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Tighter Approximation Bounds for Minimum CDS in Unit Disk Graphs

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Abstract Connected dominating set (CDS) in unit disk graphs has a wide range of applications in wireless ad hoc networks. A number of approximation algorithms for constructing a small CDS in unit disk graphs have been proposed in the literature. The majority of these algorithms follow a general two-phased approach. The first phase constructs a dominating set, and the second phase selects additional nodes to interconnect the nodes in the dominating set. In the performance analyses of these two-phased algorithms, the relation between the independence number α and the connected domination number γ_c of a unit-disk graph plays the key role. The best-known relation between them is $\alpha \leq 3\frac{2}{3}\gamma_c + 1$. In this paper, we prove that $\alpha \leq 3.4306\gamma_c + 4.8185$. This relation leads to tighter upper bounds on the approximation ratios of two approximation algorithms proposed in the literature.

Keywords Wireless ad hoc networks · Connected dominating set · Approximation algorithm · Geometric analysis

1 Introduction

Connected dominating set (CDS) has a wide range of applications in wireless ad hoc networks (cf. a recent survey [3] and references therein). Consider a wireless ad hoc network with undirected communication topology $G = (V, E)$. A CDS of G is

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a subset $U \subset V$ satisfying that each node in $V \setminus U$ is adjacent to at least one node in U and the subgraph of G induced by U is connected. A number of distributed algorithms for constructing a small CDS in wireless ad hoc networks have been proposed in the literature. The majority of these distributed algorithms follow a general two-phased approach [1, 2, 4, 7, 9–11]. The first phase constructs a dominating set, and the nodes in the dominating set are called dominators. The second phase selects additional nodes, called connectors, which together with the dominators induce a connected topology. The algorithms in [1, 2, 4, 7, 9, 10] differ in how to select the dominators and connectors. For example, the algorithm in [2] selects the dominators using the Chvatal's greedy algorithm [5] for Set Cover, the algorithms in [1, 9] select an arbitrary maximal independent set (MIS) as the dominating set, and all the algorithms in [4, 7, 10, 11] choose a special MIS with 2-hop separation property as the dominating set.

The approximation ratios of these two-phased algorithms [1, 2, 4, 7, 9, 10] have been analyzed when the communication topology is a unit-disk graph (UDG). For a wireless ad hoc network in which all nodes lie in a plane and have equal maximum transmission radii normalized to one, its communication topology $G = (V, E)$ is often modelled by a UDG in which there is an edge between two nodes if and only if their Euclidean distance is at most one. Except the algorithms in [2, 9] which have logarithmic and linear approximations ratios respectively, all other algorithms in [1, 4, 7, 10, 11] have constant approximation ratios. The algorithm in [1] targets at distributed construction of CDS in linear time and linear messages. With this objective, it trades the size of the CDS with the time complexity, and thus its approximation ratio is a large constant (but less than 192). The analyses of the algorithms in [4, 7, 10, 11] rely on the relation between the independence number (the size of a maximum independent set) α and the connected domination number (the size of a minimum connected dominating set) γ_c of a connected UDG G. A loose relation $\alpha \leq 4\gamma_c + 1$ was obtained in [10], which implies an upper bound of 8 on the approximation ratios of both algorithms in [4, 10]. A refined relation $\alpha \leq 3.8\gamma_c + 1.2$ was discovered in [12]. With such a refined relation, the upper bound on the approximation ratios of both algorithms in [4, 10] was reduced from 8 to 7.6, and an upper bound of $5.8 + \ln 5 \approx 7.41$ on the approximation ratio of the algorithms in [7] was derived (the bound $4.8 + \ln 5 \approx 6.41$ in [7] was incorrect). The best-known relation $\alpha \leq 3\frac{2}{3}\gamma_c + 1$ if G has at least two nodes was recently proven in [11]. As a result, the upper bound on the approximation ratio of the algorithm in [10] was further reduced to $7\frac{1}{3}$ in [11]. Another greedy approximation algorithm was also proposed in [11] and its approximation ratio was proven to be bounded by $6\frac{7}{18}$.

In this paper, we first prove a further improved relation $\alpha \leq 3.4306\gamma_c + 4.8185$ in Sect. 3. The proof for this bound employs an integrated area and length argument, and involves some other interesting extreme geometric problems studied in Sect. 2. Subsequently in Sect. 4, we provide tighter analyses of the approximation algorithm in [10] and the other greedy algorithm in [11]. We prove that the approximation ratio of the former algorithm is at most 6.862 and the approximation ratio of the latter algorithm is at most 6.075.

We remark that a recent paper [6] claimed that for any connected UDG G,

$$\alpha \leq 3.453\gamma_c + 8.291.$$

However, as discovered in [11], the proof for a key geometric extreme property underlying such claim was missing, and such proof is far from being apparent or easy. Such property is rigorously proved in Lemma 6. The proof for Lemma 6 is quite lengthy and delicate. Indeed, the whole Sect. 2 is part of this proof. Consequently, the bound claimed in [6] can be treated at most as a conjecture at the time of its publication rather than a proven result.

In the remaining of this section, we introduce some terms and notations. For any point u and any $r > 0$, we use $disk_r(u)$ to denote the closed disk of radius r centered at u, and $circle_r(u)$ to denote the boundary circle of $disk_r(u)$. A path or a polygon is said to be inscribed in a circle if all its vertices lie on the circle. The Lebesgue measure (or area) of a measurable set $A \subset \mathbb{R}^2$ is denoted by $|A|$. The topological boundary of a set $A \subset \mathbb{R}^2$ is denoted by ∂A. For the simplicity of presentation, the line segment between two points u and v and its length are both denoted by uv by slightly abusing the notation, but the actual meaning can be clearly told from the context.

2 A Geometric Extreme of Polygon

The technical approach to deriving improved relation between α and γ_c is an integrated area and length argument. In this section, we present some area extremes of polygons which will be used intensively in the area argument to be given in Sect. 3.

Suppose that s, o, t and t' are four points from the left to the right on a horizontal line with $os = 1$, $ot = 0.5$ and $ot' = 1/\sqrt{3}$. If P is a regular hexagon centered at o with t' as a vertex, it is easy to compute that $|P| = \sqrt{3}/2$ and $|P \cap disk_{1.5}(s)| = \sigma$, where σ is the constant

$$\frac{\sqrt{3}}{6} - \frac{1}{2} + \frac{\sqrt{8+2\sqrt{3}}}{4} + \frac{3\pi}{8} - \frac{9}{4}\arccos\frac{1+\sqrt{3}}{3} \approx 0.85505328.$$

For an arbitrary polygon P which is inscribed in $circle_{1/\sqrt{3}}(o)$ and contains $disk_{0.5}(o)$, it has the following geometric extreme property.

Theorem 1 *Suppose that P is a polygon inscribed in $circle_{1/\sqrt{3}}(o)$ satisfying that $disk_{0.5}(o) \subseteq P$. Then,*

$$|P| \geq \sqrt{3}/2,$$
$$|P \cap disk_{1.5}(s)| \geq \sigma.$$

The first inequality in Theorem 1 can be easily proved by using the property of the sine function.

Lemma 1 *Suppose that $\phi > 0$. Then, for any set of angles $\{\theta_i : 1 \leq i \leq k\}$ satisfying that $\sum_{i=1}^{k} \theta_i = \phi$ and $0 < \theta_i \leq \frac{\pi}{3}$, we have*

$$\sum_{i=1}^{k} \sin\theta_i \geq \left\lfloor \frac{\phi}{\frac{\pi}{3}} \right\rfloor \sin\frac{\pi}{3} + \sin\left(\phi - \left\lfloor \frac{\phi}{\frac{\pi}{3}} \right\rfloor \frac{\pi}{3}\right).$$

Proof Suppose that $S = \{\theta_i : 1 \le i \le k\}$ is a set of angles satisfying that $\sum_{i=1}^{k} \theta_i = \phi$, $0 < \theta_i \le \frac{\pi}{3}$ and $\sum_{i=1}^{k} \sin\theta_i$ achieves the minimum. We first claim that the sum of any pair of angles in S is greater than $\frac{\pi}{3}$. Assume to the contrary that there are two angles θ_1 and θ_2 in S with $\theta_1 + \theta_2 \le \frac{\pi}{3}$. Replacing the two angles θ_1 and θ_2 by the single angle $\theta_1 + \theta_2$ would strictly decrease the total sine values of the angles in S. This is a contradiction, and hence our first claim holds.

