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ON THE LONGEST RNG EDGE OF WIRELESS AD HOC NETWORKS

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Relative neighborhood graph (RNG) has been widely used in topology control and geographic routing in wireless ad hoc networks. Its maximum edge length is the minimum requirement on the maximum transmission radius by those applications of RNG. In this paper, we derive the precise asymptotic probability distribution of the maximum edge length of the RNG on a Poisson point process over a unit-area disk. Since the maximum RNG edge length is a lower bound on the critical transmission radius for greedy forward routing, our result also leads to an improved asymptotic almost sure lower bound on the critical transmission radius for greedy forward routing.

Keywords: Relative neighborhood graph; Poisson point process; asymptotic probability distribution.

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1. Introduction

Relative neighborhood graph (RNG) [15] of a finite planar set is a bounded-degree planar graph containing the Euclidean minimum spanning tree as a subgraph. Due to its simple construction and maintenance, RNG has found many applications in localized topology control (e.g. [6, 8, 9]) and geographic routing (e.g. [1, 7, 13]) in wireless ad hoc networks. All these applications require the maximum transmission

radius of the networking nodes be no shorter than the longest edge in the RNG. While the maximum edge length in the RNG can be easily computed in polynomial time, little is known about its random behavior when the underlying vertex set is a random point set. In this paper, we derive the precise asymptotic distribution of the maximum edge length in the RNG of a Poisson point process over a unit-area disk with density n, which is denoted by $\mathcal{P}_n$.

Denote the maximum edge length of a geometric graph G by $\lambda(G)$, and the RNG of a finite planar set V by $RNG(V)$. Denote by σ the geometric constant $\frac{2\pi}{3} - \frac{\sqrt{3}}{2}$ (whose geometric meaning will be explained in the later section). Fix a real parameter c and let

$$r_n = \sqrt{\frac{\ln n + c}{\sigma n}},$$

$$\mu = \frac{\pi}{2\sigma} e^{-c}.$$

The main result of this paper is stated in the following theorem.

Theorem 1.1. $\lim_{n \to \infty} \Pr[\lambda(RNG(\mathcal{P}_n)) \leq r_n] = e^{-\mu}$.

It is interesting to compare the maximum edge length of the RNG with the maximum edge length of the (Euclidean) minimum spanning tree (MST), which is also known the critical transmission radius for connectivity [5], and the maximum edge length of the Gabriel graph (GG) [4], which also has many applications in wireless ad hoc networks. Let $MST(V)$ and $GG(V)$ denote the MST and the GG of finite planar set V. It is well-known that for any finite planar set,

$$MST(V) \subseteq RNG(V) \subseteq GG(V).$$

Thus,

$$\lambda(MST(V)) \leq \lambda(RNG(V)) \leq \lambda(GG(V)).$$

The asymptotic distributions of $\lambda(MST(\mathcal{P}_n))$ and $\lambda(GG(\mathcal{P}_n))$ were derived in [11] (based on an earlier result [2]) and in [16] respectfully. Specifically, for any constant c,

$$\lim_{n \to \infty} \Pr\left[\lambda(MST(\mathcal{P}_n)) \leq \sqrt{\frac{\ln n + c}{\pi n}}\right] = e^{-e^{-c}},$$

$$\lim_{n \to \infty} \Pr\left[\lambda(GG(\mathcal{P}_n)) \leq 2\sqrt{\frac{\ln n + c}{\pi n}}\right] = e^{-2e^{-c}}.$$

So roughly speaking, the maximum edge length of the RNG (respectfully, GG) of a Poisson point process is asymptotically about 1.6 times (respectfully, twice) its critical transmission radius for connectivity.

Another parameter closely related to the maximum edge length of the RNG is the critical transmission radius for greedy forward routing [3, 14]. In greedy forward routing, each node discards a packet if none of its neighbors is closer to the

destination of the packet than itself, or otherwise forwards the packet to the neighbor closest to the destination of the packet. The critical transmission radius of a planar node set V for greedy forward routing, denoted by $\phi(V)$, is the smallest transmission radius by V which ensures successful delivery of any packets from any source node in V to any destination node in V. Clearly, $\lambda(RNG(V)) \leq \phi(V)$. It was recently proved in [17] that for any constant $\varepsilon > 0$, it is asymptotically almost sure (abbreviated by a.a.s.) that

$$(1 - \varepsilon)\sqrt{\frac{\ln n}{\sigma n}} \leq \phi(\mathcal{P}_n) \leq (1 + \varepsilon)\sqrt{\frac{\ln n}{\sigma n}}.$$

As the immediate consequence of Theorem 1.1, a tighter a.a.s. lower bound on $\phi(\mathcal{P}_n)$ can be obtained: For any sequence (c_n) satisfying that $\lim_{n\to\infty} c_n = \infty$ and $\lim_{n\to\infty} c_n / \ln n = 0$, it is a.s.s. that

$$\phi(\mathcal{P}_n) \geq \sqrt{\frac{\ln n - c_n}{\sigma n}}.$$

In what follows, $\mathbf{o}$ is the origin of the Euclidean plane $\mathbb{R}^2$, and $\mathbb{D}$ is the unit-area (closed) disk centered at $\mathbf{o}$. We assume that $\mathcal{P}_n$ is the Poisson point process over $\mathbb{D}$ with density n. We denote by $\mathcal{X}_n = (X_1, \ldots, X_n)$ the uniform n-point process over $\mathbb{D}$. The symbols $O, o, \sim$ always refer to the limit $n \to \infty$. To avoid trivialities, we tacitly assume n to be sufficiently large if necessary. For simplicity of notation, the dependence of sets and random variables on n will be frequently suppressed. For any finite set S, $card(S)$ denotes the cardinality of S. For any set S and positive integer k, the k-fold Cartesian product of S is denoted by S^k. The Euclidean norm of a point x is denoted by $\|x\|$, and the Euclidean distance between two points u and v is denoted by $\|uv\|$. The Lebesgue measure (or area) of a measurable set $A \subset \mathbb{R}^2$ is denoted by $|A|$. The topological boundary of a set $A \subset \mathbb{R}^2$ is denoted by ∂A. The open (respectively, closed) disk of radius r centered at x is denoted by $D(x, r)$ (respectively, $\bar{D}(x, r)$).

