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Conflict-Free channel set assignment for an optical cluster interconnection network based on rotator digraphs

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Abstract

Recently a class of scalable multi-star optical networks is proposed in [2]. In this class of networks nodes are grouped into clusters. Each cluster employs a separate pair of broadcast and select couplers. The clusters are interconnected via fiber links according to a regular topology. Self links are provided to enable connectivity among nodes in the same cluster if the cluster size is more than one. These networks can efficiently combine time and/or wavelength division with direct space division. The key design issue for these networks is the optimal conflict-free channel set assignment to the output clusters for a given cluster interconnection topology. Such conflict-free channel assignment problem has been studied for various cluster interconnection topologies [1, 2, 9–11]. In this paper, we propose the rotator digraph [5] as the cluster interconnection topology as it possesses many attractive properties. We will give an optimal conflict-free channel set assignment for this new interconnection topology. © 1998—Elsevier Science B.V. All rights reserved

Keywords: Optical network, TWDM; Vertex coloring; Rotator digraph

1. Introduction

Emerging high bandwidth applications, such as voice/video services, distributed data bases, and network super-computing, are driving the use of single-mode optical fibers as the communication media for the future [3]. Optical passive stars [6] provide a simple medium to connect nodes in a local or metropolitan area network. The single-star optical networks with time and/or wavelength division multiplexing have been extensively studied in the past [4, 7, 11]. However, the scalability of the single-star configuration is constrained by the number of wavelengths that can be coupled and separated while maintaining acceptable crosstalk and power budget levels. Recently, a multi-star configuration which efficiently combines space with time and/or wavelength

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division was proposed in [2] to overcome this limit. In this class of networks, nodes are grouped into clusters with time and/or wavelength multiplexing. Clusters are further interconnected via fiber links to form a cluster interconnection network (CIN) according to some interconnection topology. If the cluster size is more than one, self cluster links are provided to enable connectivity among nodes in the same cluster. Wavelength spatial re-use is exploited in the channel set assignment to clusters. This network class has several advantages including low link density, nice scalability and desirable reconfigurability [1].

The key design issue of this class of networks is the conflict-free channel set assignment to the output star couplers. As the channels sets are valuable resources, it's desirable to share the channel sets among the output star couplers while maintaining the conflict-free transmission. The objective of the conflict-free channel set assignment problem is to find the minimal number of disjoint channel sets required by the conflict-free communication. This optimal conflict-free channel set assignment problem has been studied for various CIN topologies, such as perfect shuffle [1], hypercube and Star graphs [9–11].

As the cluster interconnection network is multihop optical network, it's desirable to have a CIN topology with short diameter to reduce the end-to-end delay and the intermediate processing. The rotator digraph is a perfect candidate for the CIN topology as it has the sub-logarithm diameter [5]. This motivates us to propose the rotator digraph as the CIN topology. This paper will present the optimal conflict-free channel assignment for the rotator digraphs.

The rest of this paper proceeds as follows. Section 2 describes the network configuration. Section 3 introduces the interconnection and properties of rotator digraphs. Section 4 gives a graph-theoretic formulation of the conflict-free channel set assignment into a vertex coloring problem. Section 5 presents the optimal vertex coloring scheme for the vertex coloring problem described in Section 4. A conclusion discussion is presented in Section 6.

2. Network configuration

The network consists of m_1 clusters where each cluster is a set of m_0 nodes, as shown in Fig. 1, with the total network size of $M = m_1 m_0$ nodes. A node represents the lowest abstraction level and may consist of a single processor, multiple time-multiplexed processors, an interface to a space switch, or a broadband network interface unit. Each node possesses a single fixed-wavelength transmitter (light source) and a receiver that is capable to simultaneously monitor a subset of separable channels. A channel here can be a reserved time slot, a dedicated wavelength, or a reserved time slot over a given wavelength. Assume that W wavelengths are available and that the time frame is divided into T slots, a total of $C = WT$ channels are available. The receiver can be realized using a multichannel acousto-optic tunable filter or a detector array and a passive (grating based) wavelength demultiplexer [8]. Each cluster possesses its