Secondly, we claim that at most one angle in S is less than $\frac{\pi}{3}$. Assume to the contrary that there are two angles θ_1 and θ_2 in S which are less than $\frac{\pi}{3}$. By symmetry, we assume that $\theta_1 \le \theta_2$. Then $0 < \theta_1 + \theta_2 - \frac{\pi}{3} \le \theta_1 \le \theta_2 < \frac{\pi}{3}$. Since the sine function is concave over $[0, \frac{\pi}{3}]$ and $\theta_1 - (\theta_1 + \theta_2 - \frac{\pi}{3}) = \frac{\pi}{3} - \theta_2$, we have

$$\sin\frac{\pi}{3} - \sin\theta_2 < \sin\theta_1 - \sin\left(\theta_1 + \theta_2 - \frac{\pi}{3}\right).$$

Hence,

$$\sin\theta_1 + \sin\theta_2 > \sin\left(\theta_1 + \theta_2 - \frac{\pi}{3}\right) + \sin\frac{\pi}{3}.$$

So, replacing θ_1 and θ_2 by $\theta_1 + \theta_2 - \frac{\pi}{3}$ and $\frac{\pi}{3}$ would strictly decrease the total sine values of the angles in S. This is a contradiction, and hence our second claim holds.

Therefore, S must consist of $\lceil \frac{\phi}{\frac{\pi}{3}} \rceil$ angles, among which $\lfloor \frac{\phi}{\frac{\pi}{3}} \rfloor$ angles are equal to $\frac{\pi}{3}$. So, the lemma holds. □

Lemma 1 implies that for any polygon P inscribed in $circle_{1/\sqrt{3}}(o)$ satisfying that $circle_{0.5}(o) \subseteq P$, we have

$$|P| \ge 6 \cdot \frac{1}{2}\left(\frac{1}{\sqrt{3}}\right)^2 \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}.$$

Thus, the first inequality in Theorem 1 holds.

In the next, we prove the second inequality in Theorem 1. We first introduce a special type of polygons called canonical polygons. For any pair of points u and v on $circle_{1/\sqrt{3}}(o)$, let $\widehat{uv}$ be the arc in $circle_{1/\sqrt{3}}(o)$ from u to v in the counterclockwise manner. Denote by ϕ the radian of $\widehat{uv}$, and let $k = \lceil \phi/(\pi/3) \rceil$. We construct a path Q of k edges from u to v with all vertices on $\widehat{uv}$ as follows: If ϕ is an integer multiple of $\frac{\pi}{3}$, then all edges of Q are tangent to $circle_{0.5}(o)$; otherwise, all edges except the $\lceil k/2 \rceil$-th edge are tangent to $circle_{1/\sqrt{3}}(o)$ (we remark that in this case, the $\lceil k/2 \rceil$-th edge is disjoint from $circle_{1/\sqrt{3}}(o)$). The path Q is referred to as the *canonical path* inscribed in $circle_{1/\sqrt{3}}(o)$ from u to v. For any point u which lies on the right side the vertical line through t, we construct a polygon P as follows: let u_1 and u_2 be the two points on $circle_{1/\sqrt{3}}(o)$ such that the two line segments u_1u and u_2u are tangent to $circle_{1/\sqrt{3}}(o)$ and u_1 is above the line st. Then, P is surrounded by u_1u, u_2u and the canonical path from u_1 to u_2. The polygon P is referred to as the *canonical polygon* of u. The point u is called the *base vertex* of P, and the angle $\theta = \arccos\frac{1}{2ou}$ is called the *base angle* of P. Note that if u is on the ray ot, then P is symmetric with respect to the line ot, and the area of $P \cap disk_{1.5}(s)$ is a function of the base angle θ, which is denoted by $f(\theta)$. Note that $f(\frac{\pi}{6}) = \sigma$. We will derive the explicit expression of

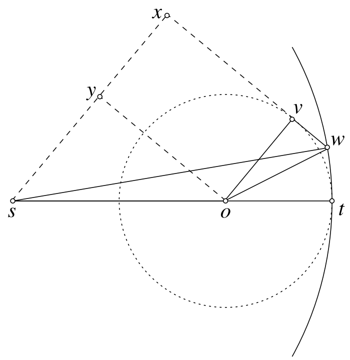


Fig. 1 Calculation of $\angle wst$ and $g(\theta)$

$f(\theta)$ and explore some useful properties of the function $f(\theta)$. We will also prove that for any canonical polygon P, $|P \cap disk_{1.5}(s)| \geq f(\theta)$ where θ is the base angle of P.

We define a geometric function g on $[0, \pi]$ as follows. For any $\theta \in [0, \pi]$, let v be a point on $circle_{1/2}(o)$ satisfying that $\angle tov = \theta$ and v is above st. Let w be the point on $circle_{1/\sqrt{3}}(o)$ satisfying that vw is tangent to $circle_{1/2}(o)$ and w lies to the right of v. Then, $g(\theta)$ is defined to be the area of the region surrounded by arc tw and the three line segments ot, ov and vw (see Fig. 1). The next lemma presents the explicit expressions of $g(\theta)$ and its first and second order derivatives.

Lemma 2 *Let* $\beta = \theta - \arccos \frac{1+2\cos\theta}{3}$. *Then*,

$$g(\theta) = \frac{9}{8}\beta - \frac{3}{4}\sin\beta + \frac{\sqrt{6}}{4}\sin\frac{\beta}{2},$$

$$g'(\theta) = \frac{13}{8} - 1.5\cos\beta,$$

$$g''(\theta) = 1.5\left(1 - \sqrt{1 - \frac{1}{2+\cos\theta}}\right)\sin\beta.$$

In addition, g *is increasing and convex on* $[0, \pi]$, *while both* g' *and* g'' *are increasing on* $[0, \frac{\pi}{2}]$.

Proof We first show that $\angle wst = \beta$. Let x be the perpendicular foot of s in the line vw, and y be the perpendicular foot of o in sx (see Fig. 1). Then, $\angle osy = \theta$ and $yx = ov = 0.5$. So

$$sx = sy + yx = 0.5 + \cos\theta.$$

Hence,

$$\angle wsx = \arccos\frac{1+2\cos\theta}{3}.$$

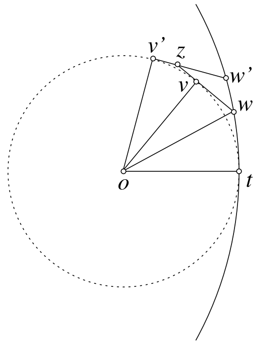


Fig. 2 Geometric proof for $g'(\theta) = \frac{ow^2}{2}$

Therefore,

$$\angle wst = \angle tsx - \angle wsx = \theta - \arccos\frac{1+2\cos\theta}{3} = \beta.$$

Now, we derive the expression of $g(\theta)$. By applying law of cosines to $\triangle osw$, $ow^2 = 13/4 - 3\cos\beta$. Hence,

$$vw = \sqrt{ow^2 - 1/4} = \sqrt{6}\sin\frac{\beta}{2}.$$

Since $g(\theta)$ is equal to the area of the sector stw minus the area of $\triangle osw$ and then plus the area of $\triangle ovw$, we have

$$g(\theta) = \frac{9}{8}\beta - \frac{3}{4}\sin\beta + \frac{1}{4}vw = \frac{9}{8}\beta - \frac{3}{4}\sin\beta + \frac{\sqrt{6}}{4}\sin\frac{\beta}{2}.$$

