Approximate Capacity Subregions of Uniform Multihop Wireless Networks

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***Abstract*—The capacity region of multihop wireless network is involved in many capacity optimization problems. However, the membership of the capacity region is NP-complete in general, and hence the direct application of capacity region is quite limited. As a compromise, we often substitute the capacity region with a polynomial approximate capacity subregion. In this paper, we construct polynomial μ-approximate capacity subregions of multihop wireless network under either 802.11 interference model or protocol interference model in which all nodes have uniform communication radii normalized to one and uniform interference radii $\rho \geq 1$. The approximation factor μ decreases with ρ in general and is smaller than the best-known ones in the literature. For example, $\mu = 3$ when $\rho \geq 2.2907$ under the 802.11 interference model or when $\rho \geq 4.2462$ under the protocol interference model. Our construction exploits a nature of the wireless interference called strip-wise transitivity of independence discovered in this paper and utilize the independence polytopes of cocomparability graphs in a spatial-divide-conquer manner. We also apply these polynomial μ-approximate capacity subregions to compute μ-approximate solutions for maximum (concurrent) multiflows.**

***Index Terms*—Capacity region, maximum (concurrent) multiflows, approximation algorithm**

I. Introduction

A multihop wireless network $\mathbf{N}$ is specified, in its most general format, by a triple $(V, A, \mathcal{I})$, where V is the set of networking nodes, A is the set of communication links among V, and $\mathcal{I}$ is the collection of sets of independent (or conflict-free) links in A specified implicitly by an interference model. The communication topology of $\mathbf{N}$ is the digraph (V, A). The *capacity region* of $\mathbf{N}$ is the convex hull $P \subset \mathbb{R}_+^A$ of the incidence vectors of all sets in $\mathcal{I}$. Alternatively, the capacity region of $\mathbf{N}$ can also be defined in terms of link schedules. A *(fractional) link schedule* Π in $\mathbf{N}$ is a set

$$\{(I_j, \lambda_j) \in \mathcal{I} \times \mathbb{R}_+ : 1 \leq j \leq k\};$$

the two values k and $\sum_{j=1}^{k} \lambda_j$ are referred to as the *size* and *length* (or *latency*) of Π respectively. Any link schedule Π in $\mathbf{N}$ of length at most one determines a link capacity function $c_\Pi \in \mathbb{R}_+^A$ given by

$$c_\Pi(e) = \sum_{1 \leq j \leq k} \lambda_j |I_j \cap \{e\}|$$

for each $e \in A$; and Π is said to be a link schedule for c_Π. Then, the capacity region P of $\mathbf{N}$ consists of link capacity functions determined by link schedules in $\mathbf{N}$ of length at most one.

The capacity region is involved in many capacity optimization problems. However, the membership of the capacity region is NP-complete in general [7], and hence the direct application of capacity region is quite limited. As a compromise, we often substitute the capacity region P with a subset Q of P which satisfies the three conditions: (1) Q has an explicit representation by a polynomial number of linear inequalities, (2) there is a polynomial algorithm which produces a link schedule of length at most one for any $d \in Q$, and (3) $P \subseteq \mu Q$ for some $\mu \geq 1$. The first condition ensures the membership of Q is polynomial, the second condition implies that $Q \subseteq P$, and the third condition ensures that Q is "close" to P. A subset Q of P satisfying these three conditions is referred to as a *polynomial μ-approximate capacity subregion*.

The polynomial constant-approximate capacity subregion plays a central role in the design and analysis of polynomial-time constant-approximation algorithms for capacity optimization problems such as **Maximum Multiflow** (**MMF**) and **Maximum Concurrent Multiflow** (**MCMF**). Suppose that we are given a set of commodities in a network $\mathbf{N}$. For any link schedule Π in $\mathbf{N}$ of length at most one, the maximum multiflow of these commodities subject to the capacity function c_Π is referred to as the *maximum multiflow subject to* Π. Suppose in addition that each commodity also has a demand associated with it. For any link schedule Π in $\mathbf{N}$ of length at most one, the maximum concurrent multiflow of these commodities subject to the capacity function c_Π is referred to as the *maximum concurrent multiflow subject to* Π. Given a set of commodities in a specified network $\mathbf{N}$, the problem **MMF** seeks a link schedule Π in $\mathbf{N}$ of length at most one such that the maximum multiflow subject to Π is maximized. Given a set of commodities with demands in a specified network $\mathbf{N}$, the problem **MCMF** seeks a link schedule Π in $\mathbf{N}$ of length at most one such that the maximum concurrent multiflow subject to Π is maximized. It was shown in [7] that if we can find a polynomial μ-approximate capacity subregion, then both **MMF** and **MCMF** admit a polynomial μ-approximation.

In this paper we construct polynomial constant-approxmatie capacity subregions of multihop wireless networks under either the 802.11 interference model or the protocol interference model respectively with uniform communication radii normalized to one and uniform interference radii equal to $\rho \geq 1$. The communication (respectively, interference) range of a node $v \in V$ is the disk centered at v of radius one and ρ respectively. These networks are collectively referred to as *uniform* multihop wireless networks. Under both interference models, an instance of a network $\mathbf{N}$ is specified by a finite planar set V of nodes and the value of the interference radius ρ. The set A of communication links consists of all pairs (u, v) satisfying that the Euclidean distance between u and v, denoted by $\|uv\|$, is at most one. Under the 802.11 interference model, two links in A conflict with each other if and only if at least one link has an end lying in the interference range of some endpoint of the other link; under the protocol interference model, two links in A conflict with each other if and only if the receiving end of at least one link lies in the interference range of the transmitting end of the other link. Under either of the two interference models, a set I of links in A are independent (i.e., $I \in \mathcal{I}$) if all links in I are mutually conflict-free. The *conflict graph* of A is the undirected graph on A in which two links are adjacent if and only if they conflict with each other. So, $\mathcal{I}$ is essentially the collection of the independent sets in the conflict graph of A.

