

$$card(\mathcal{T}_{2k}) = \frac{1}{k!}\binom{2k}{2,2,\cdots,2} = \frac{(2k)!}{k!2^k},$$

and all topologies in $\mathcal{T}_{2k}$ are isomorphic to each other, we have

$$\begin{aligned}&\frac{n^{2k}}{(2k)!}\mathbf{E}\left[\sum_{\tau\in\mathcal{T}_{2k}} B_{2k}(\tau(\mathcal{X}_{2k}))\right] \\ &= \frac{1}{k!}\left(\frac{n^2}{2}\right)^k \mathbf{E}[B_{2k}(C_k(\mathcal{X}_{2k}))].\end{aligned}$$

It is sufficient to show that

$$\left(\frac{n^2}{2}\right)^k \mathbf{E}\left[B_{2k}\left(C_k\left(\mathcal{X}_{2k}\right)\right)\right] \sim \left(\frac{\beta^2}{2}e^{-\xi}\right)^k. \tag{5}$$

For $k = 1$, by Lemma 5 we have

$$\begin{aligned}
&\frac{n^2}{2}\mathbf{E}\left[B_2\left(C_1\left(\mathcal{X}_2\right)\right)\right] \\
&= \frac{n^2}{2}\int_\Omega \Pr\left[B_2\left(C_1\left(\mathcal{X}_2\right)\right) = 1 \mid \mathcal{X}_2 = \mathbf{x}\right] d\mathbf{x} \\
&= \frac{n^2}{2}\int_\Omega e^{-n\nu(x_1x_2)}dx_1dx_2 \sim \frac{\beta^2}{2}e^{-\xi}.
\end{aligned}$$

So, the asymptotic equality (5) is true for $k = 1$. Now, suppose that $k \geq 2$, we have

$$\begin{aligned}
&\left(\frac{n^2}{2}\right)^k \mathbf{E}\left[B_{2k}\left(C_k\left(\mathcal{X}_{2k}\right)\right)\right] \\
&= \left(\frac{n^2}{2}\right)^k \int\limits_{\Gamma(C_k)} \Pr\left[B_{2k}\left(C_k\left(\mathcal{X}_{2k}\right)\right) = 1 \mid \mathcal{X}_{2k} = \mathbf{x}\right] d\mathbf{x} \\
&= \sum_{l=1}^k \left(\frac{n^2}{2}\right)^k \int\limits_{\Gamma_l(C_k)} \Pr\left[B_{2k}\left(C_k\left(\mathcal{X}_{2k}\right)\right) = 1 \mid \mathcal{X}_{2k} = \mathbf{x}\right] d\mathbf{x}.
\end{aligned}$$

By Lemma 8,

$$\begin{aligned}
&\left(\frac{n^2}{2}\right)^k \int\limits_{\Gamma_k(C_k)} \Pr\left[B_{2k}\left(C_k\left(\mathcal{X}_{2k}\right)\right) = 1 \mid \mathcal{X}_{2k} = \mathbf{x}\right] d\mathbf{x} \\
&= \left(\frac{n^2}{2}\right)^k \int\limits_{\Gamma_k(C_k)} e^{-n\nu(C_k(\mathbf{x}))}d\mathbf{x} \sim \left(\frac{\beta^2}{2}e^{-\xi}\right)^k.
\end{aligned}$$

For any $1 \leq l < k$, by Lemma 6 and 7,

$$\begin{aligned}
&\left(\frac{n^2}{2}\right)^k \int\limits_{\Gamma_l(C_k)} \Pr\left[B_{2k}\left(C_k\left(\mathcal{X}_{2k}\right)\right) = 1 \mid \mathcal{X}_{2k} = \mathbf{x}\right] d\mathbf{x} \\
&\leq \left(\frac{n^2}{2}\right)^k \int\limits_{\Gamma_l(C_k)} \chi\left(C_k\left(\mathbf{x}\right)\right) e^{-n\nu(C_k(\mathbf{x}))}d\mathbf{x} = o\left(1\right).
\end{aligned}$$

Thus, the asymptotic equality (5) is true for any $k \geq 2$.