Next, we take a geometric approach to compute $g'(\theta)$. We rotate v counterclockwise by a small angle δ to a point v' (see Fig. 2). Then the point w moves along $circle_{1.5}(s)$ to a new point denoted by w'. Denote by z the intersection point between the two lines tangent to $circle_{0.5}(o)$ at v and v' respectively. Let Δ_1, Δ_2 and Δ_3 denote the areas of the quadruple $ovzv'$, $\triangle zww'$, and the circular segment subtended by ww' respectively. Then, $g(\theta+\delta) - g(\theta) = \Delta_1 + \Delta_2 + \Delta_3$. Clearly, $\Delta_1 = \frac{1}{2}vz = \frac{1}{4}\tan\frac{\delta}{2}$. Hence, $\lim_{\delta\to 0}\frac{\Delta_1}{\delta} = \frac{1}{8}$. Since $\angle wzw' = \delta$, $\Delta_2 = \frac{1}{2}zw \cdot zw' \cdot \sin\delta$. As $\delta \to 0$, both zw and zw' converge to vw while $\frac{\sin\delta}{\delta}$ converges to 1, and hence $\lim_{\delta\to 0}\frac{\Delta_2}{\delta} = \frac{1}{2}vw^2$. Now,

$$\Delta_3 = \frac{9}{8}\left(\angle wsw' - \sin\angle wsw'\right) = \frac{9}{8}\angle wsw'\left(1 - \frac{\sin\angle wsw'}{\angle wsw'}\right).$$

As $\delta \to 0$, $\angle wsw' \to 0$ as well, and $\frac{\angle wsw'}{\delta} \to \frac{d\beta}{d\theta}$ which is bounded for any given θ. Thus,

$$\lim_{\delta\to 0}\frac{\Delta_3}{\delta} = \frac{9}{8}\frac{d\beta}{d\theta}\cdot\lim_{\delta\to 0}\left(1 - \frac{\sin\angle wsw'}{\angle wsw'}\right) = \frac{9}{8}\frac{d\beta}{d\theta}\cdot(1-1) = 0.$$

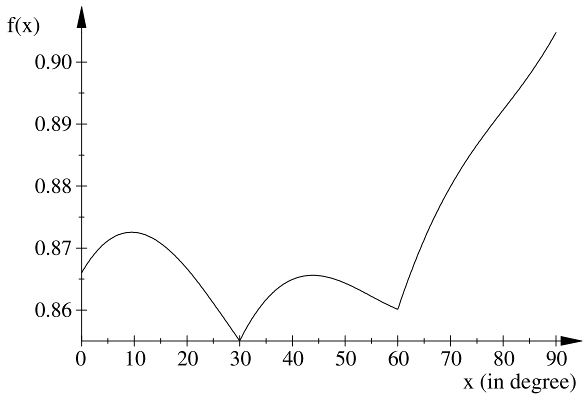


Fig. 3 The curve of f on $[0°, 90°)$

Therefore,

$$g'(\theta) = \lim_{\delta \to 0} \frac{\Delta_1 + \Delta_2 + \Delta_3}{\delta} = \frac{1/4 + vw^2}{2} = \frac{ow^2}{2} = \frac{13}{8} - 1.5 \cos\beta.$$

A straightforward calculation yields

$$\frac{d\beta}{d\theta} = 1 - \sqrt{\frac{1+\cos\theta}{2+\cos\theta}} = 1 - \sqrt{1 - \frac{1}{2+\cos\theta}}.$$

Thus,

$$g''(\theta) = 1.5 \frac{d\beta}{d\theta} \sin\beta = 1.5 \left(1 - \sqrt{1 - \frac{1}{2+\cos\theta}}\right) \sin\beta.$$

Since both g' and g'' are non-negative on $[0, \pi]$, g is increasing and convex on $[0, \pi]$ while g' is increasing on $[0, \pi]$. Clearly, $\frac{d\beta}{d\theta}$ is positive and increasing with θ on $[0, \pi]$, and consequently β is also strictly increasing with θ on $[0, \pi]$. When $\theta \in [0, \frac{\pi}{2}]$, β is acute, and hence g'' is increasing on $[0, \frac{\pi}{2}]$. $\square$

It is easy to show that $f(\theta) = 2g(\theta) + h(\theta)$, where

$$h(\theta) = \frac{1}{4\sqrt{3}} \left\lfloor 6 - \frac{\theta}{\frac{\pi}{6}} \right\rfloor + \frac{1}{6} \sin 2\left(\left(6 - \left\lfloor 6 - \frac{\theta}{\frac{\pi}{6}} \right\rfloor\right) \frac{\pi}{6} - \theta\right).$$

Figure 3 is the curve of f on $[0°, 90°)$. We observe and will prove later that f is increasing on $[60°, 90°)$. However, on either of the two intervals $[0°, 30°]$ and $[30°, 60°]$, f is neither monotone, nor concave, and nor convex. Fortunately, on either of these two intervals f has the following weak but still nice *quasi-concave* property: f is said to be quasi-concave on an interval $[a, b] \subset [0°, 90°)$ if for each triple of increasing values $\theta_1, \theta_2, \theta_3$ in $[a, b]$, $f(\theta_2) \geq \min\{f(\theta_1), f(\theta_3)\}$.

Lemma 3 *f is quasi-concave on $[0, \frac{\pi}{6}]$ and $[\frac{\pi}{6}, \frac{\pi}{3}]$ respectively, and increasing on $[\frac{\pi}{3}, \frac{\pi}{2})$.*

Proof We first show that f is concave on $[0, \frac{3\pi}{20}]$ and decreasing on $[\frac{3\pi}{20}, \frac{\pi}{6}]$. Note that on $[0, \frac{\pi}{6}]$,

$$h'(\theta) = -\frac{1}{3}\cos 2\left(\frac{\pi}{6} - \theta\right),$$
$$h''(\theta) = -\frac{2}{3}\sin 2\left(\frac{\pi}{6} - \theta\right).$$

Clearly, h'' is increasing on $[0, \frac{\pi}{6}]$. Thus, f'' is also increasing on $[0, \frac{\pi}{6}]$. For any $\theta \in [0, \frac{3\pi}{20}]$,

$$f''(\theta) \le f''\left(\frac{3}{20}\pi\right) < 0.$$

So, f is concave on $[0, \frac{3\pi}{20}]$. For any $\theta \in [\frac{3\pi}{20}, \frac{\pi}{6}]$,

$$f'(\theta) \le 2g'(\pi/6) + h'\left(\frac{3}{20}\pi\right) < 0.$$

So, f is decreasing over $[\frac{3\pi}{20}, \frac{\pi}{6}]$.

Now, we show that f is quasi-concave over $[0, \frac{\pi}{6}]$. Consider any $0 \le \theta_1 \le \theta_2 \le \theta_3 \le \frac{\pi}{6}$. If $\theta_3 \le \frac{3\pi}{20}$, then $f(\theta_2) \ge \min\{f(\theta_1), f(\theta_3)\}$ due to the concavity of f over $[0, \frac{3\pi}{20}]$. If $\theta_2 \ge \frac{3\pi}{20}$, then $f(\theta_2) \ge f(\theta_3)$ since f is decreasing over $[\frac{3\pi}{20}, \frac{\pi}{6}]$. If $\theta_2 < \frac{3\pi}{20} < \theta_3$, then $f(\theta_3) \le f(\frac{3\pi}{20})$ and $f(\theta_2) \ge \min\{f(\theta_1), f(\frac{3\pi}{20})\}$, which together imply that $f(\theta_2) \ge \min\{f(\theta_1), f(\theta_3)\}$.

By a similar argument, we can prove that f is concave over $[\frac{\pi}{6}, \frac{3\pi}{10}]$ and decreasing over $[\frac{3\pi}{10}, \frac{\pi}{3}]$. Therefore, for any $\frac{\pi}{6} \le \theta_1 \le \theta_2 \le \theta_3 \le \frac{\pi}{3}$, $f(\theta_2) \ge \min\{f(\theta_1), f(\theta_3)\}$.