The polynomial constant-approximate capacity subregions of uniform multihop wireless networks under the 802.11 interference model or the protocol interference model have been studied implicitly in [1] [3] [7] [9]. Under the 802.11 interference model, Alicherry et al. [1] gave implicitly a polynomial μ-approximate capacity subregion, where μ was claimed to grow with ρ in general and equal to $4, 8$ and 12 respectively when $\rho = 1, 2, 2.5$ respectively. Buragohain et al. [3] discovered that the claim in [1] that $\mu = 4$ when $\rho = 1$ is wrong and pointed out that μ should be 8 when $\rho = 1$. They further proposed implicitly a polynomial 3-approximate capacity subregion regardless of the value of ρ. However, their capacity subregion was shown in [7] to contain some element which does not belong to the capacity region. Hence, their capacity subregion is wrong. Wan [7] then constructed implicitly yet another polynomial 7-approximate capacity subregion regardless of ρ. Under the protocol interference model with $\rho > 1$, Wang et al. [9] proposed implicitly a polynomial $2\left\lceil 2\pi / \arcsin \frac{\rho-1}{2\rho} \right\rceil$-approximate capacity subregion, which also turned to be false as shown in [7]. Wan [7] then devised implicitly a $2\left(\left\lceil \pi / \arcsin \frac{\rho-1}{2\rho} \right\rceil - 1\right)$-approximate capacity subregion.

The approach used in this paper is totally different from those followed in [1] [3] [7] [9]. Our approach exploits a nature of the wireless interference called strip-wise transitivity of independence under either the 802.11 interference model or the protocol interference model discovered in this paper. Such nature enables us to utilize the independence polytopes of cocomparability graphs [5] in a spatial-divide-conquer manner to build a polynomial μ-approximate capacity subregion. The value of μ decreases with ρ in general and is smaller than the best-known ones obtained in [7]. For example, $\mu = 3$ when $\rho \geq 2.2907$ under the 802.11 interference model or when $\rho \geq 4.2462$ under the protocol interference model. As a result, our polynomial capacity subregions give rise to improved practical approximation algorithms for both **MMF** and **MCMF**.

At the end of this section, we introduce some basic notations and terms used throughout this paper. For any $r > 0$, a r-disk is a disk of radius r. In particular, a unit-disk is a disk of radius one. The r-disk centered at a node u is denoted by $B_r(u)$. For any set U of nodes, the union of the r-disks centered at the nodes in U is denoted by $B_r(U)$. The vertical line through a point v is denoted by l_v. Let S be a finite subset. For any real function $f \in \mathbb{R}^S$ and any subset $S' \subseteq S$, $f(S')$ denotes $\sum_{e \in S'} f(e)$.

II. Preliminaries

Consider a digraph $D = (V, A)$. For each vertex $v \in V$, we use $\delta_D^{in}(v)$ (respectively, $\delta_D^{out}(v)$) to denote the set of links in D entering (respectively, leaving) v. Consider two distinct vertices $s, t \in V$. A vector $f \in \mathbb{R}_+^A$ is called a flow from s to t, or simply a $s - t$ flow, if for each $v \in V \setminus \{s, t\}$,

$$f\left(\delta_D^{out}(v)\right) = f\left(\delta_D^{in}(v)\right)$$

This condition is called the *flow conservation law*: the amount of flow entering a vertex $v \neq s, t$ should be equal to the amount of flow leaving v. The value of a flow f from s to t is, by definition:

$$val(f) = f\left(\delta_D^{out}(s)\right) - f\left(\delta_D^{in}(s)\right).$$

So the value is the net amount of flow leaving s, which is also equal to the net amount of flow entering t.

Let $G = (V, E)$ be an undirected graph. A subset I of V is an *independent set* of G if no two vertices in I are adjacent. For any $d \in \mathbb{R}_+^V$, a *fractional (weighted) coloring* of (G, d) is a set of k pairs $(I_j, \lambda_j) \in \mathcal{I} \times \mathbb{R}_+$ for $1 \leq j \leq k$ satisfying that for each $v \in V$,

$$\sum_{1 \leq j \leq k, v \in I_j,} \lambda_j = d(v).$$

The two values k and $\sum_{j=1}^{k} \lambda_j$ are referred to as the number and weight of the fractional coloring respectively. The *fractional chromatic number* $\chi_f(G,d)$ of (G,d) is defined as the minimum weight of all fractional colorings of (G,d). The *independence polytope* P of G is the convex hull of the incidence vectors of the independent sets in G. Equivalently, it consists of all $d \in \mathbb{R}_+^V$ with $\chi_f(G,d) \leq 1$. A polytope Q is said to be a *μ-approximation* of P for some $\mu \geq 1$ if $Q \subseteq P \subseteq \mu Q$.

A graph $G = (V,E)$ is said to be a *cocomparability graph* [5] if there is a vertex ordering $\langle v_1, v_2, \cdots, v_n \rangle$ of V satisfying that if $i < j < k$ and $v_i v_k \in E$ then either $v_i v_j \in E$ or $v_j v_k \in E$. Such ordering is referred to as its *cocomparability ordering*. Then, a minimum-weighted fractional coloring of (G,d) can be computed in polynomial time [4], [6]; moreover, the independence polytope of G can be represented by a linear number of inequalities [2]. Suppose that $G = (V,E)$ is a cocomparability graph with a cocomparability ordering $\langle v_1, v_2, \cdots, v_n \rangle$ of its vertices. Let s and t be two vertices not in V, and define an auxiliary digraph $G^* = (V \cup \{s,t\}, A^*)$ in which

$$\begin{aligned} A^* = \{(v_i, v_j) : 1 \leq i < j \leq n, v_i v_j \notin E\} \\ \cup \{(s,v) : v \in V\} \cup \{(v,t) : v \in V\}. \end{aligned}$$

Its construction can be interpreted as follows: We take the complement graph G^c of G and orient each edge $v_i v_j$ with $i < j$ in G^c to an arc (v_i, v_j). After that we add an auxiliary source s and an auxiliary sink t, and then add an arc from s to every node in V and an arc from every node in V to t. Thus, G^* is referred to as the *augmented complementary digraph* of G. Clearly, any independent set in G defines a s-t path in G^* and vice versa. We use $\mathcal{F}$ to denote the set of s–t flows f in G^*. For any flow $f \in \mathcal{F}$, we define a function $d_f \in \mathbb{R}^V$ by $d_f(v) = f\left(\delta_{G^*}^{out}(v)\right)$. Then, the independence polytope of G is

$$P = \{d_f : f \in \mathcal{F}, val(f) \leq 1\}.$$

For any $f \in \mathcal{F}$ with $val(f) \leq 1$, a link schedule for d_f can be computed as follows. We first decompose the flow f into path flows using the standard flow decomposition method [4]. Suppose that

$$\{(p_j, \lambda_j) : 1 \leq j \leq k\}$$

is a path flow decomposition of f. Let I_j be the set of vertices other than s and t in the path p_j for each $1 \leq j \leq k$. Then, I_j is an independent set of G. Hence,

$$\{(I_j, \lambda_j) : 1 \leq j \leq k\}$$

is a link schedule of d_f.