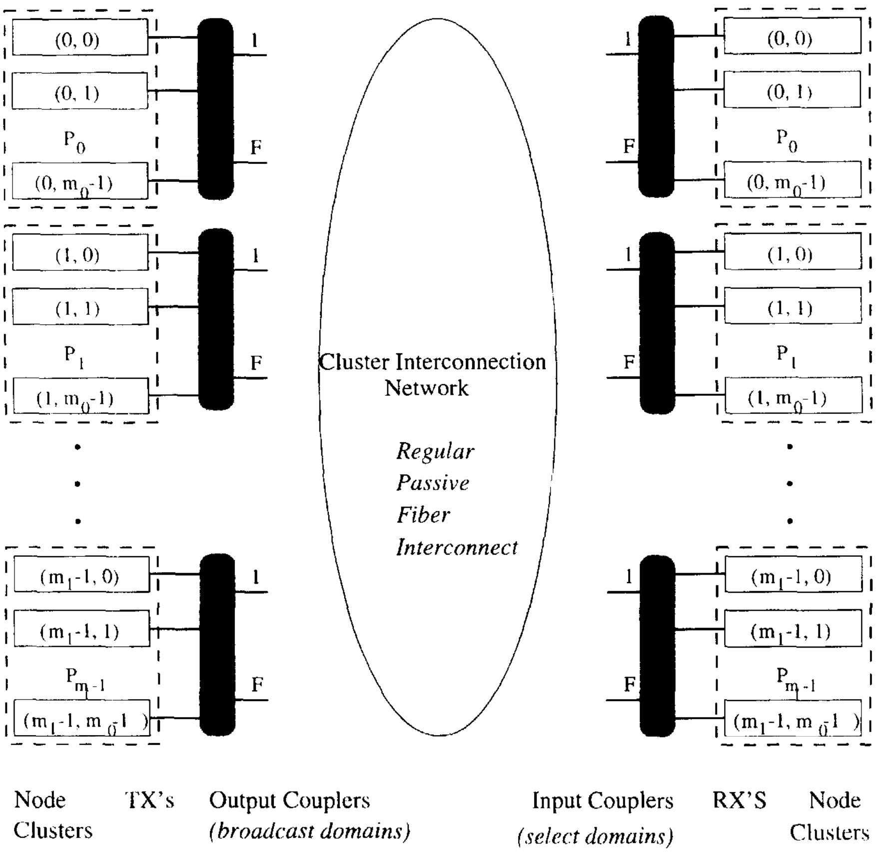

Fig. 1. Multi-sar network with discrete broadcast-select domains: m_1 clusters, each with m_0 nodes transmitting through output couplers and receiving through input couplers, interconnected via a regular CIN topology.

own broadcast and select domains realized by an output and an input star couplers, respectively. The cluster interconnection network (CIN) refers to the fiber connection pattern from output couplers to input couplers. Each cluster is provided with an extra self link to enable connectivity among nodes in the same cluster. The dimension of the output coupler is $m_0 : F$ and that of the input coupler is $F : m_0$, where

$$F = 1 + \text{ the degree in the CIN topology.}$$

Nodes in a cluster transmit over an ordered set of m_0 distinct channels through the output broadcast star coupler. At the input coupler side, several distinct channel sets are monitored depending on the CIN topology. Transmit channel sets are assigned to the output couplers such that no conflicts may happen at the input coupler. That is, the assignment is such that the channel sets which can be listened to through any input coupler are disjoint to provide a collision-less environment.

The network configuration efficiently combines space with time and/or wavelength division. It has several advantages including low link density, nice scalability and desirable reconfigurability. It reduces to an all-spaced network when $m_0 = 1$ node per cluster and to a time and wavelength division multiplexed (TWDM) network when $m_1 = 1$ cluster with m_0 nodes.

3. Rotator digraphs

As the cluster interconnection network is multihop optical network, it's desirable to have a CIN topology with short diameter to reduce the end-to-end delay and the intermediate processing. Various topologies have been proposed as the CIN topologies including perfect shuffle [1], hypercube and Star graph [9–11]. The rotator digraph is a perfect candidate for the CIN topology as it has the sub-logarithm diameter [5] which has shorter diameter than the perfect shuffle, hypercube and star graph with the same network size. This motivates us to propose the rotator digraph as the CIN topology.

The rotator digraph is a member of a class of graphs called *Cayley digraphs*. This class of graphs uses a group-theoretic approach as a basis for defining digraphs. Let G be a finite group and S a set of generators for G. The *Cayley graph of* G *with generating set* S, denoted by $Cay(S : G)$, is defined as follows.

1. Each element of G is a vertex of $Cay(S : G)$.
2. For x and y in G, there is a link from x to y if and only if $xs = y$ for some $s \in S$.