Next, we prove that f is increasing over $[\frac{\pi}{3}, \frac{\pi}{2})$ by showing that f' is positive over $[\frac{\pi}{3}, \frac{\pi}{2})$. Note that $h'(\theta) = \frac{1}{3}\cos 2\theta$ for $\theta \in [\frac{\pi}{3}, \frac{\pi}{2})$. For any $\theta \in [\frac{\pi}{3}, \frac{5\pi}{12}]$,

$$f'(\theta) \ge 2g'(\pi/3) + h'\left(\frac{5}{12}\pi\right) > 0.$$

For any $\theta \in [\frac{5\pi}{12}, \frac{\pi}{2})$,

$$f'(\theta) \ge 2g'\left(\frac{5}{12}\pi\right) - \frac{1}{3} > 0. \qquad \square$$

It is easy to verify that $f(0) = \sqrt{3}/2$, $f(\frac{\pi}{6}) = \sigma$, and

$$f\left(32^\circ\right) < f\left(\frac{\pi}{3}\right) < f\left(34^\circ\right).$$

So, by Lemma 3 we have the following corollary.

Corollary 1 *The minimum of f on the interval* $[0^\circ, 90^\circ)$ (*respectively*, $[32^\circ, 90^\circ)$ *and* $[34^\circ, 90^\circ)$) *is achieved at* 30° (*respectively*, 32° *and* 60°).

Next, we prove the following extreme property of the canonical polygons.

Lemma 4 *For any canonical polygon P with base angle θ, $|P \cap disk_{1.5}(s)| \geq f(\theta)$.*

Proof Denote by u the base vertex of P, and let v_1 and v_2 be the two points on $circle_{0.5}(o)$ satisfying that uv_1 and uv_2 are tangent to $circle_{0.5}(o)$. Denote by θ_1 and θ_2 the radians of the two arcs $\widehat{v_1 t}$ and $\widehat{v_2 t}$ respectively. Then $\theta_1 + \theta_2 = 2\theta$, and $|P \cap disk_{1.5}(s)| = g(\theta_1) + g(\theta_2) + h(\theta)$. By Lemma 2, g is convex, and hence $g(\theta_1) + g(\theta_2) \geq 2g(\frac{\theta_1+\theta_2}{2}) = 2g(\theta)$. Thus, $|P \cap disk_{1.5}(s)| \geq 2g(\theta) + h(\theta) = f(\theta)$. □

Corollary 1 and Lemma 4 imply that $|P \cap disk_{1.5}(s)| \geq \sigma$ for any canonical polygon P. We move on to prove the same inequality holds for any polygon P inscribed in $circle_{1/\sqrt{3}}(o)$ satisfying that $circle_{0.5}(o) \subseteq P$. All vertices of any such polygon P can be classified into two categories. A vertex of P is called an *inner* vertex if it belongs to $disk_{1.5}(s)$, and an *outer* vertex otherwise. Let u_1, respectively u_2, be the rightmost inner vertex of P above, respectively below st. By Lemma 1, replacing all the internal vertices of P by the vertices of the canonical path from u_1 to u_2 would not increase $|P \cap disk_{1.5}(s)|$. Thus, from now on we assume that all internal vertices of P are the vertices of the canonical path from u_1 to u_2. Let u' be the point such that the two line segments u_1u' and u_2u' are both tangent to $circle_{0.5}(o)$. Then, u' is on the right side of the vertical line through t. Let P' be the canonical polygon of u'. If no side of P is a secant of $disk_{1.5}(s)$, then

$$|P \cap disk_{1.5}(s)| \geq \left|P' \cap disk_{1.5}(s)\right| \geq \sigma.$$

So, we further assume that at least one side of P is a secant of $disk_{1.5}(s)$. We will prove that $|P \cap disk_{1.5}(s)| > \sigma$.

Let p, respectively p', be the upper, respectively lower, intersection point between $circle_{1/\sqrt{3}}(o)$ and $circle_{1.5}(s)$, and let q, respectively q', be the upper, respectively lower, intersection point between the vertical line through t and $circle_{1.5}(s)$ (see Fig. 4). A straightforward calculation yields

$$\angle pot = \arccos \frac{11\sqrt{3}}{24} \approx 37.453°,$$

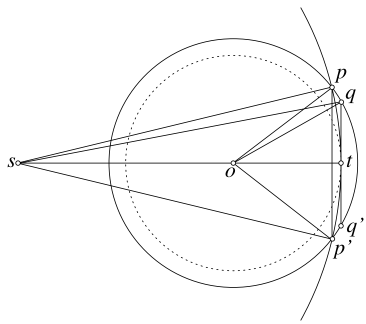


Fig. 4 The basic configuration

$$\angle poq = \angle pot - \frac{\pi}{6} \approx 7.453°,$$

$$\angle ops = \arccos \frac{19\sqrt{3}}{36} \approx 23.916°.$$

Thus, any side of P which is a secant of $circle_{1.5}(s)$ must have its left endpoint on either the minor arc $\widehat{pq}$ or the minor arc $\widehat{p'q'}$, and its right endpoint on the minor arc $\widehat{qq'}$. Consequently, there are at most two sides of P which are secants of $circle_{1.5}(s)$. Next, we show that at most one side of P is a secant of $circle_{1.5}(s)$ using the following lemma.

Lemma 5 *Suppose that e is a side of P which is a secant of $circle_{1.5}(s)$. Then, the central angle of e at o is greater than $2\angle ops$, and the central angle of the chord $e \cap disk_{1.5}(s)$ at s is at most $\frac{\pi}{3} - 2\angle ops$.*

Proof Let u be the left endpoint of e and assume by symmetry that u lies on the arc $\widehat{pq}$. Denote by a the chord $e \cap disk_{1.5}(s)$ of $circle_{1.5}(s)$. Let v be the midpoint of e and w be the midpoint of a (see Fig. 5(a)). Then, v and w are the perpendicular feet of o and s respectively on e, v lies above st, and w lies between u and v on e. Let x be the perpendicular foot of o in sw. Then

$$sw = sx + xw = \cos \angle wst + ov = \cos \angle vot + ov = \cos (\angle uot - \angle uov) + ov.$$

Suppose we fix the length of e but allow u to freely move along the arc $\widehat{pq}$ towards p. When u moves towards p, both $\angle uov$ and ov are fixed but $\angle uot$ increases, hence sw decreases and e remains as secant of $circle_{1.5}(s)$. Similarly, suppose we fix u but allow e to increase its length. When the length of e increases, both $\angle uov$ and ov decrease, hence sw decreases and e remains as secant of $circle_{1.5}(s)$.

Now, we show that the central angle of e at o must be greater than $2\angle ops$. Assume to the contrary that the central angle of e at o is at most $2\angle ops$. We first move u to p while fixing the length of e, and then subsequently increase the central angle of e to $2\angle ops$ while fixing u to the point p (see Fig. 5(b)). Then e would still be a secant of $circle_{1.5}(s)$. On the other hand, the central angle of e is $2\angle pov$ in this case and hence $\angle pov = \angle ops$. So, sp is parallel to ov and is thus perpendicular to e. This means that

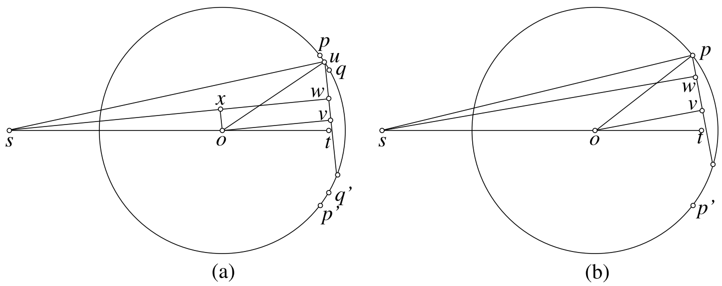


Fig. 5 A side e of P which is a secant of $circle_{1.5}(s)$

e is tangent to $circle_{1.5}(s)$ at p, which is a contradiction. Therefore, the central angle of e at o must be greater than $2\angle ops$.