III. Strip-wise Transitivity of Independence

In this section, we prove a strip-wise transitivity of independence under the 802.11 interference model described in Theorem 1, and a similar strip-wise transitivity of independence under the protocol interference model described in Theorem 2.

Theorem 1: Consider a multihop wireless network under the 802.11 interference model in which each node has communication radius equal to one and interference radius equal to ρ. Suppose that a_1, a_3 and a_2 are three links whose midpoints lie in a horizontal strip of height

$$h(\rho) = \sqrt{\rho^2 - \frac{1}{4}} \cos\left(\frac{\pi}{6} + \arcsin\frac{1}{2\rho}\right)$$

from left to right. If both a_1 and a_2 are independent with a_3, then a_1 and a_2 are also independent with each other.

Theorem 2: Consider a multihop wireless network under the protocol interference model in which each node has communication radius equal to one and interference radius equal to ρ. Suppose that a_1, a_3 and a_2 are three links whose transmitting endpoints lie in a horizontal strip of height

$$h(\rho) = (\rho - 1) \sin\left(\arccos\frac{\rho - 1}{2\rho} - \arcsin\frac{1}{\rho}\right)$$

from left to right. If both a_1 and a_2 are independent with a_3, then a_1 and a_2 are also independent with each other.

The next two subsections are devoted to the proofs of the above two theorems respectively.

A. Proof of Theorem 1

The height function $h(\rho)$ defined in Theorem 1 has the following geometric interpretation. Consider four points v_1, u_1, u_2, v_2 counterclockwise on a circle of radius ρ centered at a point p satisfying that $\|u_1 v_1\| = \|u_2 v_2\| = 1$ and $\|u_1 u_2\| = \rho$ (see Fig. 1). Let z_1 and z_2 be the midpoints of $u_1 v_1$ and $u_2 v_2$ respectively, and q be the perpendicular foot of p on $z_1 z_2$. Then, $\|pq\|$ is exactly $h(\rho)$. Indeed, in the right triangle $p u_1 z_1$, we have

$$\|p z_1\| = \sqrt{\rho^2 - \frac{1}{4}}, \angle u_1 p z_1 = \arcsin\frac{1}{2\rho}.$$

Thus, in the right triangle $p q z_1$, we have

$$\angle z_1 p q = \angle u_1 p q + \angle u_1 p z_1 = \frac{\pi}{6} + \arcsin\frac{1}{2\rho},$$

and hence

$$\begin{aligned} \|pq\| &= \|p z_1\| \cos \angle z_1 p q \\ &= \sqrt{\rho^2 - \frac{1}{4}} \cos\left(\frac{\pi}{6} + \arcsin\frac{1}{2\rho}\right) \\ &= h(\rho). \end{aligned}$$

In addition, $\|z_1 q\| = g(\rho)$, where

$$g(\rho) = \sqrt{\rho^2 - \frac{1}{4}} \sin\left(\frac{\pi}{6} + \arcsin\frac{1}{2\rho}\right).$$

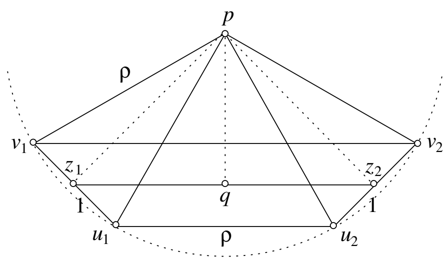

Fig. 1. The distance between p and $z_1 z_2$ is exactly $h(\rho)$.

From the above geometric interpretation, we immediately have $h(\rho) < \frac{\sqrt{3}}{2}\rho$. From the algebraic expression of $h(\rho)$, it is easy to see that $h(\rho)$ strictly increases with ρ and hence

$$h(\rho) \geq h(1) = \frac{\sqrt{3}}{4}.$$

A straightforward calculation yields

$$g(\rho) = \frac{1}{2}\sqrt{1 - \frac{1}{4\rho^2}} \left(\sqrt{\rho^2 - \frac{1}{4}} + \frac{\sqrt{3}}{2}\right).$$

So, $g(\rho)$ also strictly increases with ρ, and hence

$$g(\rho) \geq g(1) = \frac{3}{4}.$$

Consider a horizontal strip S of height $h(\rho)$, and let L_1 and L_2 be its top boundary and bottom boundary respectively. Suppose that u and v are two points. We introduce three (closed) rectangles with two sides aligned with the vertical lines through u and v respectively.

- $S(u, v)$: the portion of S between the two vertical lines through u and v.
- $R(u, v)$: the rectangle surrounded by the vertical lines through u and v respectively and the two horizontal lines through u and v respectively. Note that uv is a diagonal of $R(u, v)$.
- $T(u, v)$: the rectangle whose top side is the upper one of the top sides $R(u, v)$ and $S(u, v)$, and whose bottom sides are the lower one of the bottom sides of $R(u, v)$ and $S(u, v)$. Clearly, it is the minimal rectangle with vertical and horizontal sides which contains $S(u, v)$ and $R(u, v)$.

We begin with two technical geometric lemmas, whose proofs are omitted due the space limitation.

Lemma 3: Suppose that uv is a line segment satisfying that its length is at most one and its midpoint z is in S.

1) Let p and q be the intersection points of l_z with L_1 and L_2 respectively. Let Ω be the intersection of the two $\sqrt{\rho^2 - \frac{1}{4}}$-disks centered at p and q respectively (see Figure 2(a)). Then, $\Omega \subseteq B_\rho(u, v)$ and the length of the line segment $\Omega \cap L_1$ is $2g(\rho)$.
2) Suppose that u is below L_1. Let p_1 and p_2 be the two points on L_1 such that $\triangle u p_1 p_2$ is regular (see Figure 2(b)). Then, the rectangle $R(u, p_1) \cup R(u, p_2) \subseteq B_\rho(u, v)$.
3) $T(u, v) \subseteq B_\rho(u, v)$.