The remaining of this paper is organized as follows. The proof for Theorem 1.1 is presented in Sec. 3. Some preliminary results to be used in the proof for Theorem 1.1 are established in Sec. 2. The proofs for two technical lemmas on the limits of several relevant integrals are postponed in Sec. 4.

2. Preliminaries

The lune of a line segment uv, denoted by $L(uv)$, is the intersection of the disks $D(u, \|uv\|)$ and $D(v, \|uv\|)$ (see Fig. 1); the line segment uv is called the waist of $L(uv)$; the two intersection points of $\partial D(u, \|uv\|)$ and $\partial D(v, \|uv\|)$ are called the vertices of $L(uv)$. It is easy to verify that

$$|L(uv)| = \sigma\|uv\|^2.$$

Thus, σ is the area of the lune whose waist has unit length. For a finite planar set V, $RNG(V)$ consists of all edges uv satisfying that $L(uv) \cap V = \emptyset$. For a finite
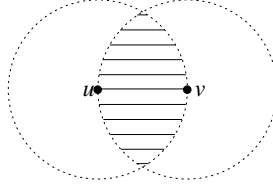
Fig. 1. The lune $L(uv)$ of a line segment uv.

planar set V and any number $r > 0$, the r-disk graph of V consists of all edges uv satisfying that $\|uv\| \leq r$.

For any line segment e, we define

$$\nu(e) = |L(e) \cap \mathbb{D}|.$$

A geometric graph is a graph on a finite planar set whose edges are line segments. Let H be a geometric graph. We use $V(H)$ and $E(H)$ to denote the vertex set and edge set of H respectively. Let $\chi(H)$ be the indicator for $H \subseteq RNG(V(H))$ and define

$$\nu(H) = |(\cup_{e \in E(H)} L(e)) \cap \mathbb{D}|.$$

For any positive number n, define

$$f_n(H) = \chi(H) e^{-n\nu(H)},$$
$$g_n(H) = e^{-n \sum_{e \in E(H)} \nu(e)}.$$

Let $\varepsilon = 0.001$. Fix a sequence (c_n) of real numbers satisfying that $c_n > c$, $c_n = o(\ln n)$ and $c_n \to \infty$. Let

$$R_n = \sqrt{\frac{\ln n + c_n}{\sigma n}},$$
$$R'_n = \left(1 + \frac{\varepsilon}{2}\right)\sqrt{\frac{\ln n}{\sigma n}}.$$

Then, for sufficiently large n, we have $r_n < R_n < R'_n < (1+\varepsilon)r_n$. Define

$$\Omega = \{(x_1, x_2) \in \mathbb{D}^2 : r_n < \|x_1 x_2\| \leq R_n\},$$
$$\Omega' = \{(x_1, x_2) \in \mathbb{D}^2 : R_n < \|x_1 x_2\| \leq R'_n\}.$$

The following Lemma gives the limits of two fundamental integrals, whose proof is postponed to Sec. 4.

Lemma 2.1. *The following are true*:

$$n^2 \int_\Omega e^{-nv(x_1 x_2)} dx_1 dx_2 \sim 2\mu,$$
$$n^2 \int_{\Omega'} e^{-nv(x_1 x_2)} dx_1 dx_2 = o(1).$$

A topology with numbered vertices is specified by a collection of the pairs of the indices of the numbered vertices. For any integer $m \geq 2$, denote $\mathcal{T}_m$ set of topologies on m numbered vertices without isolated vertex. For any $\tau \in \mathcal{T}_m$, and any sequence U of m planar points, $\tau(U)$ denotes the geometric graph on U with topology τ. For any $\tau \in \mathcal{T}_m$, we denote by $\Gamma(\tau)$ the set of $\mathbf{x} \in \mathbb{D}^m$ satisfying that all edges of $\tau(\mathbf{x})$ have length in $(r_n, R_n]$. Note that for each $\mathbf{x} \in \Gamma(\tau)$, the $\sqrt{3}R_n$-disk graph on the midpoints of the edges in any connected component of $\tau(\mathbf{x})$ is connected. Thus, the $\sqrt{3}R_n$-disk graph on the midpoints of the edges in $\tau(\mathbf{x})$ has no more connected components than $\tau(\mathbf{x})$ itself. For any positive integer l no more than the number of connected components of τ, we denote by $\Gamma_l(\tau)$ the set of $\mathbf{x} \in \Gamma(\tau)$ such that the $\sqrt{3}R_n$-disk graph on the midpoints of the edges in $\tau(\mathbf{x})$ has l connected components.

Lemma 2.2. *Suppose that $2 < m \leq 2k$ and τ is a forest in $\mathcal{T}_m$ with k edges. Then, for any positive integer $l \leq \min\{m - k, k - 1\}$,*

$$n^m \int_{\Gamma_l(\tau)} f_n(\tau(\mathbf{x})) d\mathbf{x} = o(1),$$

$$n^m \int_{\Gamma_l(\tau)} g_n(\tau(\mathbf{x})) d\mathbf{x} = o(1).$$

The proof for Lemma 2.2 is also postponed to Sec. 4. Lemma 2.2 implies the following two corollaries.

Corollary 2.3. *Suppose that $\tau \in \mathcal{T}_{2k}$ is a perfect matching for some $k \geq 2$. Then,*

$$n^{2k} \int_{\Gamma_k(\tau)} g_n(\tau(\mathbf{x})) d\mathbf{x} \sim (2\mu)^k.$$

Proof. We denote by C_k the perfect matching of $2k$ numbered vertices $v_1, v_2, \ldots, v_{2k}$ which consists of k edges $v_{2i-1} v_{2i}$ for $1 \leq i \leq k$. By symmetry, we only have to prove the lemma holds for $\tau = C_k$. Note that

$$\Gamma_k(C_k) = \Gamma(C_k) \backslash \cup_{l=1}^{k-1} \Gamma_l(C_k).$$

Since $\Gamma(C_k) = \Omega^k$, we have

$$n^{2k} \int_{\Gamma(C_k)} g_n(C_k(\mathbf{x})) d\mathbf{x} = \prod_{i=1}^{k} \left(n^2 \int_{\Omega} e^{-n\nu(x_{2i-1} x_{2i})} dx_{2i-1} dx_{2i} \right)$$

$$\sim (2\mu)^k.$$

By Lemma 2.2 for each $1 \leq l \leq k - 1$,

$$n^{2k} \int_{\Gamma_l(C_k)} g_n(C_k(\mathbf{x})) d\mathbf{x} = o(1).$$

Thus, the corollary holds. $\qquad\square$

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Corollary 2.4. *Suppose that $\tau \in \mathcal{T}_m$ is not a perfect matching. Then,*