The rotator digraph is defined through the permutation group. Let P_n be the set consisting of all permutations on n symbols $\{1,2,\ldots,n\}$. A permutation $\sigma \in P_n$ is represented by $\sigma(1)\sigma(2)\cdots\sigma(n)$. For example, the permutation $\sigma = 2143$ represents that $\sigma(1) = 2, \sigma(2) = 1, \sigma(3) = 4$ and $\sigma(4) = 3$. The permutation $1234\cdots n$ is called the identity permutation. A special class of cycles is the rotations. The permutation

$$\alpha_k = 23\cdots k1(k+1)\cdots n = (1,2,\ldots,k)$$

is called the *left rotation* of length k, where $2 \leqslant k \leqslant n$. The permutation

$$\beta_k = k12\cdots(k-1)(k+1)\cdots n = (k,k-1,\ldots,1)$$

is called the *right rotation* of length k, where $2 \leqslant k \leqslant n$.

Traditionally, the product of two permutations has two different definitions. The two definitions differ at the order in which the multiplication is taken.

1. The multiplication is taken from left to right. We use "·" to denote this multiplication operation. The product of any two permutations α and β is given by

   $$\alpha \cdot \beta(i) = \beta(\alpha(i))$$

   for any $1 \leqslant i \leqslant n$.
2. The multiplication is taken from right to left. We use "∘" to denote this multiplication operation. The product of any two permutations α and β is given by

   $$\alpha \circ \beta(i) = \alpha(\beta(i))$$

   for any $1 \leqslant i \leqslant n$.

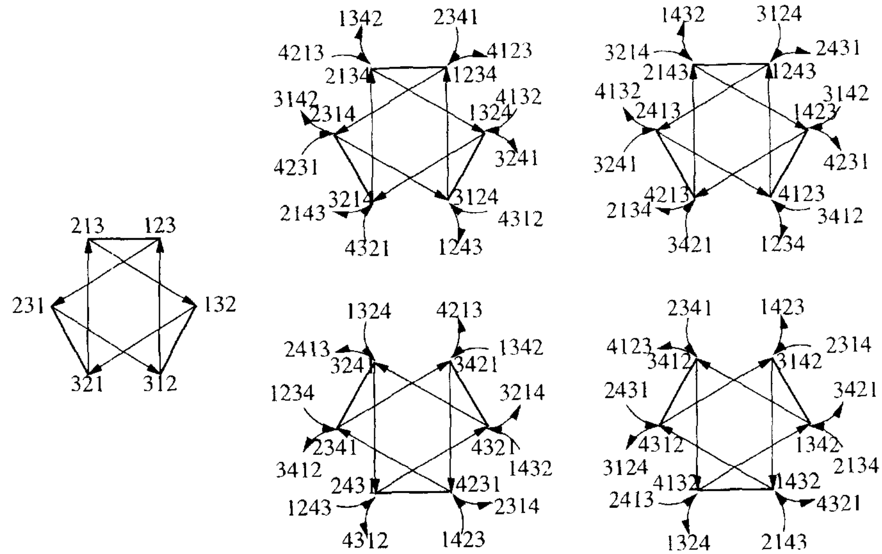

Fig. 2. The 3-dimensional and 4-dimensional left rotator digraphs.

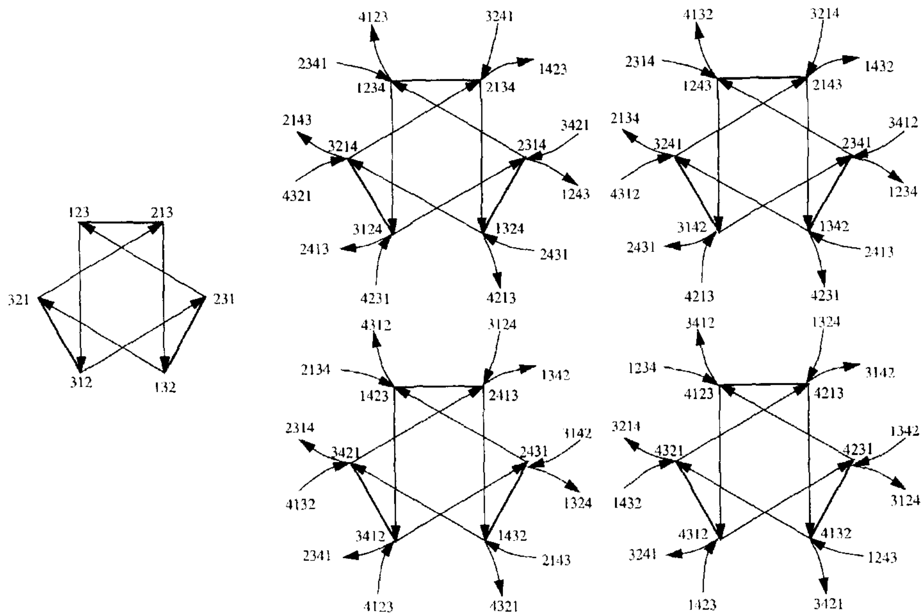

Fig. 3. The 3-dimensional and 4-dimensional right rotator digraphs.