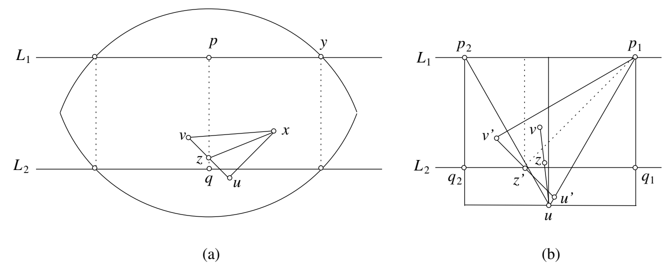

Fig. 2. Two subregions of $B_\rho(u, v)$.

Lemma 4: Consider two line segments $u_i v_i$ for $1 \leq i \leq 2$ of length at most one whose midpoints lying in S from the left to right. If $\|u_1 u_2\| \leq \rho$ and u_1 lies to the left of u_2, then

$$T(u_1, u_2) \subseteq B_\rho(u_1, v_1, u_2, v_2).$$

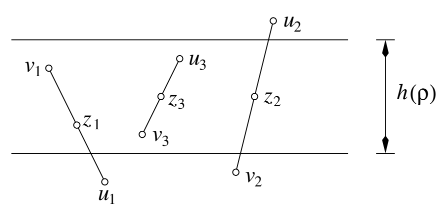

Fig. 3. If $u_1 v_1$ and $u_3 v_3$ conflict with each other, then at least one of them conflicts with $u_2 v_2$.

Now, we are ready to prove Theorem 1. For each $1 \leq i \leq 3$, let u_i and v_i be the two endpoints of a_i and z_i be the midpoint of a_i (see Figure 3). We first claim that both u_3 and v_3 lie on the right side of l_{z_1}. Assume to the contrary that our claim does not hold. Since z_3 lies to the right of z_1, u_3 and v_3 lie on different sides of z_1. By symmetry, we assume that u_3 is on the left side of z_1, and u_3 is below v_3 (see Figure 4). Then, u_3 is below L_1. Since the distance between u_3 and l_{z_1} is at most $1/2$, u_3 cannot be in S. Hence, u_3 is below L_2. Let q be the intersection point of l_{z_1} and L_2. Then,

$$\begin{aligned}\|q u_3\| &> \sqrt{\rho^2 - \frac{1}{4}} - h(\rho) \\ &= \sqrt{\rho^2 - \frac{1}{4}} \left(1 - \cos\left(\frac{\pi}{6} + \arcsin\frac{1}{2\rho}\right)\right).\end{aligned}$$

Hence,

$$\begin{aligned}\|qz_3\|^2 &\le \|u_3z_3\|^2 - \|qu_3\|^2 \\ &< \frac{1}{4} - \left(\rho^2 - \frac{1}{4}\right)\left(1 - \cos\left(\frac{\pi}{6} + \arcsin\frac{1}{2\rho}\right)\right)^2 \\ &= \frac{1}{4} - \left(1 - \frac{1}{4\rho^2}\right)\left(\rho - \frac{\sqrt{3}}{2}\sqrt{\rho^2 - \frac{1}{4}} + \frac{1}{4}\right)^2.\end{aligned}$$

The last expression decreases with ρ when $\rho \ge 1$ because both $1 - \frac{1}{4\rho^2}$ and $\rho - \frac{\sqrt{3}}{2}\sqrt{\rho^2 - \frac{1}{4}} + \frac{1}{4}$ are positive and increase with ρ. Hence, the last expression achieves its maximum $1/16$ at $\rho = 1$. So, $\|qz_3\| < 1/4$. On the other hand, since v_3 must be above L_2, $\|qv_3\| > g(\rho)$. Therefore,

$$\|v_3z_3\| \ge \|qv_3\| - \|qz_3\| > g(\rho) - \frac{1}{4} \ge \frac{3}{4} - \frac{1}{4} = \frac{1}{2},$$

which is a contradiction. So, our claim holds. Similarly, we can show that both u_3 and v_3 lie on left side of l_{z_2}. So, both u_3 and v_3 lie between l_{z_1} and l_{z_2}.

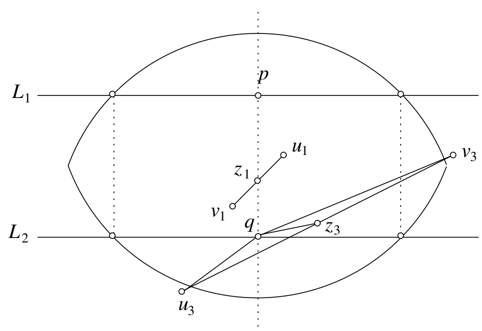

Fig. 4. If u_3 and v_3 lie on different sides of l_{z_1}, then $\|u_3v_3\| > 1$.

Next, we claim that both u_1 and v_1 are apart from l_{z_2} by a distance greater than $1/2$. Assume to the contrary that our claim does not hold. By symmetry, we assume that the distance between u_1 and l_{z_2} is at most $1/2$. Then, the distance between l_{z_1} and l_{z_2} is at most one. By Lemma 3(1) ,

$$S(z_1, z_2) \subseteq B_\rho(u_1, v_1) \cup B_\rho(u_2, v_2) \subseteq B_\rho(u_1, v_1, u_2, v_2).$$

Thus, u_3 and v_3 lie in the two pieces of the vertical strip between l_{z_1} and l_{z_2} separated by $B_\rho(u_1, v_1, u_2, v_2)$. On the other hand, both chords $B_\rho(u_1) \cap l_{z_1}$ and $B_\rho(u_1) \cap l_{z_2}$ have length at least $\sqrt{3}$. Hence, $\|u_3v_3\| \ge \sqrt{3}$, which is a contradiction. Thus, our claim holds. Similarly, we can show that both u_2 and v_2 are apart from l_{z_1} by a distance greater than $1/2$. So, both u_1 and v_1 lie to the left side of both u_2 and v_2.