$$n^m \int_{\Gamma(\tau)} f_n(\tau(\mathbf{x}))d\mathbf{x} = o(1).$$

Proof. Clearly, $m > 2$. Let τ' be a maximal spanning forest of τ. Then, $\tau' \in \mathcal{T}_m$ and τ' is not a perfect matching. In addition, $\Gamma(\tau') \supseteq \Gamma(\tau)$, and for any $\mathbf{x} \in \Gamma(\tau)$,

$$f_n(\tau'(\mathbf{x})) \geq f_n(\tau(\mathbf{x})).$$

Hence,

$$\int_{\Gamma(\tau)} f_n(\tau(\mathbf{x}))d\mathbf{x} \leq \int_{\Gamma(\tau)} f_n(\tau'(\mathbf{x}))d\mathbf{x} \leq \int_{\Gamma(\tau')} f_n(\tau'(\mathbf{x}))d\mathbf{x}.$$

Thus, it is sufficient to show that

$$n^m \int_{\Gamma(\tau')} f_n(\tau'(\mathbf{x}))d\mathbf{x} = o(1).$$

Let k be the number of edges in τ'. Then, τ has $m - k$ tree components and hence

$$\Gamma(\tau') = \cup_{l=1}^{m-k} \Gamma_l(\tau').$$

Since τ' is not a perfect matching, we have $m < 2k$, which implies $m - k \leq k - 1$. By Lemma 2.2, for any $1 \leq l \leq m - k$,

$$n^m \int_{\Gamma_l(\tau')} f_n(\tau'(\mathbf{x}))d\mathbf{x} = o(1).$$

Therefore,

$$n^m \int_{\Gamma(\tau')} f_n(\tau'(\mathbf{x}))d\mathbf{x} = \sum_{l=1}^{m-k} n^m \int_{\Gamma_l(\tau')} f_n(\tau'(\mathbf{x}))d\mathbf{x} = o(1). \qquad \square$$

For any geometric graph H, let $B_n(H)$ be the indicator for $H \subseteq RNG(V(H) \cup \mathcal{P}_n)$ and all edges of H have length in $(r_n, R_n]$.

Lemma 2.5. *For any $\tau \in \mathcal{T}_m$, if τ is a perfect matching then*

$$n^m \mathbf{E}[B_n(\tau(\mathcal{X}_m))] \sim (2\mu)^{m/2};$$

otherwise,

$$n^m \mathbf{E}[B_n(\tau(\mathcal{X}_m))] = o(1).$$

Proof. For any $\tau \in \mathcal{T}_m$,

$$\mathbf{E}[B_n(\tau(\mathcal{X}_m))] = \int_{\Gamma(\tau)} \Pr[B_n(\tau(\mathbf{x})) = 1]d\mathbf{x}.$$

For any $\mathbf{x} \in \Gamma(\tau)$,

$$\Pr[B_n(\tau(\mathbf{x})) = 1] \leq f_n(\tau(\mathbf{x})).$$

In addition, if τ is a perfect matching then for any $\mathbf{x} \in \Gamma_{m/2}(\tau)$,

$$\Pr[B_n(\tau(\mathbf{x})) = 1] = g_n(\tau(\mathbf{x})).$$

If τ is not a perfect matching, then by Corollary 2.4, we have

$$n^m \mathbf{E}[B_n(\tau(\mathcal{X}_m))] \le n^m \int_{\Gamma(\tau)} f_n(\tau(\mathbf{x})) d\mathbf{x} = o(1).$$

In the next, we assume that τ is a perfect matching. Let $m = 2k$. For $k = 1$, $\Gamma(\tau) = \Omega$ and hence by Lemma 2.1 we have

$$\begin{aligned}
n^2 \mathbf{E}[B_n(\tau(\mathcal{X}_2))] &= n^2 \int_\Omega \Pr[B_n(\tau(\mathbf{x})) = 1] d\mathbf{x} \\
&= n^2 \int_\Omega e^{-n\nu(x_1 x_2)} dx_1 dx_2 \\
&\sim 2\mu.
\end{aligned}$$

So, the Lemma holds for $k = 1$. So, we further assume that $k \ge 2$. Note that

$$\begin{aligned}
n^{2k} \mathbf{E}[B_n(\tau(\mathcal{X}_{2k}))] &= n^{2k} \int_{\Gamma(\tau)} \Pr[B_n(\tau(\mathbf{x})) = 1] d\mathbf{x} \\
&= \sum_{l=1}^k n^{2k} \int_{\Gamma_l(\tau)} \Pr[B_n(\tau(\mathbf{x})) = 1] d\mathbf{x}.
\end{aligned}$$

By Corollary 2.3 we have

$$n^{2k} \int_{\Gamma_k(\tau)} \Pr[B_n(\tau(\mathbf{x})) = 1] d\mathbf{x} = n^{2k} \int_{\Gamma_k(\tau)} g_n(\tau(\mathbf{x})) d\mathbf{x} \sim (2\mu)^k,$$

and for any $1 \le l < k$,

$$n^{2k} \int_{\Gamma_l(\tau)} \Pr[B_n(\tau(\mathbf{x})) = 1] d\mathbf{x} \le n^{2k} \int_{\Gamma_l(\tau)} f_n(\tau(\mathbf{x})) d\mathbf{x} = o(1).$$

Thus,

$$n^{2k} \mathbf{E}[B_n(\tau(\mathcal{X}_{2k}))] \sim (2\mu)^k.$$

So, the Lemma holds in this case. $\qquad\square$

For any $\mathbf{x} = (x_1, x_2) \in \mathbb{D}^2$, let $B'_n(\mathbf{x})$ be the indicator for $x_1 x_2 \in RNG(\{x_1, x_2\} \cup \mathcal{P}_n)$ and $R_n < \|x_1 x_2\| \le R'_n$.