Since the central angle of a decreases with sw, it achieves its maximum when $u = p$ and the central angle of e is $\frac{\pi}{3}$ (see Fig. 5(b)). In this case, the central angle of a at s is

$$2\angle psw = 2\left(\frac{\pi}{2} - \angle spw\right) = 2\left(\frac{\pi}{2} - \left(\angle ops + \frac{\pi}{3}\right)\right) = \frac{\pi}{3} - 2\angle ops.$$

Thus, the central angle of a at s is at most $\frac{\pi}{3} - 2\angle ops$ in general. $\square$

Since

$$\angle pop' = 2\angle pot = 2\,(\angle ops + \angle pso) < 2 \cdot 2\angle ops = 4\angle ops.$$

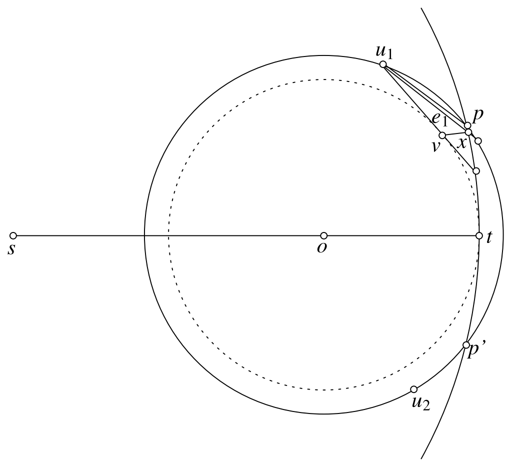
Lemma 5 implies that exactly one side of P is a secant of $circle_{1.5}(s)$. We denote such side by e and assume by symmetry that the left endpoint of e lies on the arc pq. The area of the circular cap determined by the chord $e \cap disk_{1.5}(s)$ in $disk_{1.5}(s)$ is referred to as the *outer loss*. Denote by ℓ_1 the constant

$$\frac{9}{8}\left(\frac{\pi}{3} - 2\angle ops - \sin\left(\frac{\pi}{3} - 2\angle ops\right)\right) \approx 1.7915347 \times 10^{-3}.$$

By Lemma 5, the outer loss is at most ℓ_1. Let e_1 be the side of P between u_1 and the adjacent outer vertex, φ be the central angle of e_1, and δ be the central angle of pu_1. Denote by x the intersection point between e_1 and $circle_{1.5}(s)$. Since the right endpoint of e_1 is on the arc pq, we have $\delta < \varphi \le \delta + \angle poq$. We consider two cases according to whether δ is at least 3° or not.

Case 1: $\delta \ge 3°$. Let v be the point on $circle_{0.5}(o)$ at which $u'u_1$ is tangent to $circle_{0.5}(o)$ (see Fig. 6). The area of the arc triangle surrounded by e_1, the ray u_1v and $circle_{1.5}(s)$ is referred to as the *inner gain*. Clearly, $|P \cap disk_{1.5}(s)|$ is at least

Fig. 6 Inner gain



$|P' \cap disk_{1.5}(s)|$ plus the inner gain and minus the outer loss. Since the inner gain is more than $|\triangle u_1 vx|$, we have

$$|P \cap disk_{1.5}(s)| \geq \left|P' \cap disk_{1.5}(s)\right| + |\triangle u_1 vx| - \ell_1.$$

First, we show that if $\delta \in [3°, 48°]$ then $|\triangle u_1 vx| > \ell_1$, from which we have

$$|P \cap disk_{1.5}(s)| > \left|P' \cap disk_{1.5}(s)\right| \geq \sigma$$

by Corollary 1 and Lemma 4. Clearly, $u_1 v = \frac{1}{2\sqrt{3}}$. Since

$$u_1 x \geq u_1 p = \frac{2}{\sqrt{3}} \sin \frac{\delta}{2},$$

$$\angle v u_1 x \geq \angle v u_1 q = \frac{1}{2}\left(\frac{\pi}{3} - \delta - \angle poq\right),$$

we have

$$\begin{aligned}
|\triangle u_1 vx| &= \frac{1}{2} u_1 v \cdot u_1 x \cdot \sin \angle v u_1 x \\
&\geq \frac{1}{6} \sin \frac{\delta}{2} \cdot \sin\left(\frac{\pi}{6} - \frac{\angle poq}{2} - \frac{\delta}{2}\right) \\
&= \frac{1}{12}\left[\cos\left(\frac{\pi}{6} - \frac{\angle poq}{2} - \delta\right) - \cos\left(\frac{\pi}{6} - \frac{\angle poq}{2}\right)\right].
\end{aligned}$$

Since $\frac{\pi}{6} - \frac{\angle poq}{2} \approx 26.274°$, the last expression above is concave with $\delta \in [3°, 48°]$ and its minimum over $[3°, 48°]$ is achieved at boundary. A simple calculation yields that its minimum is achieved at $\delta = 3°$ and is greater than ℓ_1. Hence, $|\triangle u_1 vx| \geq \ell_1$ when $\delta \in [3°, 48°]$.

Now, we show that if $\varphi \in [12°, 57°]$, then $|\triangle u_1 vx| \geq \ell_1$, from which we have

$$|P \cap disk_{1.5}(s)| > \left|P' \cap disk_{1.5}(s)\right| \geq \sigma$$

by Corollary 1 and Lemma 4. Since

$$u_1 x \geq u_1 p = \frac{2}{\sqrt{3}} \sin \frac{\delta}{2} \geq \frac{2}{\sqrt{3}} \sin \frac{\varphi - \angle poq}{2},$$

$$\angle v u_1 x = \frac{\pi}{6} - \frac{\varphi}{2},$$

we have

$$\begin{aligned}
|\triangle u_1 vx| &= \frac{1}{2} u_1 v \cdot u_1 x \cdot \sin \angle v u_1 x \\
&\geq \frac{1}{6} \sin\left(\frac{\varphi}{2} - \frac{\angle poq}{2}\right) \sin\left(\frac{\pi}{6} - \frac{\varphi}{2}\right)
\end{aligned}$$

$$= \frac{1}{12}\left[\cos\left(\varphi - \frac{\pi}{6} - \frac{\angle poq}{2}\right) - \cos\left(\frac{\pi}{6} - \frac{\angle poq}{2}\right)\right]$$
$$= \frac{1}{12}\left[\cos\left(\frac{\pi}{6} - \frac{\angle poq}{2} - \left(\frac{\pi}{3} - \varphi\right)\right) - \cos\left(\frac{\pi}{6} - \frac{\angle poq}{2}\right)\right].$$

Since $\frac{\pi}{3} - \varphi \in [3°, 48°]$ when $\varphi \in [12°, 57°]$, the above expression is also greater than ℓ_1. Hence, $|\triangle u_1vx| > \ell_1$ when $\varphi \in [12°, 57°]$.

Next, we assume that $\delta \notin [3°, 48°]$ and $\varphi \notin [12°, 57°]$. Since $\delta \geq 3°$, we have $\varphi > \delta > 48°$, which implies that $\varphi > 57°$. Hence, the base angle of P' is $\frac{\angle u_1ou_2}{2} - \frac{\pi}{6}$. Since

$$\frac{\angle u_1ou_2}{2} - \frac{\pi}{6} \geq \frac{\delta + \angle pop'}{2} - \frac{\pi}{6} = \frac{\delta}{2} + \angle poq$$
$$\geq \frac{\varphi - \angle poq}{2} + \angle poq = \frac{\varphi + \angle poq}{2} \geq 32°,$$

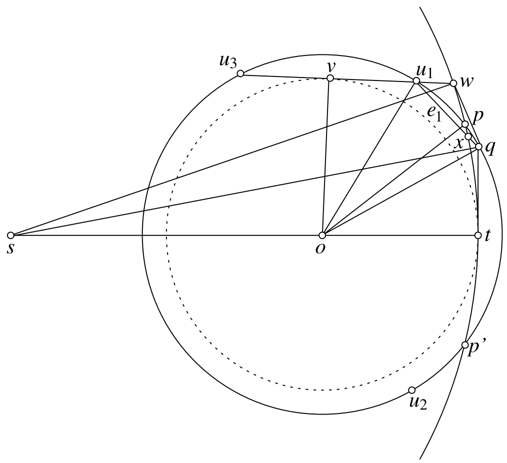
by Corollary 1 and Lemma 4 we have

$$\left|P' \cap disk_{1.5}(s)\right| \geq f\left(32°\right) > \sigma + \ell_1.$$

Consequently, $|P \cap disk_{1.5}(s)| > \sigma$.