Finally, we prove by contradiction that a_1 and a_2 are independent. Assume to the contrary that a_1 and a_2 are not independent. By symmetry, we assume that $\|u_1u_2\| \le \rho$. By Lemma 3 and Lemma 4,

$$T(u_1, v_1) \cup T(u_1, u_2) \cup T(u_2, v_2)$$

contains $S(z_1, z_2)$ and is singly-connected (i.e., contains no holes). So, u_3 and v_3 lie in the two pieces of the vertical strip between l_{z_1} and l_{z_2} separated by

$$T(u_1, v_1) \cup T(u_1, u_2) \cup T(u_2, v_2).$$

Hence, u_3v_3 must cross the path $v_1u_1u_2v_2$ consisting of the three line segments v_1u_1, u_1u_2 and u_2v_2. We consider three cases:

Case 1: u_3v_3 crosses v_1u_1. Then,

$$\|u_1u_3\| + \|v_1v_3\| \le \|u_1v_1\| + \|u_3v_3\| \le 2.$$

Hence,

$$\min\{\|u_1u_3\|, \|v_1v_3\|\} \le 1 \le \rho,$$

which is a contradiction.

Case 2: u_3v_3 crosses u_2v_2. Using the same argument as in **Case 1**, we can reach a contradiction.

Case 3: u_3v_3 crosses u_1u_2. Then,

$$\|u_1u_3\| + \|u_2v_3\| \le \|u_1u_2\| + \|u_3v_3\| \le \rho + 1.$$

Hence,

$$\min\{\|u_1u_3\|, \|u_2v_3\|\} \le \frac{\rho + 1}{2} \le \rho,$$

which is a contradiction.

Therefore, in either case, we get a contradiction. This completes the proof of Theorem 1.

B. Proof of Theorem 2

The function $h(\rho)$ defined in Theorem 2 has the following geometric interpretation. Consider a convex quadruple $puqv$ in which $\angle quv = \frac{\pi}{2}$, $\|qu\| = 1$, $\|qv\| = \|qp\| = \rho$ and $\|pv\| = \rho - 1$ (see Figure 5). Then, the distance between p and uv is exactly $h(\rho)$. Such geometric interpretation immediately implies that

$$2\sqrt{(\rho - 1)^2 - h(\rho)^2} \le \sqrt{\rho^2 - 1}$$

and the equality holds if and only if $\rho = 1$. Indeed, when $\rho = 1$, p is identical to v and hence $h(\rho) = 0$, which implies both sides of the above inequality equal to zero. When $\rho > 1$, we have

$$\|pu\| > \|pq\| - \|qu\| = \rho - 1 = \|pv\|,$$

which implies

$$\begin{aligned}\sqrt{\rho^2 - 1} &= \|uv\| \\ &= \sqrt{\|pu\|^2 - h(\rho)^2} + \sqrt{\|pv\|^2 - h(\rho)^2} \\ &> 2\sqrt{(\rho-1)^2 - h(\rho)^2}.\end{aligned}$$

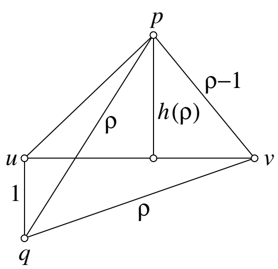

Fig. 5. The distance between p and uv is exactly $h(\rho)$

Clearly, $h(\rho)$ is continuous. It was proved in [8] that $\arccos\frac{\rho-1}{2\rho} - \arcsin\frac{1}{\rho}$ strictly increases with ρ. Thus, $h(\rho)$ is also strictly increasing. Since

$$\lim_{\rho\to\infty}\left(\arccos\frac{\rho-1}{2\rho} - \arcsin\frac{1}{\rho}\right) = \frac{\pi}{3}.$$

we have

$$\arccos\frac{\rho-1}{2\rho} - \arcsin\frac{1}{\rho} < \frac{\pi}{3}.$$

Thus, when $\rho > 1$,

$$\begin{aligned}h(\rho) &= (\rho-1)\sin\left(\arccos\frac{\rho-1}{2\rho} - \arcsin\frac{1}{\rho}\right) \\ &< (\rho-1)\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}(\rho-1).\end{aligned}$$

So,

$$h(\rho) \le \frac{\sqrt{3}}{2}(\rho-1)$$

and the equality holds if and only if $\rho = 1$.

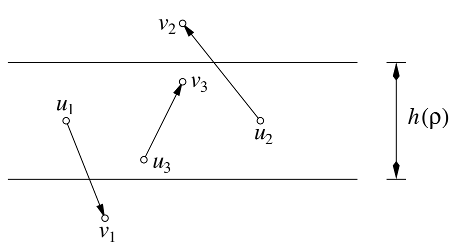

Fig. 6. If both (u_1, v_1) and (u_2, v_2) are independent with (u_3, v_3), then they are independent with each other.

Now, we are ready to prove Theorem 2. For each $1 \le i \le 3$, let $a_i = (u_i, v_i)$. It is easy to prove that Theorem 2 holds when $\rho = 1$. So, we assume that $\rho > 1$. Since both a_1 and a_2 are independent with a_3, $\|u_1u_3\| > \rho - 1$ and $\|u_2u_3\| > \rho - 1$. Since $h(\rho) < \frac{\sqrt{3}}{2}(\rho-1)$, the three vertical lines l_{u_i} for $1 \le i \le 3$ are distinct. Let w be the intersection point between the segment u_1u_2 and l_{u_3}. Then,

$$\|u_3w\| \le h(\rho) < \frac{\sqrt{3}}{2}(\rho-1).$$

So u_3w is shorter than both u_1u_3 and u_2u_3, which implies that both $\angle u_3u_1u_2$ and $\angle u_3u_2u_1$ are acute. Furthermore, the distance between u_3 and u_1u_2 is no more than $\|u_3w\|$, and hence is at most $h(\rho)$. Thus,

$$\|u_1u_2\| > 2\sqrt{(\rho-1)^2 - h(\rho)^2} > \rho - 1.$$

For $i = 1$ and 2, let p_i be the point on $\partial B_{\rho-1}(u_i)$ satisfying that (1) the distance between p_i and the line u_1u_2 is $h(\rho)$, (2) p_i lies on the same side of u_1u_2 as u_3, and (3) the perpendicular foot p_i' of p_i on the line u_1u_2 is on the segment u_1u_2 (see Figure 7). Since

$$\|u_1u_2\| > 2\sqrt{(\rho-1)^2 - h(\rho)^2} = \|u_1p_1'\| + \|u_2p_2'\|,$$

p_1' is closer to u_1 than p_2'. Thus,

$$\|u_1p_2\| > \|u_1p_1\| = \rho - 1.$$

This implies that p_2 is outside $B_{\rho-1}(u_1)$. Similarly, p_1 is outside $B_{\rho-1}(u_2)$. Then, u_3 lies inside the rectangle $p_1p_1'p_2'p_2$.