Lemma 2.6. $n^2 \mathbf{E}[B'_n(\mathcal{X}_2)] = o(1).$

Proof. For any $\mathbf{x} = (x_1, x_2) \in \Omega'$,

$$\Pr[B_n'(\mathbf{x}) = 1] = e^{-nv(x_1 x_2)}.$$

By Lemma 2.1,

$$
\begin{aligned}
n^2 \mathbf{E}[B_n'(\mathcal{X}_2)] &= n^2 \int_{\Omega'} \Pr[B_n'(\mathbf{x}) = 1] d\mathbf{x} \\
&= \frac{n^2}{2} \int_{\Omega'} e^{-nv(x_1 x_2)} dx_1 dx_2 \\
&= o(1). \qquad\qquad \square
\end{aligned}
$$

3. Proof for Theorem 1.1

We first give a brief overview on our approach to prove Theorem 1.1. Let M_n (respectively, M_n' and M_n'') denote the number of edges in $RNG(\mathcal{P}_n)$ with length in $(r_n, R_n]$ (respectively, $(R_n, R_n']$ and $(R_n', +\infty)$). Then, $\lambda(RNG(\mathcal{P}_n)) \leq r_n$ if and only if $M_n + M_n' + M_n'' = 0$. In Lemma 3.1, we will show that $M_n'' = 0$ is a.a.s. In Lemma 3.4, we will prove that $E[M_n'] = o(1)$, which implies that $M_n' = 0$ is a.a.s. by Markov's inequality. In Lemma 3.6, we will prove that M_n is asymptotically Poisson with mean μ. Consequently,

$$
\begin{aligned}
\lim_{n \to \infty} \Pr[\lambda(RNG(\mathcal{P}_n)) \leq r_n] &= \lim_{n \to \infty} \Pr[M_n + M_n' + M_n'' = 0] \\
&= \lim_{n \to \infty} \Pr[M_n = 0] \\
&= e^{-\mu}.
\end{aligned}
$$

We first utilize the tool of minimal scan statistics developed in [17] to prove that $M_n'' = 0$ is a.a.s.

Lemma 3.1. $\Pr[M_n'' > 0] = o(1).$

Proof. For any finite point set $V \subset \mathbb{D}$ and any $r > 0$, define

$$\mathcal{S}(V, r) = \min_{\{u,v\} \subset \mathbb{D}, \|uv\| = r} |V \cap L(uv)|.$$

For any pair of $\{u, v\} \subset \mathbb{D}$ with $\|uv\| = R_n'$,

$$n|L(uv)| = n \cdot \sigma(R_n')^2 = \left(1 + \frac{\varepsilon}{2}\right)^2 \ln n > (1 + \varepsilon) \ln n.$$

Using this fact and following the same argument in the proof of Lemma 9 in [17], we can prove that $\mathcal{S}(\mathcal{P}_n, R_n') > \Theta(\ln n)$ is a.a.s. In particular, $\mathcal{S}(\mathcal{P}_n, R_n') > 0$ a.a.s. Next, we claim that the event $M_n'' > 0$ implies the event $\mathcal{S}(\mathcal{P}_n, R_n') = 0$. Suppose that $M_n'' > 0$. Then there are a pair of nodes $\{X, Y\} \subset \mathcal{P}_n$ such that $\|XY\| > R_n'$ and $L(XY) \cap \mathcal{P}_n$ is empty. Let Y' be the point on the line segment XY satisfying that $\|XY'\| = R_n'$. Thus $L(XY') \subset L(XY)$. So $L(XY') \cap \mathcal{P}_n$ is empty. This implies

that $\mathcal{S}(\mathcal{P}_n, r_n) = 0$. Therefore, our claim holds. Consequently,

$$\Pr[M_n'' > 0] \leq \Pr[\mathcal{S}(\mathcal{P}_n, R_n') = 0] = o(1). \qquad \square$$

Two key techniques used in our remaining proof are the Palm theory for Poisson processes (see, e.g. Theorem 1.6 in [12]) and the Brun's sieve (see, e.g. Theorem 10 in [16]), which are stated below.

Theorem 3.2. *Suppose that N is a non-negative integer random variable, and $B_1, \ldots, B_N$ are N Bernoulli random variables. If there is a constant μ such that for every fixed positive integer k,*

$$\mathbf{E}\left[\sum_{I \subseteq \{1,\ldots,N\}, |I|=k} \prod_{i \in I} B_i\right] \sim \frac{\mu^k}{k!},$$

then $\sum_{i=1}^{N} B_i$ is asymptotically Poisson with mean μ.

Theorem 3.3. *Suppose that $h(U, V)$ is a bounded measurable function defined on all pairs of the form (U, V) with V being a finite planar set and U being a subset of V. Then for any positive integer k,*

$$\mathbf{E}\left[\sum_{U \subseteq \mathcal{P}_n, |U|=k} h(U, \mathcal{P}_n)\right] = \frac{n^k}{k!}\mathbf{E}[h(\mathcal{X}_k, \mathcal{X}_k \cup \mathcal{P}_n)].$$

Now, we apply Palm theory to show that $E[M_n']$ is vanishing.

Lemma 3.4. $E[M_n'] = o(1)$.

Proof. For any pair (U, V) with V being a finite planar set and U being a subset of V, define $h'(U, V)$ to be the number of edges in the subgraph of $RNG(V)$ induced by U with length in $(R_n, R_n']$. By applying the Palm theory, we have

$$\mathbf{E}[M_n'] = \mathbf{E}\left[\sum_{U \subset \mathcal{P}_n, |U|=2} h'(U, \mathcal{P}_n)\right]$$

$$= \frac{n^2}{2}\mathbf{E}[h'(\mathcal{X}_2, \mathcal{X}_2 \cup \mathcal{P}_n)]$$

$$= \frac{n^2}{2}\mathbf{E}[B_n'(\mathcal{X}_2)].$$

By Lemma 2.6, the lemma follows. $\qquad \square$

For any positive integer k, denote by $\mathcal{H}_{n,k}$ the collection of k-edge subgraphs of $RNG(\mathcal{P}_n)$ in which all edges have length in $(r_n, R_n]$ and no vertex is isolated. Next, we apply Palm theory to compute the asymptotic average of $card(\mathcal{H}_{n,k})$.