Next, We prove the asymptotic equality (4) for any $\tau \in \mathcal{T}_m$ with $k^* \leq m < 2k$. Since such τ does not exist for $k = 1$, we assume that $k \geq 2$. Let F be any maximal spanning forest of τ. Then, the maximum degree of F is at least two and the minimum degree of F is at least one. By Lemma 9, we have

$$\begin{aligned}
&n^m\mathbf{E}\left[B_m\left(\tau\left(\mathcal{X}_m\right)\right)\right] \leq n^m\mathbf{E}\left[B_m\left(F\left(\mathcal{X}_m\right)\right)\right] \\
&= n^m \int\limits_{\Gamma(F)} \Pr\left[B_m\left(F\left(\mathcal{X}_m\right)\right) = 1 \mid \mathcal{X}_m = \mathbf{x}\right] d\mathbf{x} \\
&\leq n^m \int\limits_{\Gamma(F)} \chi\left(F\left(\mathbf{x}\right)\right) e^{-n\nu(F(\mathbf{x}))}d\mathbf{x} = o\left(1\right).
\end{aligned}$$

So, the asymptotic equality ((4) is true for any $\tau \in \mathcal{T}_m$ with $k^* \leq m < 2k$. ∎

ACKNOWLEDGEMENT

This work is supported in part by NSF grant CCF-0515088 of USA, a grant from the Research Grants Council of Hong Kong SAR under Project No. CityU 122807, CityU 114307, National Basic Research Program of China Grant 2007CB807900 and 2007CB807901, the NSC Grant No. NSC95-2221-E-009-059-MY3, and by the MOE ATU Program.

REFERENCES

[1] J. Cartigny, F. Ingelrest, D. Simplot-Ryl, I. Stojmenovic: Localized LMST and RNG based minimum energy broadcast protocols in ad hoc networks, *IEEE INFOCOM* 2003; also appeared in *Ad Hoc Networks* 3(1):1-16, 2005.

[2] H. Dette and N. Henze: The limit distribution of the largest nearest-neighbour link in the unit d-cube, *Journal of Applied Probability* 26: 67–80, 1989.

[3] G. Finn: Routing and addressing problems in large metropolitan-scale internetworks, technical Report ISI Research Report ISU/RR-87-180, March 1987.

[4] K. Gabriel and R. Sokal: A new statistical approach to geographic variation analysis, *Systematic Zoology* 18:259–278, 1969.

[5] P. Gupta and P.R. Kumar: Critical power for asymptotic connectivity in wireless networks, in *Stochastic Analysis, Control, Optimization and Applications: A Volume in Honor of W. H. Fleming*, W.M. McEneaney, G.Yin, and Q. Zhang, Eds. Birkhauser, March 1998, pp. 547–566.

[6] A. A.-K. Jeng, R.-H. Jan: The r-Neighborhood Graph: An Adjustable Structure for Topology Control in Wireless Ad Hoc Networks, *IEEE Transactions on Parallel and Distributed Systems* 18(4): 536-549, 2007.

[7] B. Karp and H. T. Kung. GPSR: greedy perimeter stateless routing for wireless networks, *ACM MOBICOM* 2000.

[8] X.-Y. Li, P.-J. Wan, W. Yu: Power Efficient and Sparse Spanner for Wireless Ad Hoc Networks, *IEEE ICCCN 2001*.

[9] N. Li, J. C. Hou: Localized topology control algorithms for heterogeneous wireless networks. *IEEE/ACM Transactions on Networking* 13(6): 1313-1324, 2005.

[10] G. Kozma, Z. Lotker, M. Sharir, and G. Stupp: Geometrically aware communication in random wireless networks, *ACM PODC* 2004.

[11] M. D. Penrose: The longest edge of the random minimal spanning tree, *The annals of applied probability* 7(2):340–361, 1997.

[12] M. Penrose: *Random Geometric Graphs*, Oxford University Press, 2003.

[13] M. Seddigh, J. Solano and I. Stojmenovic: RNG and internal node based broadcasting in one-to-one wireless networks, *ACM Mobile Computing and Communications Review* 5(2):37-44, April 2001.

[14] H. Takagi and L. Kleinrock: Optimal transmission ranges for randomly distributed packet radio terminals, IEEE Transactions on Communications 32(3): 246–257, 1984.

[15] G. Toussaint: The relative neighborhood graph of a finite planar set, *Pattern Recognition* 12(4):261-268, 1980.

[16] P.-J. Wan, and C.-W. Yi: On The Longest Edge of Gabriel Graphs in Wireless Ad Hoc Networks, *IEEE Transactions on Parallel and Distributed Systems* 18(1):111-125, 2007.

[17] P.-J. Wan, C.-W. Yi, F. Yao, and X. Jia: Asymptotic Critical Transmission Radius for Greedy Forward Routing in Wireless Ad Hoc Networks, *ACM MOBIHOC* 2006, pp 25-36.