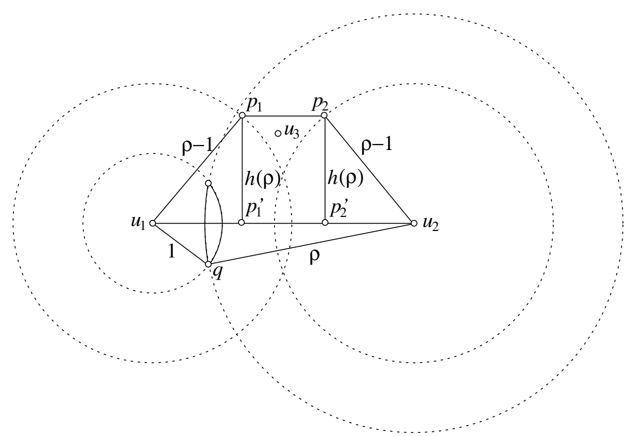

Fig. 7. Figure for the proof of Theorem 2.

Next, we claim that

$$B_1(u_1) \cap B_\rho(u_2) \subset B_\rho(u_3).$$

The claim holds trivially if $\|u_1u_2\| > \rho + 1$. So, we assume that $\|u_1u_2\| \le \rho + 1$. Since $\|u_1u_2\| > \rho - 1$, $\partial B_1(u_1)$ and $\partial B_\rho(u_2)$ intersect at two (possibly identical) points. Let q be the intersection point of $\partial B_1(u_1)$ and $\partial B_\rho(u_2)$ which lies on different side of u_1u_2 from u_3 (see Figure 7). Consider the triangle $\triangle p_2qu_2$. Since p_2u_2 and qu_2 have fixed length, $\|p_2q\|$ strictly increases with $\angle p_2u_2q$ by the law of cosine.

Since $\angle p_2u_2q = \angle p_2u_2u_1 + \angle qu_2u_1$ and $\angle p_2u_2u_1$ is also fixed, $\|p_2q\|$ also strictly increases with $\angle qu_2u_1$. Since

$$2\sqrt{(\rho-1)^2 - h(\rho)^2} \leq \sqrt{\rho^2-1}.$$

Thus, $\angle qu_2u_1$, and hence $\|p_2q\|$, achieves its maximum when qu_1 is perpendicular to u_1u_2. By the geometric interpretation of $h(\rho)$, $\|p_2q\| = \rho$ when qu_1 is perpendicular to u_1u_2. Thus, $\|p_2q\| \leq \rho$ in general. Since both q and p_1 lie on the same side of the perpendicular bisector of u_1u_2, which is also the perpendicular bisector of p_1p_2, we have

$$\|p_1q\| \leq \|p_2q\| \leq \rho.$$

Clearly, for $i = 1$ and 2,

$$\|p_i'q\| \leq \max\{\|qu_1\|, \|qu_2\|\} = \rho.$$

Thus, the distances from q to the four vertices of the rectangle $p_1p_1'p_2'p_2$ are all at most ρ. So, $\|qu_3\| \leq \rho$ and hence

$$B_1(u_1) \cap B_\rho(u_2) \subset B_\rho(u_3).$$

So, our claim holds. Similarly, we can show that

$$B_1(u_2) \cap B_\rho(u_1) \subset B_\rho(u_3).$$

Finally, we prove by contradiction that a_1 and a_2 are independent. Assume to the contrary that a_1 and a_2 are not independent. Then, either $v_1 \in B_1(u_1) \cap B_\rho(u_2)$, or $v_2 \in B_1(u_2) \cap B_\rho(u_1)$. By symmetry, we assume the former holds. Then, $v_1 \in B_\rho(u_3)$. Hence, a_3 conflicts with a_1, which is a contradiction. This completes the proof of Theorem 2.

IV. Approximate Capacity Subregions

In this section, we exploit the strip-wise transitivity of independence under either the 802.11 interference model or the protocol interference model to construct approximate capacity subregions of uniform multihop wireless networks under either the 802.11 interference model or the protocol interference model. We assume that all nodes have unit communication radius, and have interference radius equal to ρ. Under the 802.11 interference model, the height function $h(\rho)$ is as defined in Theorem 1; under the protocol interference model, we assume that $\rho > 1$ and the height function $h(\rho)$ is as defined in Theorem 2. We define the representative of the communication links as follows. Under the 802.11 interference model, the *representative* of a link is its midpoint; under the protocol interference model, the *representative* of a link is its transmitting endpoint. A link is said to be associated with a horizontal strip if its representative lies in this strip.

Now, we describe the construction of a polynomial μ-approximate capacity subregion Q, where

$$\mu = \left\lceil \frac{\rho+1}{h(\rho)} \right\rceil + 1.$$

We first compute the minimal axis-parallel rectangle surrounding all the networking nodes. Then, we partition such rectangle into top-closed bottom-open horizontal strips in the manner that the upper boundary of the top-most strip aligns with the top of the rectangle, the heights of all strips except the bottom-most one are all equal to $(\rho+1)/(\mu-1)$, and the height of the bottom-most strip is at most $(\rho+1)/(\mu-1)$ (see Figure 8). Let ℓ denote the total number of strips. We number the successive strips from top to bottom using integers $0, 1, \cdots, \ell-1$, and let A_i denote the set of links in A associated with the strip i. Then, $A_0, A_1, \cdots, A_{\ell-1}$ form a partition of A. For each $0 \leq i \leq \ell-1$, let G_i be the conflict graph of A_i. By Theorem 1 and Theorem 2, each G_i is a cocomparability graph with the lexicographic ordering of the representatives of A_i being its cocomparability ordering. Let P_i be the independence polytope of G_i as constructed in Section II. We define Q to be $\frac{1}{\mu}\prod_{i=0}^{\ell-1} P_i$.