Lemma 3.5. *For any fixed positive integer k,*

$$\mathbf{E}[card(\mathcal{H}_{n,k})] \sim \frac{\mu^k}{k!}.$$

Proof. For any pair (U, V) with V being a finite planar set and U being a subset of V, define $h(U, V)$ to be the number of k-edge subgraphs of $RNG(V)$ on U in which all edges have length in $(r_n, R_n]$ and no vertex is isolated. By applying the Palm theory, we have

$$\mathbf{E}[card(\mathcal{H}_n)] = \mathbf{E}\left[\sum_{m=2}^{2k} \sum_{U \subset \mathcal{P}_n, |U|=m} h(U, \mathcal{P}_n)\right]$$

$$= \sum_{m=2}^{2k} \mathbf{E}\left[\sum_{U \subset \mathcal{P}_n, |U|=m} h(U, \mathcal{P}_n)\right]$$

$$= \sum_{m=2}^{2k} \frac{n^m}{m!} \mathbf{E}[h(\mathcal{X}_m, \mathcal{X}_m \cup \mathcal{P}_n)].$$

Thus, it is sufficient to show that

$$\frac{n^m}{m!} \mathbf{E}[h(\mathcal{X}_m, \mathcal{X}_m \cup \mathcal{P}_n)] = \begin{cases} 0, & \text{if } 2 \leq m < 2k; \\ \dfrac{\mu^k}{k!}, & \text{if } m = 2k. \end{cases}$$

For any positive integer k and any $2 \leq m \leq 2k$, denote $\mathcal{T}_{m,k}$ the set of topologies in $\mathcal{T}_m$ with exactly k edges. Note that any topology in $\mathcal{T}_{2k,k}$ is a perfect matching, and

$$card(\mathcal{T}_{2k,k}) = \frac{1}{k!}\binom{2k}{2, 2, \ldots, 2} = \frac{(2k)!}{k! 2^k}.$$

By Lemma 2.5, we have

$$\frac{n^{2k}}{(2k)!} \mathbf{E}[h(\mathcal{X}_{2k}, \mathcal{X}_{2k} \cup \mathcal{P}_n)] = \frac{1}{(2k)!} \sum_{\tau \in \mathcal{T}_{2k,k}} n^{2k} \mathbf{E}[B(\tau(\mathcal{X}_{2k}))]$$

$$\sim \frac{1}{(2k)!} \cdot \frac{(2k)!}{k! 2^k} \cdot (2\mu)^k$$

$$= \frac{\mu^k}{k!}.$$

Now suppose that $2 \leq m < 2k$. Then any topology in $\mathcal{T}_{m,k}$ is not a perfect matching. By Lemma 2.5,

$$n^m \mathbf{E}[h(\mathcal{X}_m, \mathcal{X}_m \cup \mathcal{P}_n)] = \sum_{\tau \in \mathcal{T}_{m,k}} n^m \mathbf{E}[B_n(\tau(\mathcal{X}_m))] = o(1). \qquad \square$$

Finally, we apply the Brun's sieve together with Lemma 3.5 to prove M_n is asymptotically Poisson.

Lemma 3.6. *M_n is asymptotically Poisson with mean μ.*

Proof. Let $\mathcal{E}_n$ be the set of edges of $RNG(\mathcal{P}_n)$. For any edge $e \in \mathcal{E}_n$, define $\bar{B}(e)$ to be the Bernoulli random variable which equals to one if and only if $r_n < \|e\| \leq R_n$. Then

$$M_n = \sum_{e \in \mathcal{E}_n} \bar{B}(e).$$

For subset F of $\mathcal{E}_n$, $\prod_{e \in F} \bar{B}(e) = 1$ if and only if F is the edge set of a subgraph of $RNG(\mathcal{P}_n)$ in which all edges have length in $(r_n, R_n]$ and no vertex is isolated. Fix a positive integer k. By treating each k-subset F of $\mathcal{E}_n$ as an k-edge subgraphs of $RNG(\mathcal{P}_n)$, we have that

$$\sum_{F \subseteq \mathcal{E}_n, |F|=k} \prod_{e \in F} \bar{B}(e) = card(\mathcal{H}_{n,k}).$$

Hence, by Lemma 3.5,

$$\mathbf{E}\left[\sum_{F \subseteq \mathcal{E}_n, |F|=k} \prod_{e \in F} \bar{B}(e) \right] \sim \frac{\mu^k}{k!}.$$

By Theorem 3.2, M_n is asymptotically Poisson with mean μ. $\qquad \square$

4. Proofs for Lemma 2.1 and Lemma 2.2

In this section, we prove Lemma 2.1 and Lemma 2.2. We first present some useful geometric results. For $x \in \mathbb{D}$, let $t(x)$ denote the distance between x and $\partial\mathbb{D}$, which is equal to $\frac{1}{\sqrt{\pi}} - \|x\|$. For any $0 < \rho < \frac{1}{\sqrt{\pi}}$, define

$$\mathbb{D}_\rho(0) = \left\{ x \in \mathbb{D} : t(x) \geq \rho \right\},$$

$$\mathbb{D}_\rho(1) = \left\{ x \in \mathbb{D} : \sqrt{\frac{1}{\pi} - \rho^2} \leq t(x) < \rho \right\},$$

$$\mathbb{D}_\rho(2) = \left\{ x \in \mathbb{D} : t(x) < \sqrt{\frac{1}{\pi} - \rho^2} \right\}.$$

With this notation, the midpoint of any line segment $xy \subset \mathbb{D}$ is not in $\mathbb{D}_{\|xy\|/2}(2)$. For $x \in \mathbb{D}$ and $0 < \rho < \frac{1}{\sqrt{\pi}}$, define $\theta(x, \rho)$ as follows. If $x \in \mathbb{D}_\rho(0)$, then $\theta(x, \rho) = 2\pi$. If $x \in \mathbb{D}_\rho(2)$, then $\theta(x, \rho) = 0$. If $x \in \mathbb{D}_\rho(1)$, let u and v be the two intersection points of $\partial B(x, \rho)$ and $\partial\mathbb{D}$, and define $\theta(x, \rho) = 2\pi - \angle uxv$ (see Fig. 2).

Lemma 4.1. *For $x \in \mathbb{D}$ and $0 < \rho < \frac{1}{\sqrt{\pi}}$,*
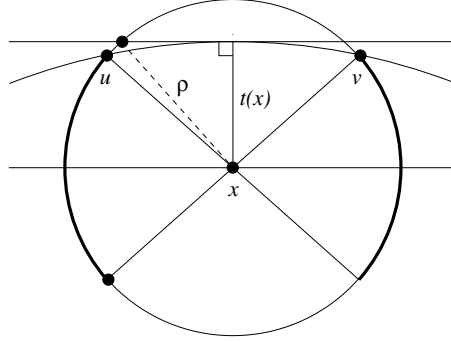
$$\rho\theta(x, \rho) \leq 2\pi t(x).$$

Fig. 2. If $x \in \mathbb{D}_\rho(1)$, then $\theta(x, \rho) = 2\pi - 2\angle uxv$.