P_n is a group under both multiplication operations. It is well-known that the set of $n-1$ left rotations $\{\alpha_2, \alpha_3, \ldots, \alpha_n\}$ is a generator set of P_n under both multiplication operations.

An n-dimensional rotator digraph R_n, also refered to as n-rotator, is the Cayley digraph $Cay(S : G)$ where G is P_n, and S consists of $n-1$ left rotations $\{\alpha_2, \alpha_3, \ldots, \alpha_n\}$.

The topology of R_n under the multiplication "·" is different from that under the multiplication "$\circ$". Fig. 2 illustrates the 4-star under the multiplication "·", and Fig. 3 illustrates the 4-star under the multiplication "$\circ$".

It is easy to see that the n-rotator R_n consists of $n!$ nodes, and each node has in-degree $n-1$ and out-degree $n-1$. The rotator digraphs have a lot of attractive properties. All rotator digraphs are vertex and edge symmetric. In [5], it is proved that the diameter of the n-rotator is $n-1$, and it has a simple and optimal routing algorithm. They share the fault handling capacity of hierarchical Cayley graphs, including the Star and Pancake graphs and the binary hypercubes, with good performance possible because of the small diameter and average routing distance.

4. Conflict-free channel set assignment: A coloring problem

It is easy to verify that for any CIN topology a channel set assignment is conflict-free if and only if for any input clusters, all the output clusters it connects from and itself have different channel sets form each other. If we regard clusters as vertices and the channel sets as colors, we can formulate the conflict-free channel set assignment problem to the following vertex coloring problem.

$\bar{2}$-VC Given a regular digraph G, a vertex coloring scheme is called a $\bar{2}$-*VC* of G if for any vertex v, its parents and itself have different colors from each other.

The minimal number of colors required by any $\bar{2}$-VC of G is denoted by $\chi_{\bar{2}}(G)$. Then $\chi_{\bar{2}}(G)$ represents the minimal number of disjoint channel sets to satisfy the conflict-free communication for a given CIN topology G.

We first find the lower bound for $\chi_{\bar{2}}(G)$.

Lemma 1 (Lower-bound). *For any regular digraph with nodal degree d, $\chi_{\bar{2}}(G) \geqslant d+1$.*

Proof. Since the nodal degree of G is d, each node has d parents. As all these d parents and the node itself must have distinct colors, the minimum number of colors required by any $\bar{2}$-VC of G is at least $d+1$. □

Since the n-rotator R_n has nodal degree of $n-1$, so as a direct application of the above lemma, we can get the lower bound for $\chi_{\bar{2}}(R_n)$ in the following corollary.

Corollary 1. $\chi_{\bar{2}}(R_n) \geqslant n$.

In the next section, we will show that the lower bound in the above corollary is actually achievable for any rotator digraph.

5. Coloring scheme for rotator digraphs

In this section, we will give a $\bar{2}$-VC for R_n under each multiplication which uses exactly n colors. Therefore, such $\bar{2}$-VC is optimal according to Corollary 1.

5.1. Multiplication is taken from left to right

In this section, we will give an optimal $\bar{2}$-VC for R_n under the multiplication "·".

Scheme-I Coloring the vertex α with color $\alpha^{-1}(1)$.

We first prove that the Scheme-I is a $\bar{2}$-VC for R_n under multiplication "·".

Lemma 2. *Scheme-I is a $\bar{2}$-VC for R_n under the multiplication "·".*

Proof. Let α and β be two vertices in R_n that have distance of at most two. We consider two cases.

Case 1: The distance between α and β is one. Then there exist some $2 \leqslant i \leqslant n$ such that $\beta = \alpha \cdot s_i$. In this case,

$$\beta^{-1} = (\alpha \cdot s_i)^{-1} = s_i^{-1} \cdot \alpha^{-1} = s_i \cdot \alpha^{-1}.$$

So we have

$$\beta^{-1}(1) = (s_i \cdot \alpha^{-1})(1) = \alpha^{-1}(s_i(1)) = \alpha^{-1}(i) \neq \alpha^{-1}(1).$$

This implies that α and β have different colors.