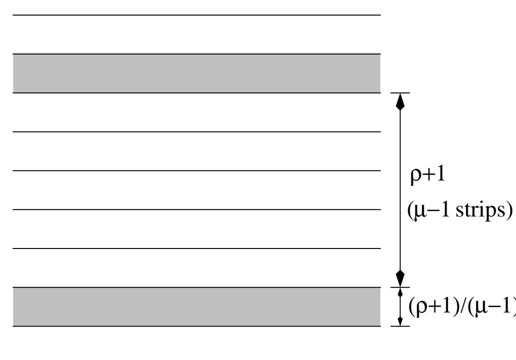

Fig. 8. Partition of the plane into half-open half-closed strips of height $(\rho+1)/(\mu-1)$ where $\mu = \lceil(\rho+1)/h(\rho)\rceil + 1$. Two strips separated by $\mu - 1$ strips can spatially share the time-slots.

Next, we show that Q is indeed a polynomial μ-approximate capacity subregion. Let P be the capacity region. It is trivial that

$$P \subseteq \prod_{i=0}^{\ell-1} P_i = \mu Q.$$

On the other hand, we give a polynomial algorithm which produces a fractional schedule of length at most one for any $d \in Q$. As a result, $Q \subseteq P$ and hence Q is a polynomial μ-approximation of P. For each $0 \leq i < \ell$, let G_i^* be the augmented complementary digraph of G_i, and $\mathcal{F}_i^*$ denote the set of flows from the auxiliary source to the the auxiliary sink in G_i^*. Any $d \in Q$ is represented by ℓ flows $f_i \in \mathcal{F}_i^*$ with $val(f_i) \leq 1/\mu$ for $0 \leq i < \ell$ such that the restriction of d on A_i is d_{f_i} for $0 \leq i < \ell$. Using the method described in Section II, we can obtain in polynomial time a link schedule Π_i for d_{f_i} of length $val(f_i)$, which is at most $1/\mu$. Note that any pair of links $a \in A_i$ and $a' \in A_{i'}$ are independent if $0 \leq i < i' < \ell$ and $i = i' \pmod{\mu}$. Thus, for each $0 \leq j < \mu$ we can merge all link schedules Π_i with $0 \leq i < \ell$ and $i = j \bmod \mu$ into a link schedule of length at most $1/\mu$. The concatenation of these μ link schedules is a link schedule for d of length at most $\mu \cdot (1/\mu) = 1$.

In the remaining of this section, we compute the value of μ. We first compute μ under the 802.11 interference model. Note that

$$\frac{\rho+1}{h(\rho)} = \frac{1+\frac{1}{\rho}}{\sqrt{1-\frac{1}{4\rho^2}}\cos\left(\frac{\pi}{6}+\arcsin\frac{1}{2\rho}\right)}.$$

Since $1+\frac{1}{\rho}$ strictly decreases with ρ and both $\sqrt{1-\frac{1}{4\rho^2}}$ and $\cos\left(\frac{\pi}{6}+\arcsin\frac{1}{2\rho}\right)$ strictly increase with ρ, $\frac{\rho+1}{h(\rho)}$ strictly decreases with ρ. When $\rho = 1$,

$$\frac{\rho+1}{h(\rho)} = \frac{2}{\frac{\sqrt{3}}{4}} = \frac{8}{\sqrt{3}}.$$

On the other hand,

$$\lim_{\rho\to\infty}\frac{\rho+1}{h(\rho)} = \frac{1}{\cos\frac{\pi}{6}} = \frac{2}{\sqrt{3}}.$$

Hence,

$$\frac{2}{\sqrt{3}} < \frac{\rho+1}{h(\rho)} \leq \frac{8}{\sqrt{3}},$$

which implies that $\mu - 1$ is an integer between 2 and 5. The root of $\frac{\rho+1}{h(\rho)} = 4$ (respectively, 3 and 2) in $[1,\infty)$ can be translated as a root of a quartic equation and is equal to 1.0891 (respectively, 1.3609 and 2.2907) by applying the quartic formula. Therefore,

$$\mu = \begin{cases} 6 & \text{if } \rho \in [1, 1.0891); \\ 5 & \text{if } \rho \in [1.0891, 1.3609); \\ 4 & \text{if } \rho \in [1.3609, 2.2907); \\ 3 & \text{if } \rho \in [2.2907, \infty). \end{cases}$$

Next, we compute μ under the protocol interference model. Note that

$$\frac{\rho+1}{h(\rho)} = \frac{\rho+1}{\rho-1} \cdot \frac{1}{\sin\left(\arccos\frac{\rho-1}{2\rho} - \arcsin\frac{1}{\rho}\right)}.$$

Since $\frac{\rho+1}{\rho-1} = 1 + \frac{2}{\rho-1}$ strictly decreases with ρ and $\arccos\frac{\rho-1}{2\rho} - \arcsin\frac{1}{\rho}$ strictly increases with ρ [8], $\frac{\rho+1}{h(\rho)}$ strictly decreases with ρ. Since $h(\rho) < \frac{\sqrt{3}}{2}(\rho-1)$, we have

$$\frac{\rho+1}{h(\rho)} > \frac{\rho+1}{\frac{\sqrt{3}}{2}(\rho-1)} > \frac{2}{\sqrt{3}}.$$

Let $\rho_1 = \infty$, and for each integer $k \geq 2$ let ρ_k be the unique root of $(\rho+1)/h(\rho) = k$ in $[1,\infty)$. Then, $\mu = k+1$ over $[\rho_k, \rho_{k-1})$ for any $k \geq 2$. By a straightforward algebraic calculation, ρ_k is the unique root of the following quartic polynomial in $(1,\infty)$:

$$\left(4-3k^2\right)\rho^4 + 4\left(k^2+k+2\right)\rho^3 + 2\left(3k^2-2k+2\right)\rho^2 - 4k(3k+1)\rho + \left(5k^2+4k\right).$$

The numeric values of ρ_k can be computed with the quartic formula. Table I lists the numeric values of ρ_k for $2 \leq k \leq 11$.

k	ρ_k	k	ρ_k
2	4.2462	7	1.5715
3	2.5689	8	1.5009
4	2.0632	9	1.4476
5	1.8167	10	1.4058
6	1.6697	11	1.3721

TABLE I
NUMBERIC VALUES OF ρ_k FOR $2 \leq k \leq 11$.