Proof. The lemma holds trivially if $x \in \mathbb{D}_\rho(0)$ or $x \in \mathbb{D}_\rho(2)$. So, we consider the case that $x \in \mathbb{D}_\rho(1)$. It is easy to see that

$$\theta(x, \rho) \leq 4 \arcsin \frac{t(x)}{\rho}.$$

Using the equality $\sin \alpha \geq \frac{2}{\pi} \alpha$ for any $\alpha \in [0, \pi/2]$, we obtain

$$\theta(x, \rho) \leq 4 \cdot \frac{\pi}{2} \cdot \frac{t(x)}{\rho} = \frac{2\pi t(x)}{\rho}.$$

Thus, $\rho\theta(x, \rho) \leq 2\pi t(x)$. □

If $e \subset \mathbb{D}$ and the midpoint of e is apart from $\partial\mathbb{D}$ by at least $\frac{\sqrt{3}}{2}\|e\|$, then $L(e) \subset \mathbb{D}$. The next lemma gives a lower bound on $|L(e) \cap \mathbb{D}|$ if otherwise.

Lemma 4.2. *Consider a line segment $e \subset \mathbb{D}$ with midpoint z. If $t(z) \leq \frac{\sqrt{3}}{2}\|e\|$, then*

$$|L(e) \cap \mathbb{D}| \geq \frac{1}{2}|L(e)| + \frac{\|e\|}{2}t(z).$$

Proof. Let a and b be the two endpoints of e, and c_1 and c_2 be the two vertices of $L(e)$ with c_1 being farther away from the center of $\mathbb{D}$ (see Fig. 3). Then, the half lune abc_2 is fully contained in $\mathbb{D}$. If $c_1 \in \partial\mathbb{D}$, then the triangle abc_1 is contained in $\mathbb{D}$ and its area is $\frac{\|e\|}{2}\|az\| \geq \frac{\|e\|}{2}t(z)$. So, the lemma holds if $c_1 \in \partial\mathbb{D}$. Now assume that $c_1 \notin \partial\mathbb{D}$. Let u be the intersection point of c_1z and $\partial\mathbb{D}$. Then, the triangle abu is contained in $\mathbb{D}$ and its area is $\frac{\|e\|}{2}\|uz\| \geq \frac{\|e\|}{2}t(z)$. So, the lemma also holds if $c_1 \notin \partial\mathbb{D}$. □

For any geometric graph H on a finite subset of $\mathbb{D}$, an edge e of H is called an outermost edge of H if its midpoint is the nearest to $\partial\mathbb{D}$.

Lemma 4.3. *Suppose that H is a geometric graph over a finite subset of $\mathbb{D}$ with at least two edges satisfying that all the edges have length between $R/(1+\varepsilon)$ and R for*
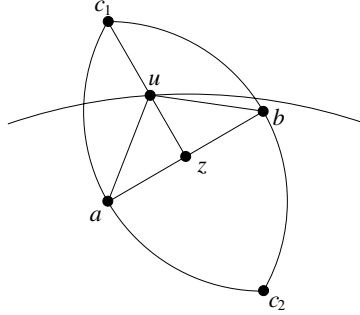
Fig. 3. $L(e) \cap \mathbb{D}$ contains the triangle abc_2 and the half lune abu.

$R \leq \frac{1}{200\sqrt{\pi}}$ *and the midpoints of its edges induce a connected $\sqrt{3}R$-disk graph. Let e' be an outermost edge of H, and ℓ be the largest distance between the midpoint of e' and the midpoints of other edges of H. Then,*

$$\sum_{e \in E(H)} \nu(e) \geq \nu(e') + 0.0026R\ell.$$

In addition, if $\chi(H) = 1$ then

$$\nu(H) \geq \nu(e') + 0.0026R\ell.$$

The proof of this lemma is very lengthy and complicated. We omit the proof due to the limitation on the space.

Now it is ready to give the proofs for Lemma 2.1 and Lemma 2.2. We will frequently change the integral variables using a technique introduced in [16]. Consider a tree topology on k planar points $x_1, x_2, \ldots, x_k$, and assume without loss of generality that $x_{k-1}x_k$ is an edge in this tree. Let z_{k-1}, ρ, and ω be the midpoint, half-length and the slope of $x_{k-1}x_k$ respectively. We root the tree at x_k. For $1 \leq i \leq k-2$, let z_i be the midpoint of the edge between x_i and its parent in such rooted tree. Then, we replace $x_1, x_2, \ldots, x_k$ by $z_1, \ldots, z_{k-1}, \rho, \omega$. The Jacobian determinant of this change is $4^{k-1}\rho$.

We begin with the proof for Lemma 2.1. Let $\rho = \rho(x_1, x_2)$ be the half-length of x_1x_2, and $z = z(x_1, x_2)$ be the midpoint of x_1x_2. Let Ω_1 be the set of $(x_1, x_2) \in \Omega$ satisfying that $z \in \mathbb{D}_{\sqrt{3}\rho}(0)$, and let $\Omega_2 = \Omega \setminus \Omega_1$. First, we calculate the integration over Ω_1. If $(x_1, x_2) \in \Omega_1$, $L(x_1x_2)$ is fully contained in $\mathbb{D}$ and $v(x_1x_2) = 4\sigma\rho^2$. Changing the integration variable x_1 and x_2 by z, ρ, and the slope of x_1x_2 yields

$$n^2 \int_{\Omega_1} e^{-nv(x_1x_2)} dx_1 dx_2 = 8\pi n^2 \int_{\frac{r_n}{2}}^{\frac{R_n}{2}} e^{-4n\sigma\rho^2} \rho \, d\rho \int_{\mathbb{D}_{\sqrt{3}\rho}(0)} dz$$

$$\sim 8\pi n^2 \int_{\frac{r_n}{2}}^{\frac{R_n}{2}} e^{-4n\sigma\rho^2} \rho \, d\rho$$