Case 2: $\delta < 3°$. Let u_3 be the inner vertex of P adjacent to u_1. Then, u_1u_3 is tangent to $circle_{0.5}(o)$ at the midpoint, denoted by v, of u_1u_3. Denote by θ_1 the angle $\angle vot$. Then, $\theta_1 = \angle pot + \delta + \frac{\pi}{6} < \frac{\pi}{2}$. Let u'' be the point such that the two line segments u_3u'' and u_2u'' are both tangent to $circle_{0.5}(o)$. Then, u'' is on the right side of the vertical line through t, and u_1 is on the line segment u_3u''. Let P'' be the canonical polygon of u''. The base angle of P'' is $\frac{\angle u_1ou_2}{2}$, which is greater than $\frac{\angle pop'}{2} = \angle pot \approx 37.453°$. By Corollary 1 and Lemma 4, $|P'' \cap disk_{1.5}(s)| \geq f(\frac{\pi}{3})$. Let w be the intersection point between vu_1 and $circle_{1.5}(s)$ (see Fig. 7). The area of the arc triangle u_1wx surrounded by the arc wx and the two line segments u_1w and

Fig. 7 Inner loss



u_1x is referred to as the *inner loss* and is denoted by ℓ_2. Then,

$$|P \cap disk_{1.5}(s)| \ge \left|P'' \cap disk_{1.5}(s)\right| - \ell_1 - \ell_2 \ge f\left(\frac{\pi}{3}\right) - \ell_1 - \ell_2.$$

So, it is sufficient to show that $\ell_2 < f(\frac{\pi}{3}) - \sigma - \ell_1$.

We first claim that $\ell_2 \le |\triangle u_1wq|$. Indeed, while u_1 is fixed the arc triangle u_1wx grows when the right endpoint of e_1 moves toward q. Thus we only need to prove that the claim holds when the right endpoint of e_1 is q. Note that when $\theta_1 < \frac{\pi}{2}$,

$$\angle wst < \arcsin\frac{1}{3} < 2\arcsin\frac{1}{3\sqrt{3}} = 2\angle qst.$$

Thus, $\angle qsw = \angle wst - \angle qst < \angle qst$. By the law of cosines, we have $qw < qt$. Hence,

$$qs^2 = st^2 + qt^2 > sw^2 + qw^2,$$

which implies that $\angle swq$ is obtuse. So, the arc triangle u_1wx is contained in $\triangle u_1wq$, and consequently the inner loss is at most $|\triangle u_1wq|$. Hence the claim holds. Therefore,

$$|P \cap disk_{1.5}(s)| \ge \left|P'' \cap disk_{1.5}(s)\right| - |\triangle u_1wq| - \ell_1 \ge f\left(\frac{\pi}{3}\right) + |\triangle u_1vx| - \ell_1.$$

Now, we claim that $|\triangle u_1wq|$ increases with θ_1 and hence with δ when $\delta \le 3°$. Note that

$$|\triangle u_1wq| = \frac{1}{2}u_1w \cdot u_1q \cdot \sin\angle qu_1w.$$

Since ow increases with θ_1 and

$$u_1w = vw - \frac{1}{2\sqrt{3}} = \sqrt{ow^2 - 1/4} - \frac{1}{2\sqrt{3}},$$

u_1w increases with θ_1. Since

$$u_1q = \frac{2}{\sqrt{3}}\sin\left(\frac{\theta_1}{2} - \frac{\pi}{6}\right)$$

u_1q also increases with θ_1. Since

$$\angle qu_1w = \pi - \frac{\pi}{3} - \angle ou_1q = \frac{2\pi}{3} - \frac{\pi - \angle qou_1}{2} = \frac{\frac{\pi}{3} + \angle qou_1}{2} = \frac{\theta_1}{2}$$

$\angle qu_1w$ increases with θ_1. Thus, our claim holds.

Denote by ℓ_2 the area of $\triangle u_1wq$ when $\delta = 3°$. It is easy to verify that when $\delta = 3°$, $\theta_1 = \angle pot + \frac{11\pi}{60}$ and

$$|\triangle u_1wq| = \left(\frac{\sqrt{2}}{4}\sin\frac{\theta_1 - \arccos\frac{1+2\cos\theta_1}{3}}{2} - \frac{1}{12}\right)\left(\frac{\sqrt{3}}{2} - \cos\left(\theta_1 - \frac{\pi}{6}\right)\right)$$

$$< f\left(\frac{\pi}{3}\right) - \sigma - \ell_1.$$

Therefore, $\ell_2 < f(\frac{\pi}{3}) - \sigma - \ell_1$, which implies $|P \cap disk_{1.5}(s)| > \sigma$.

3 Independence Number vs. Connected Domination Number

In this section, we present an improved upper bound on the independence number in terms of the connected domination number.

Theorem 2 *Let α and γ_c be the independence number and connected domination number of a connected UDG G. Then,*

$$\alpha \leq 3.4306\gamma_c + 4.8185.$$

We prove the above theorem by an integrated area and length argument. Let U be a minimum CDS of G, and define

$$\Omega = \bigcup_{u \in U} disk_{1.5}(u).$$

Consider a maximum independent set I of G. We construct the Voronoi diagram defined by I. For each $o \in I$, we use $Vor(o)$ to denote its Voronoi cell and call the set $Vor(o) \cap \Omega$ as the *truncated Voronoi cell* of o. Clearly, $|\Omega|$ is the total area of truncated Voronoi cells of all nodes in I. We partition I into two subsets I_1 and I_2 defined by

$$I_1 = \left\{o \in I : disk_{1/\sqrt{3}}(o) \subset \Omega\right\},$$
$$I_2 = I \setminus I_1.$$

Denote by α_1 and α_2 the size of I_1 and I_2 respectively. The next lemma provides a lower bound on each truncated Voronoi cell.

Lemma 6 *For each o in I_1 (respectively, I_2), the area of its truncated Voronoi cell is at least $\sqrt{3}/2$ (respectively, σ).*

Proof Since the pairwise distances of the points in I are at least one, the distance between o and each side of $Vor(o)$ is at least 0.5 and consequently $disk_{0.5}(o) \subset Vor(o)$. Next, we show that no vertex of $Vor(o)$ is inside $disk_{1/\sqrt{3}}(o)$. Let v be a vertex of $Vor(o)$, and e_1 and e_2 be the two sides of $Vor(o)$ incident to v (see Fig. 8). Let o_1 (respectively, o_2) be the point which is symmetric to o with respect to e_1 (respectively, e_2). Then, both o_1 and o_2 belong to I, and hence the three sides of $\triangle oo_1o_2$ are all at least 1. Clearly, v is the center of $\triangle oo_1o_2$. Since at least one of the three central angles of $\triangle oo_1o_2$ is at most 120°, the circumscribing radius of $\triangle oo_1o_2$ is at least $1/\sqrt{3}$. Thus, $ov \geq 1/\sqrt{3}$.

Let s be the node in the MCDS U closest to o. Then, $o \in disk_1(s)$. If $disk_{1/\sqrt{3}}(o) \subseteq Vor(o)$, then $|Vor(o) \cap \Omega| \geq |disk_{1/\sqrt{3}}(o) \cap disk_{1.5}(s)| > \sqrt{3}/2$. So, we assume

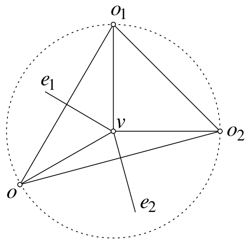


Fig. 8 Any vertex of *Vor*(o) is apart from o by at least $1/\sqrt{3}$

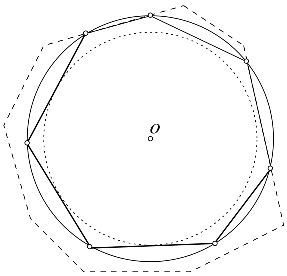


Fig. 9 Inserting operations

$disk_{1/\sqrt{3}}(o)$ is not fully contained in $Vor(o)$. Then $Vor(o)$ intersects $circle_{1/\sqrt{3}}(o)$. We construct a polygon $P \subseteq Vor(o)$ satisfying that P is inscribed in $circle_{1/\sqrt{3}}(o)$ and $disk_{0.5}(o) \subseteq P \subseteq Vor(o)$. Let Q be the sequence of intersecting points between $Vor(o)$ and $circle_{1/\sqrt{3}}(o)$ in the counterclockwise order. For each pair of successive u and v in Q, if $\angle uov \le \frac{\pi}{3}$, we add to P a side between u and v; otherwise, we add to P a path inscribed in the arc from u to v satisfying that each edge in this path is either tangent to or disjoint from $circle_{1/\sqrt{3}}(o)$ (see Fig. 9). The resulting polygon P meets the requirement. By Theorem 1, $|P| \ge \sqrt{3}/2$.