Case 2: The distance between α and β is two. Then there are two different integers $2 \leqslant i, j \leqslant n$ such that $\beta = \alpha \cdot s_i \cdot s_j$. In this case,

$$\beta^{-1} = (\alpha \cdot s_i \cdot s_j)^{-1} = s_j^{-1} \cdot s_i^{-1} \cdot \alpha^{-1} = s_j \cdot s_i \cdot \alpha^{-1}.$$

So we have

$$\begin{aligned}\beta^{-1}(1) &= (s_j \cdot s_i \cdot \alpha^{-1})(1) = \alpha^{-1}(s_i(s_j(1))) \\ &= \alpha^{-1}(s_i(j)) = \alpha^{-1}(j) \neq \alpha^{-1}(1).\end{aligned}$$

This also implies that α and β have different colors.

Thus in both cases α and β have different colors. Therefore, Scheme-I is a $\bar{2}$-VC for R_n under the multiplication "·". □

The number of colors used by Scheme-I is n. From Corollary 1 we can immediately obtain the following theorem.

Theorem 1. *Under the multiplication "·",*

$$\chi_{\bar{2}}(R_n) = n,$$

and Scheme-I is an optimal $\bar{2}$-VC for R_n.

A vertex coloring scheme is said to be *blanced* if each color is assigned to the same number of vertices. Balance is a desired property for vertex colorings. The next lemma shows that the Scheme-I has this nice property.

Lemma 3. *Scheme-I assigned each color to exactly $(n-1)!$ vertices.*

Proof. It is easy to see that for any $1 \leqslant i \leqslant n$, all the $(n-1)!$ vertices which map i to 1 have the color i. □

5.2. Multiplication is taken from right to left

In this section, we will give an optimal $\bar{2}$-VC for R_n under the multiplication "$\circ$".

Scheme-II Coloring the vertex π with color $\pi(1)$.

We first prove that the Scheme-I is a $\bar{2}$-VC for R_n under multiplication "$\circ$".

Lemma 4. *Scheme-II is a $\bar{2}$-VC for R_n under the multiplication* "$\circ$".

Proof. Let π and σ be two vertices in R_n. We consider two cases.

Case 1: σ is a parent of π. Then there exist some $2 \leqslant i \leqslant n$ such that $\pi = \sigma \circ \alpha_i$. In this case,

$$\pi(1) = (\sigma \circ \alpha_i)(1) = \sigma(\alpha_i(1)) = \sigma(2) \neq \sigma(1).$$

This implies that π and σ have different colors.

Case 2: σ and π have a common child. Then there are two different intergers $2 \leqslant i, j \leqslant n$ such that $\pi = \sigma \circ \alpha_i \circ \beta_j$. In this case,

$$\begin{aligned}\pi(1) &= (\sigma \circ \alpha_i \circ \beta_j)(1) = \sigma(\alpha_i(\beta_j(1))) \\ &= \sigma(\alpha_i(j)) \neq \sigma(\alpha_i(i)) = \sigma(1).\end{aligned}$$

This also implies that α and β have different colors.

Thus in both cases α and β have different colors. Therefore, Scheme-II is a $\bar{2}$-VC for R_n under the multiplication "$\circ$". □

The number of colors used by Scheme-II is also n. From Lemma 4 we can immediately obtain the following theorem.

Theorem 2. *Under the multiplication* "$\circ$",

$$\chi_{\bar{2}}(R_n) = n,$$

and Scheme-II is an optimal $\bar{2}$-VC for R_n.

Scheme-II also has the nice balance property as Scheme-I.

Lemma 5. *Scheme-I assigned each color to exactly $(n-1)!$ vertices.*

Proof. It is easy to see that for any $1 \leqslant i \leqslant n$, all the $(n-1)!$ vertices which map 1 to i have the color i. □

Another interesting property is that the $n-1$ children of any node has the same color.

Lemma 6. *In the coloring given by Scheme-II, the $n-1$ children of any node has the same color.*

Proof. We only need to prove that for any vertex π and any $2 \leqslant i \leqslant n$, the value of $(\pi \circ \alpha_i)(1)$ depends only on π rather than on i. In fact,

$$(\pi \circ \alpha_i)(1) = \pi(\alpha_i(1)) = \pi(2).$$

Therefore, the lemma is true. □

6. Conclusion

This paper studied a passive optical network based on rotator digraphs. The considered configuration is cluster-based and has the potential of combining both time and wavelength division multiplexing with a space-connected structure to achieve efficient scalability. The focus of this paper is on deriving optimal conflict-free channel sets assignments. For each of the two models of rotator digraph, we find an optimal conflict-free channel set assignment. Both schemes are simple and balanced.

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