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$$= -\frac{\pi}{\sigma} n e^{-4\sigma\rho^2} \Big|_{\frac{r_n}{2}}^{\frac{R_n}{2}}$$

$$\sim \frac{\pi}{\sigma} e^{-c}$$

$$= 2\mu.$$

Next, we calculate the integration over Ω_2. Let $t = t(z)$ be the distance between z and $\partial\mathbb{D}$. By Lemmas 4.1 and 4.2, we have

$$\rho\theta(z,\rho) \leq 2\pi t,$$

and

$$v(x_1 x_2) \geq 2\sigma\rho^2 + \rho t.$$

Changing the integration variable as above yields

$$n^2 \int_{\Omega_2} e^{-nv(x_1 x_2)} dx_1 dx_2 \leq n^2 \int_{\Omega_2} e^{-n(2\sigma\rho^2 + \rho t)} dx_1 dx_2.$$

$$= 4n^2 \int_{\frac{r_n}{2}}^{\frac{R_n}{2}} d\rho \int_{\mathbb{D}_{\sqrt{3}\rho}(1)\setminus\mathbb{D}_\rho(2)} e^{-n(2\sigma\rho^2 + \rho t)} \rho\theta(z,\rho) dz$$

$$= 4n^2 \int_{\frac{r_n}{2}}^{\frac{R_n}{2}} d\rho \int_{\mathbb{D}_{\sqrt{3}\rho}(1)\setminus\mathbb{D}_\rho(2)} e^{-n(2\sigma\rho^2 + \rho t)} \rho\theta(z,\rho) dz$$

$$\leq 8\pi n^2 e^{-2n\sigma\rho^2} \int_{\frac{r_n}{2}}^{\frac{R_n}{2}} d\rho \int_{\mathbb{D}} e^{-n\rho t} t \, dz$$

$$= O(1) n^{1.5} \int_{\frac{r_n}{2}}^{\frac{R_n}{2}} d\rho \int_0^{1/\sqrt{\pi}} e^{-n\rho t} t \, dt$$

$$\leq O(1) n^{1.5} \int_{\frac{r_n}{2}}^{\frac{R_n}{2}} d\rho \int_0^\infty e^{-n\rho t} t \, dt$$

$$= O(1) \frac{1}{\sqrt{n}} \int_{\frac{r_n}{2}}^{\frac{R_n}{2}} \rho^{-2} d\rho$$

$$\leq O(1) \frac{1}{\sqrt{n}} r_n^{-2} R_n$$

$$= O(1) \frac{1}{\sqrt{n} r_n}$$

$$= O(1) \frac{1}{\sqrt{\ln n}}$$

$$= o(1).$$

Therefore,

$$n^2 \int_{\Omega} e^{-nv_{x_1 x_2}} dx_1 dx_2 \sim 2\mu.$$

Note that $\Omega \cup \Omega'$ consists of $(x_1, x_2) \in \mathbb{D}^2$ satisfying that $r_n < \|x_1 x_2\| \leq R'_n$. Using the same argument as above, we can show that

$$n^2 \int_{\Omega \cup \Omega'} e^{-nv(x_1 x_2)} dx_1 dx_2 \sim 2\mu.$$

Thus, the second asymptotic equality in Lemma 2.1 holds.

We move to prove Lemma 2.2 by establishing the following two lemmas.

Lemma 4.4. *Suppose that $2 < m \leq 2k$ and τ is a forest in $\mathcal{T}_m$ with k edges. Then,*

$$n^m \int_{\Gamma_1(\tau)} f_n(\tau(\mathbf{x})) d\mathbf{x} = o(1),$$

$$n^m \int_{\Gamma_1(\tau)} g_n(\tau(\mathbf{x})) d\mathbf{x} = o(1).$$

Proof. Enumerate the edges of τ arbitrarily by $e_1, \ldots, e_k$. For any $\mathbf{x} \in \Gamma_1(\tau)$, let z_i denote the middle point of e_i in $F(\mathbf{x})$ for each $1 \leq i \leq k$. For any pair of distinct integers p and q between 1 and k, let S_{pq} denote the set of $\mathbf{x} \in \Gamma_1(\tau)$ satisfying that e_p is an outermost edge in $\tau(\mathbf{x})$ and z_q is the farthest from z_p among all $z_1, \ldots, z_k$. Then, it suffices to prove for any such p and q,

$$n^m \int_{S_{pq}} f_n(\tau(\mathbf{x})) d\mathbf{x} = o(1),$$

$$n^m \int_{S_{pq}} g_n(\tau(\mathbf{x})) d\mathbf{x} = o(1).$$

Fix a pair of distinct integers p and q between 1 and k. Let p' and p'' be the indices of the two endpoints of the edges e_p. By Lemma 4.3 for any $\mathbf{x} \in S_{pq}$,

$$f_n(\tau(\mathbf{x})) \leq e^{-n(v(x_{p'} x_{p''}) + \eta R_n \|z_p z_q\|)},$$

$$g_n(\tau(\mathbf{x})) \leq e^{-n(v(x_{p'} x_{p''}) + \eta R_n \|z_p z_q\|)},$$

in which η is a positive constant. Thus, we only need to show that

$$n^m \int_{S_{pq}} e^{-n(v(x_{p'} x_{p''}) + \eta R_n \|z_p z_q\|)} d\mathbf{x} = o(1).$$

We change the integral variables $x_1, \ldots, x_m$ as follows. For the tree component containing e_p, we replace the x_i's in this tree by the midpoints of the edges in this tree except z_p and $x_{p'}$, $x_{p''}$ (both of which are kept). For any other tree component, we use the method introduced at the beginning of this section: pick an arbitrary edge as the rooted edge. We replace x_i's in this tree by the midpoints of all the

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edges in this tree together with the half-length and slope of the root edge. Such change of integration variables yields

$$n^m \int_{S_{pq}} e^{-n(v(x_{p'}x_{p''})+\eta R_n\|z_p z_q\|)}d\mathbf{x}$$

$$\leq O(1)n^m \left(\int_\Omega e^{-nvv(x_{p'}x_{p''})}dx_{p'}dx_{p''}\right)\left(\int_{\frac{r_n}{2}}^{\frac{R_n}{2}} \rho d\rho\right)^{m-k-1}$$

$$\cdot \left(\int_{\mathbb{R}^2} e^{-\eta n R_n\|z_p z_q\|}dz_q\right)\left(\int_{\bar{D}(z_p,\|z_p z_q\|)} dz\right)^{k-2}$$

$$\sim O(1)n^{m-2}(R_n^2 - r_n^2)^{m-k-1}\int_{\mathbb{R}^2} e^{-\eta n R_n\|z_p z_q\|}\|z_p z_q\|^{2(k-2)}dz_q$$

$$\leq O(1)n^{m-2}(R_n^2 - r_n^2)^{m-k-1}\int_0^\infty e^{-\eta n R_n\mu}\mu^{2k-3}d\mu$$

$$= O(1)\frac{n^{m-2}(R_n^2 - r_n^2)^{m-k-1}}{(nR_n)^{2(k-1)}}$$

$$= O(1)\frac{(nR_n^2 - nr_n^2)^{m-k-1}}{(nR_n^2)^{k-1}}$$

$$= O(1)\frac{(c_n - c)^{m-k-1}}{(\ln n)^{k-1}}$$

$$= o(1),$$

where the asymptotic equality follows from Lemma 2.1, and the last equality follows from $c_n = o(\ln n)$ and $k \geq 2$. □