If $o \in I_1$, then

$$P \subseteq Vor(o) \cap disk_{1/\sqrt{3}}(o) \subseteq Vor(o) \cap \Omega,$$

hence

$$|Vor(o) \cap \Omega| \ge |P| \ge \sqrt{3}/2.$$

Now, we assume that $o \in I_2$. Note that $|P \cap disk_{1.5}(s)|$ grows when moving o away from s along a fixed radius of $disk_{1.5}(s)$. By Theorem 1, $|P \cap disk_{1.5}(s)| \ge \sigma$. Since

$$P \cap disk_{1.5}(s) \subseteq Vor(o) \cap \Omega,$$

we have

$$|Vor(o) \cap \Omega| \ge |P \cap disk_{1.5}(s)| \ge \sigma. \qquad \square$$

We define

$$\Omega' = \bigcup_{v \in U} disk_{1.5-1/\sqrt{3}}(v).$$

The next lemma gives an upper bound on the length of $\partial\Omega'$.

Lemma 7 *The length of* $\partial\Omega'$ *is at most* $2(1-1/\sqrt{3})\alpha_2$.

Proof For each $o \in I_2$, let o' be a point in U which is closest to o. Then,

$$1.5 - 1/\sqrt{3} < oo' \leq 1.$$

Let o'' be the point which is the intersection of the segment oo' and $circle_{1.5-1/\sqrt{3}}(o')$. Then, $o'' \in \partial\Omega'$ and $oo'' = oo' - o'o'' \leq 1 - (1.5 - 1/\sqrt{3}) = 1/\sqrt{3} - 0.5$. We call o'' the projection of o on $\partial\Omega'$. Consider two points o_1 and o_2 in I. Then,

$$o_1''o_2'' \geq o_1o_2 - o_1o_1'' - o_2o_2'' > 1 - 2\left(1/\sqrt{3} - 0.5\right) = 2\left(1 - 1/\sqrt{3}\right).$$

Now we decompose $\partial\Omega'$ into disjoint arc-polygons, each of which is a maximally connected piece of $\partial\Omega'$. We claim that if a piece contains $k \geq 1$ projections of points in I_2, then its length is at least $2(1-1/\sqrt{3})k$. Such a claim leads to the lemma immediately. The claim is true if $k \geq 2$. So we assume that $k = 1$. Suppose that a piece Q contains the projection o'' of a point $o \in I_2$ on $\partial\Omega'$. Let P be the region surrounded by the piece. Then exactly one of o and o' is inside P. If $o' \in P$, then the whole disk $disk_{1.5-1/\sqrt{3}}(o')$ is contained in P, and hence the length of Q is at least $(3-2/\sqrt{3})\pi$, which is greater than $2(1-1/\sqrt{3})$. So, we further assume that $o \in P$. We prove by contradiction that $P \subsetneq \Omega$. Assume to the contrary that $P \subseteq \Omega$. Since $o \in I_2$, there is a point $p \in disk_{1/\sqrt{3}}(o) \setminus \Omega$. Clearly, $p \notin P$ and hence op intersects with Q. Let q be an intersection point between op and Q, and v be the center of the arc in Q which contains q. Then, $vq = 1.5 - 1/\sqrt{3}$ and $pq \leq op \leq 1/\sqrt{3}$. Hence,

$$vp \leq vq + pq \leq 1.5 - 1/\sqrt{3} + 1/\sqrt{3} = 1.5.$$

So, $p \in disk_{1.5}(v) \subseteq \Omega$, which is a contradiction. Therefore, $P \subsetneq \Omega$. Consider an arbitrary point $x \in P \setminus \Omega$. Then, the distance between x and any point in U is greater than 1.5, which implies that the distance between x and any arc in Q is more than $1.5 - (1.5 - 1/\sqrt{3}) = 1/\sqrt{3}$. Hence. the disk $disk_{1/\sqrt{3}}(x)$ is contained in P, and as a result $|P| > \pi/3$. By the well-known isodiametric inequality (see, e.g., in [8]), the length of Q is more than $2\pi/\sqrt{3}$, which is greater than $2(1-1/\sqrt{3})$. Thus, the claim also holds when $k = 1$. □

By Lemma 6,

$$|\Omega| \geq \frac{\sqrt{3}}{2}\alpha_1 + \sigma\alpha_2 = \frac{\sqrt{3}}{2}\alpha - \left(\frac{\sqrt{3}}{2} - \sigma\right)\alpha_2,$$

which implies

$$\alpha \leq \frac{|\Omega|}{\frac{\sqrt{3}}{2}} + \left(1 - \frac{\sigma}{\frac{\sqrt{3}}{2}}\right)\alpha_2. \tag{1}$$

It is easy to prove by induction on γ_c that

$$|\Omega| \leq \frac{9}{2}\left((\gamma_c - 1)\left(\arcsin\frac{1}{3} + \frac{\sqrt{8}}{9}\right) + \frac{\pi}{2}\right), \tag{2}$$

and the length of $\partial\Omega'$ is at most

$$2\left(3 - \frac{2}{\sqrt{3}}\right)\left((\gamma_c - 1)\arcsin\frac{1}{3 - \frac{2}{\sqrt{3}}} + \frac{\pi}{2}\right).$$

By Lemma 7,

$$\begin{aligned}\alpha_2 &\leq \frac{2(3 - \frac{2}{\sqrt{3}})((\gamma_c - 1)\arcsin\frac{1}{3-\frac{2}{\sqrt{3}}} + \frac{\pi}{2})}{2(1 - \frac{1}{\sqrt{3}})} \\ &= \frac{\sqrt{3}+7}{2}\left((\gamma_c - 1)\arcsin\frac{1}{3 - \frac{2}{\sqrt{3}}} + \frac{\pi}{2}\right). \end{aligned} \tag{3}$$

The three inequalities (1), (2) and (3) imply altogether that α is at most

$$\begin{aligned}&(\gamma_c - 1)\left(\sqrt{27}\left(\arcsin\frac{1}{3} + \frac{\sqrt{8}}{9}\right) + \frac{(1 - \frac{\sigma}{\frac{\sqrt{3}}{2}})(\sqrt{3}+7)}{2}\arcsin\frac{1}{3 - \frac{2}{\sqrt{3}}}\right) \\ &\quad + \frac{\pi}{2}\left(\sqrt{27} + \left(1 - \frac{\sigma}{\frac{\sqrt{3}}{2}}\right)\frac{\sqrt{3}+7}{2}\right) \\ &\approx 3.4305176\gamma_c + 4.8184688.\end{aligned}$$

Thus, Theorem 2 follows.

4 Tighter Approximation Ratios

In this section, we derive tighter bounds on the approximation ratio of the distributed algorithm proposed in [10] and the other greedy algorithm proposed in [11]. For the convenience of presentation, we call them **WAF** and **WWY** respectively. Let $G = (V, E)$ be a unit-disk graph. We denote by α and γ_c the independence number and connected domination number of G respectively. For any finite set S, we use $|S|$ to denote the cardinality of S.

The CDS produced by the algorithm **WAF** consists of a maximal independent set I and a set C of connectors. Specifically, let T be an arbitrary rooted spanning tree of G. The set I is selected in the first-fit manner in the breadth-first-search ordering in T as follows. Initially I is empty. For each node visited in the BFS ordering of T, it is added to I if and only if it is not adjacent to any node in the current I. Let s be a neighbor of the root of T which is adjacent in G to the most nodes in I. Then, C consists of s and the parents (in T) of the nodes in $I \diagdown I(s)$. It was proved in [10] that $I \cup C$ is a CDS and $|I \cup C| \leq 8\gamma_c - 1$. Later on, two progressively improved tighter bounds $7.6\gamma_c + 1.4$ and $7\frac{1}{3}\gamma_c$ were obtained in [12] and [11] respectively. The next theorem further improves the bound on $|I \cup C|$.