V. MAXIMUM (CONCURRENT) MULTIFLOW

Consider a wireless network $\mathbf{N}$ given by a triple $(V, A, \mathcal{I})$ and let P denote its capacity region. Suppose that we are given k commodities with s_j, t_j being the source and sink, respectively, for commodity j. We use $\mathcal{F}_j$ to denote the set of s_j–t_j flows in the digraph (V, A). A k-flow is a sequence of flows $\langle f_1, f_2, \cdots, f_k\rangle$ with $f_j \in \mathcal{F}_j$ for each $1 \leq i \leq k$. The maximum multiflow is defined by the following linear program (LP):

$$\begin{array}{ll} \max & \sum_{j=1}^k val(f_j) \\ s.t. & f_j \in \mathcal{F}_j, \forall 1 \leq j \leq k; \\ & \sum_{j=1}^k f_j \in P. \end{array}$$

Similarly, the maximum concurrent multiflow with demands $d(j)$ for $1 \leq j \leq k$ is defined by the following LP:

$$\begin{array}{ll} \max & \phi \\ s.t. & f_j \in \mathcal{F}_j, \forall 1 \leq j \leq k; \\ & val(f_j) \geq \phi d(j), \forall 1 \leq j \leq k; \\ & \sum_{j=1}^k f_j \in P. \end{array}$$

However, the membership of the capacity region P is NP-complete in general [7].

For developing a practical μ-approximation algorithm, we replace the capacity region P by a polynomial μ-approximate capacity subregion Q. Suppose that Q is a polynomial μ-approximate capacity subregion. A k-flow $\langle f_1, f_2, \cdots, f_k\rangle$ is said to be *Q-restricted* if $\sum_{j=1}^k f_j \in Q$. The maximum Q-restricted multiflow is defined by the following LP:

$$\begin{array}{ll} \max & \sum_{j=1}^k val(f_j) \\ s.t. & f_j \in \mathcal{F}_j, \forall 1 \leq j \leq k; \\ & \sum_{j=1}^k f_j \in Q. \end{array}$$

This LP is of polynomial size and we solve this LP in polynomial time to obtain a k-flow $\langle f_1, f_2, \cdots, f_k\rangle$. Then we compute a fractional link schedule of length at most one for $\sum_{j=1}^k f_j$. Such link schedule is a μ-approximate solution. Similarly, the maximum concurrent Q-restricted multiflow with demands $d(j)$ for $1 \le j \le k$ is defined by the following LP:

$$\begin{array}{ll} \max & \phi \\ s.t. & f_j \in \mathcal{F}_j, \forall 1 \le j \le k; \\ & val(f_j) \ge \phi d(j), \forall 1 \le j \le k; \\ & \sum_{j=1}^k f_j \in Q. \end{array}$$

We first solve this LP of polynomial size in polynomial time to obtain a k-flow $\langle f_1, f_2, \cdots, f_k \rangle$, and then compute a fractional link schedule of length at most one for $\sum_{j=1}^k f_j$. This link schedule is a μ-approximate solution.

Let Q be the approximate capacity subregion defined in the previous section. We describe an actual implementation of computing the maximum (concurrent) Q-restricted multiflow. We compute μ and a partition $A_0, A_1, \cdots, A_{\ell-1}$ of A as in the previous section. For each $0 \le i < \ell$, let G_i be the conflict graph of A_i. Then, each G_i is a cocomparability graph with the lexicographic ordering of the representatives of A_i being its cocomparability ordering. Let G_i^* be the augmented complementary digraph of G_i, and $\mathcal{F}_i^*$ denote the set of flows from the auxiliary source to the auxiliary sink in G_i^*. Then, the maximum Q-restricted multiflow can be rewritten as the following LP:

$$\begin{array}{ll} \max & \sum_{j=1}^k val(f_j) \\ s.t. & f_j \in \mathcal{F}_j, \forall 1 \le j \le k; \\ & f_i^* \in \mathcal{F}_i^*, \forall 0 \le i < \ell; \\ & val(f_i^*) \le 1/\mu, \forall 0 \le i < \ell; \\ & \sum_{j=1}^k f_j(e) = f_i^* \left(\delta_{G_i^*}^{out}(a) \right), \forall a \in A_i, \forall 0 \le i < \ell. \end{array}$$

We remark that two groups of flows $\{f_1, f_2, \cdots, f_k\}$ and $\{f_0^*, f_1^*, \cdots, f_{\ell-1}^*\}$ are involved in this LP. We solve this LP and obtain these two groups of flows. A link schedule of length at most one for $\sum_{j=1}^k f_j$ is then constructed directly from the second group of flows $\{f_0^*, f_1^*, \cdots, f_{\ell-1}^*\}$ using the algorithm described in Section IV. Similarly, the maximum concurrent Q-restricted multiflow with demands $d(j)$ for $1 \le j \le k$ can be rewritten as the following LP:

$$\begin{array}{ll} \max & \phi \\ s.t. & f_j \in \mathcal{F}_j, \forall 1 \le j \le k; \\ & f_i^* \in \mathcal{F}_i^*, \forall 0 \le i < \ell; \\ & val(f_j) \ge \phi d(j), \forall 1 \le j \le k; \\ & val(f_i^*) \le 1/\mu, \forall 0 \le i < \ell; \\ & \sum_{j=1}^k f_j(e) = f_i^* \left(\delta_{G_i^*}^{out}(a) \right), \forall a \in A_i, \forall 0 \le i < \ell. \end{array}$$

By solving this LP, we obtain two groups of flows $\{f_1, f_2, \cdots, f_k\}$ and $\{f_0^*, f_1^*, \cdots, f_{\ell-1}^*\}$ and compute a link schedule for $\sum_{j=1}^k f_j$ from the second group of flows.

VI. Conclusion

In this paper, we have discovered a nature of the wireless interference called strip-wise transitivity of independence under either the 802.11 interference model or the protocol interference model. We exploit such nature and utilize the independence polytopes of cocomparability graphs in a spatial-divide-conquer manner to construct a polynomial μ-approximate capacity subregion, where μ decreases with ρ in general and is smaller than the best-known ones obtained in [7]. For example, $\mu = 3$ when $\rho \ge 2.2907$ under the 802.11 interference model or when $\rho \ge 4.2462$ under the protocol interference model. We also apply these polynomial μ-approximate capacity subregions to compute μ-approximate solutions for **MMF** and **MCMF** respectively.

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