Lemma 4.5. *Suppose that $2 < m \leq 2k$ and τ is a forest in $\mathcal{T}_m$ with k edges. For any integer $2 \leq l \leq \min\{m-k, k-1\}$,*

$$n^m \int_{\Gamma_l(\tau)} f_n(\tau(\mathbf{x}))d\mathbf{x} = o(1),$$

$$n^m \int_{\Gamma_l(\tau)} g_n(\tau(\mathbf{x}))d\mathbf{x} = o(1).$$

Proof. τ has $m-k$ tree components on m numbered vertices $v_1, v_2, \ldots, v_m$. Enumerate them arbitrarily by $T_1, \ldots, T_{m-k}$. For each $1 \leq q \leq m-k$, let

$$I_q = \{1 \leq i \leq m : v_i \text{ is a vertex of } T_q\}.$$

Fix an integer $2 \leq l \leq \min\{m-k, k-1\}$. Consider any nontrivial l-partition $\Pi = \{Q_1, Q_2, \ldots, Q_l\}$ of $\{1, 2, \ldots, m-k\}$. It induces a partition $\Pi' = \{P_1, P_2, \ldots, P_l\}$ of

$\{1, 2, \ldots, m\}$ in which $P_j = \cup_{q \in Q_j} I_q$ for each $1 \leq j \leq l$. Let $S(\Pi)$ denote the set of $\mathbf{x} \in \Gamma_l(\tau)$ such that for each $1 \leq j \leq l$, the set of midpoints of the subgraph of $\tau(\mathbf{x})$ induced by $\{x_i : i \in P_j\}$ is a connected component of the $\sqrt{3}R_n$-disk graph on the midpoints of the edges in $\tau(\mathbf{x})$. Then $\Gamma_l(\tau)$ is the union of $S(\Pi)$ over all nontrivial l-partitions Π of $\{1, 2, \ldots, m-k\}$. So, it is sufficient to show that for any l-partition Π of $\{1, 2, \ldots, k\}$,

$$n^m \int_{S(\Pi)} f_n(\tau(\mathbf{x}))d\mathbf{x} = o(1),$$

$$n^m \int_{S(\Pi)} g_n(\tau(\mathbf{x}))d\mathbf{x} = o(1).$$

Now, fix an l-partition $\Pi = \{Q_1, Q_2, \ldots, Q_l\}$ of $\{1, 2, \ldots, m-k\}$. Let $\Pi' = \{P_1, P_2, \ldots, P_l\}$ be the partition of $\{1, 2, \ldots, m\}$ induced by Π. For each $1 \leq j \leq l$, let $m_j = card(P_j)$, τ_j be the topology on m_j numbered vertices which is a subgraph of τ induced by the subset of vertices $\{v_i : i \in P_j\}$. Then, at least one $m_j \geq 2$. For any $\mathbf{x} = (x_1, x_2, \ldots, x_m) \in \mathbb{D}^m$ and each $1 \leq j \leq l$, let

$$\mathbf{x}^{(j)} = (x_{i_1}, x_{i_2}, \ldots, x_{i_{m_j}})$$

where $i_1, i_2, \ldots, i_{m_j}$ are the m_j indices in P_j in the increasing order. Clearly, for each $\mathbf{x} \in S(\Pi)$ and each $1 \leq j \leq l$, $\mathbf{x}^{(j)} \in \Gamma_1(\tau_j)$. Hence,

$$S(\Pi) \subseteq \{\mathbf{x} \in \mathbb{D}^m : \mathbf{x}^{(j)} \in \Gamma_1(\tau_j), 1 \leq j \leq l\}.$$

For any $\mathbf{x} \in S(\Pi)$,

$$\nu(\tau(\mathbf{x})) = \sum_{j=1}^{l} \nu(\tau_j(\mathbf{x}^{(j)})),$$

$$\chi(\tau(\mathbf{x})) \leq \prod_{j=1}^{l} \chi(\tau_j(\mathbf{x}^{(j)})),$$

which imply

$$f_n(\tau(\mathbf{x})) \leq \prod_{j=1}^{l} f_n(\tau_j(\mathbf{x}^{(j)})).$$

It is obvious that for any $\mathbf{x} \in S(\Pi)$,

$$g_n(\tau(\mathbf{x})) = \prod_{j=1}^{l} g_n t(\tau_j(\mathbf{x}^{(j)})).$$

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Thus,

$$
n^m \int_{S(\Pi)} f_n(\tau(\mathbf{x})) d\mathbf{x} \leq n^m \int_{S(\Pi)} \prod_{j=1}^{l} f_n(\tau_j(\mathbf{x}^{(j)})) d\mathbf{x}
$$

$$
\leq n^m \int_{\{\mathbf{x} \in \mathbb{D}^m : \mathbf{x}^{(j)} \in \Gamma_1(\tau_j), 1 \leq j \leq l\}} \prod_{j=1}^{l} f_n(\tau_j(\mathbf{x}^{(j)})) d\mathbf{x}
$$

$$
= \prod_{j=1}^{l} \left(n^{m_j} \int_{\Gamma_1(\tau_j)} f_n(\tau_j(\mathbf{x}^{(j)})) d\mathbf{x}^{(j)} \right)
$$

$$
= o(1),
$$

where the last equality follows from Lemma 2.1, Lemma 4.4 and the fact that at least one $m_j \geq 2$. Similarly, we can show that

$$
n^m \int_{S(\Pi)} g_n(\tau(\mathbf{x})) d\mathbf{x} = o(1).
$$

So, the Lemma follows. □

Acknowledgment

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