Theorem 3 *The CDS produced by the algorithm* ***WAF*** *has size at most* $6.862\gamma_c + 8.637$.

Proof Let I and C be the set of nodes selected by the algorithm **WAF** in the first phase and the second phase respectively. Since $|C| \leq |I| - 1$, we have

$$|I \cup C| \leq 2|I| - 1 \leq 2(3.4306\gamma_c + 4.8185) - 1 \leq 6.862\gamma_c + 8.637.$$

So, the theorem follows. □

In the next, we study the algorithm **WWY**. The first phase of this algorithm is the same as the algorithm **WAF**, and we let I be the selected maximal independent set. But the second phase selects the connectors in a more economic way. For any subset $U \subseteq V \setminus I$, let $q(U)$ be the number of connected components in $G[I \cup U]$. For any $U \subseteq V \setminus I$ and any $w \in V \setminus I$, we define

$$\triangle_w q(U) = q(U) - q(U \cup \{x\}).$$

The value $\triangle_w q(U)$ is referred to as the *gain* of w with respect to U. The following lemma was proved in [11].

Lemma 8 *Suppose that* $q(U) > 1$ *for some* $U \subseteq V \setminus I$. *Then, there exists a* $w \in V \setminus (I \cup U)$ *such that* $\triangle_w q(U) \geq \max\{1, \lceil q(U)/\gamma_c \rceil - 1\}$.

The second phase of the algorithm **WWY** runs as follows. We use C to denote the sequence of selected connectors. Initially C is empty. While $q(C) > 1$, choose a node $w \in V \setminus (I \cup C)$ with *maximum* gain with respect to C and add w to C. When $q(C) = 1$, then $I \cup C$ is a CDS. It was proved in [11] that $|I \cup C| \leq 6\frac{7}{18}\gamma_c$. We derive a tighter bound on the output CDS in the theorem below.

Theorem 4 *The CDS produced by the algorithm* ***WWY*** *has size at most* $6.075\gamma_c + 5.425$.

Proof Let I and C be the set of nodes selected by the algorithm **WWY** in the first phase and the second phase respectively. If $\gamma_c = 1$, then $|I| \leq 5$ and $|C| \leq 1$, hence $|I \cup C| \leq 6$. Thus, the theorem holds trivially if $\gamma_c = 1$. If $|I| \leq 3\gamma_c + 2$, then $|I \cup C| \leq 2|I| - 1 \leq 6\gamma_c + 3$, and the theorem also holds. From now on, we assume that $\gamma_c \geq 2$ and $|I| > 3\gamma_c + 2$.

We break C into three contiguous (and possibly empty) subsequences C_1, C_2 and C_3 as follows. C_1 is the shortest prefix of C satisfying that $f(C_1) \leq 3\gamma_c + 2$, and

$C_1 \cup C_2$ is the shortest prefix of C satisfying that $f(C_1 \cup C_2) \leq 2\gamma_c + 1$. We will prove that

$$|C_1| \leq \begin{cases} \frac{|I|}{3} - \gamma_c, & \text{if } f(C_1) \leq 3\gamma_c + 1, \\ \frac{|I|-2}{3} - \gamma_c, & \text{if } f(C_1) = 3\gamma_c + 2; \end{cases}$$

$$|C_2| \leq \begin{cases} \frac{\gamma_c}{2}, & \text{if } f(C_1) \leq 3\gamma_c + 1, \\ \frac{\gamma_c+1}{2}, & \text{if } f(C_1) = 3\gamma_c + 2; \end{cases}$$

$$|C_3| \leq 2\gamma_c - 1.$$

From the first two inequalities, we have

$$|C_1 \cup C_2| \leq \frac{|I|}{3} - \frac{\gamma_c}{2}.$$

Using the third inequality, we have

$$|C| \leq \frac{|I|}{3} - \frac{\gamma_c}{2} + 2\gamma_c - 1 = \frac{|I|}{3} + \frac{3}{2}\gamma_c - 1.$$

So,

$$\begin{aligned} |I \cup C| &\leq \frac{4|I|}{3} + \frac{3}{2}\gamma_c - 1 \leq \frac{4}{3}(3.4306\gamma_c + 4.8185) + \frac{3}{2}\gamma_c - 1 \\ &\leq 6.075\gamma_c + 5.425. \end{aligned}$$

Thus, the theorem follows.

First, we prove the first inequality. This inequality holds trivially if $C_1 = \emptyset$. So we assume that $C_1 \neq \emptyset$ and let u be the last node in C_1. Then,

$$f(C_1 \setminus \{u\}) \geq 3\gamma_c + 3.$$

By Lemma 8, each node in C_1 has gain at least three. We consider two cases:

Case 1: $f(C_1) \leq 3\gamma_c + 1$. Then,

$$3(|C_1| - 1) \leq |I| - f(C_1 \setminus \{u\}) \leq |I| - (3\gamma_c + 3),$$

which implies $|C_1| \leq \frac{|I|}{3} - \gamma_c$.

Case 2: $f(C_1) = 3\gamma_c + 2$. Then,

$$3|C_1| \leq |I| - f(C_1) = |I| - (3\gamma_c + 2),$$

which implies $|C_1| \leq \frac{|I|-2}{3} - \gamma_c$.

Now, we prove the second inequality. This inequality holds trivially if $|C_2| \leq 1$. So, we assume that $|C_2| \geq 2$ and let v be the last node in C_2. Then,

$$f(C_1 \cup C_2 \setminus \{v\}) \geq 2\gamma_c + 2.$$

By Lemma 8, each node in C_2 has gain at least two. We consider three cases.

Case 1: $f(C_1) \le 3\gamma_c$. Then

$$2(|C_2|-1) \le f(C_1) - f(C_1 \cup C_2 \setminus \{v\}) \le 3\gamma_c - (2\gamma_c + 2),$$

which implies that $|C_2| \le \gamma_c/2$.

Case 2: $f(C_1) = 3\gamma_c + 1$. By Lemma 8, the first node in C_2 has gain at least three. Thus,

$$3 + 2(|C_2|-2) \le f(C_1) - f(C_1 \cup C_2 \setminus \{v\}) \le 3\gamma_c + 1 - (2\gamma_c + 2),$$

which also implies that $|C_2| \le \gamma_c/2$. Therefore, $|C_2| \le \gamma_c/2$.

Case 3: $f(C_1) = 3\gamma_c + 2$. By Lemma 8, the first node in C_2 has gain at least three. Thus,

$$3 + 2(|C_2|-2) \le f(C_1) - f(C_1 \cup C_2 \setminus \{v\}) \le 3\gamma_c + 2 - (2\gamma_c + 2),$$

which implies $|C_2| \le \frac{\gamma_c+1}{2}$.

Finally, we prove $|C_3| \le 2\gamma_c - 1$. By Lemma 8 each node in C_3 has gain at least one.

Case 1: $f(C_1 \cup C_2) \le 2\gamma_c$. Then

$$|C_3| \le f(C_1 \cup C_2) - 1 \le 2\gamma_c - 1.$$

Case 2: $f(C_1 \cup C_2) = 2\gamma_c + 1$. By Lemma 8, the first node in C_3 has gain at least two. Thus,

$$2 + (|C_3|-1) \le f(C_1 \cup C_2) - 1 = 2\gamma_c + 1 - 1,$$

which implies that $|C_3| \le 2\gamma_c - 1$. □

5 Discussions

In this paper, we obtained a tighter relation between the independence number and connected domination number of a connected UDG. We actually proved the following stronger result on packing. Let V be a set of n nodes of a connected dominating set, and Γ be the unions of unit-disks centered at V. Then, we can pack in Γ at most $3.4306n + 4.8185$ points whose pairwise distances are greater than or equal to one. We'd like to emphasize that here we allow two points packed in Γ to have distance equal to one. On the other hand, a packing of $3n + 3$ points in Γ whose pairwise distances are greater than one was presented in [11]. It was also conjectured $3n + 3$ is the exact bound. Thus, there is still a gap between the bound $3.4306n + 4.8185$ derived in this paper and the conjectured bound $3n + 3$.

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